CS109B Notes for Lecture 4/12/95

Single-Source Shortest Paths

Given a directed or undirected graphs with non-negative "lengths" of edges/arcs (= numeric labels), and given a *source* node s, find for each node v the shortest "distance" (= least sum of labels) of any path from s to v.

Dijkstra's Algorithm

Grows a region of *settled* nodes whose shortest distance from s is known.

- Inductive computation: For each node v, $dist(v)$ is the length of the shortest path to v that goes only through settled nodes (called a *special* path).
  - □ If v is settled, then $dist(v)$ is the correct shortest distance to v.

Basis: Initially, only s is settled.

- $dist(s) = 0$, and $dist(v)$ for other nodes v is either the length of an arc $s \rightarrow v$ or ∞ if there is no such arc.

Induction: Find the least $dist(v)$ for any v that is *not* settled.

1. Make v settled.
2. For every unsettled node u, see if there is now a shorter special path that goes through v, the newly settled node.
   - □ Compare $dist(u)$ with $dist(v)$ + the length of arc $v \rightarrow u$.
   - □ Replace $dist(u)$ with the latter, if the latter is smaller.

Why Does It Work? (FCS, pp. 504ff)

Intuition: if there were a shorter path from s to v, then it would first leave the settled region to some other node w.

- Thus, $dist(w) < dist(v)$.
- Note needed assumption that lengths are ≥ 0.

$O(n^2)$ Implementation

There are $n-1$ "rounds" in which a node is settled. In each round:

- $O(n)$ time to pick the smallest *dist* among unsettled nodes.
- $O(n)$ time to consider if other *dist* values need to be lowered.

$O(m \log n)$ Implementation (FCS, pp, 506ff)

Better if $m << n^2$ (i.e., the graph is *sparse*) and adjacency lists are used. Key ideas:

1. Keep *dist* in a priority queue, so we can find and delete the least distance of an unsettled node in $O(\log n)$ time.
   - ☐ Actually, "priority" is lowest-first here, not greatest-first.
   - ☐ When we lower $dist(u)$, the position of u in the PQ may change, so it will take $O(\log n)$ time to "bubbleup."
2. Count the work of updating successors u of the settled node v more carefully.
   - ☐ If v has m_v successors, then work is $O(m_v \log n)$ ($\log n$ for bubbling up for each of m_v nodes).
   - ☐ Thus, total update work $= \sum_v m_v \log n$ $= O(m \log n)$.
   - ☐ That is also the dominant term of the whole algorithm.

Class Problem

Suppose we have already computed $dist(v)$ for all nodes v. Now, we add another arc $y \to z$ with some length. Do we have to recompute all the distances, or can we take advantage of the old distances?