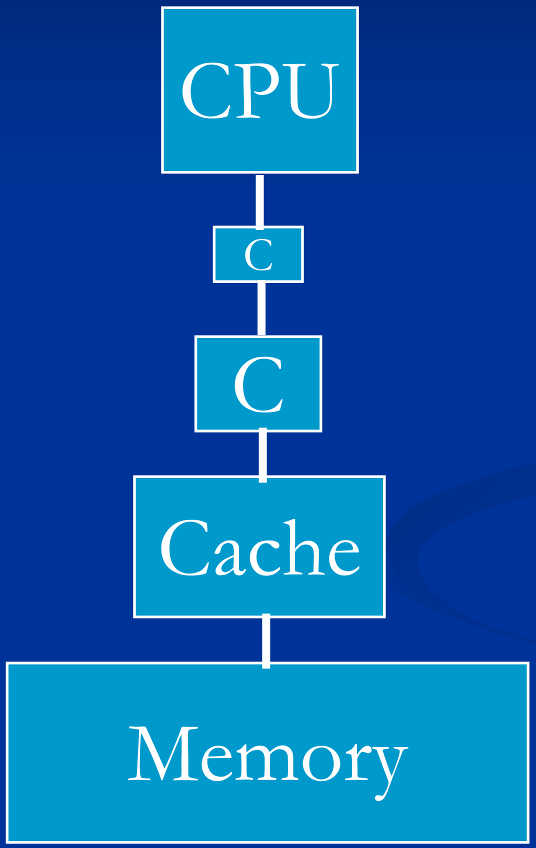
# Loop Transformations and Locality

# Agenda

- *Introduction*
- Loop Transformations
- Affine Transform Theory

# Memory Hierarchy



# Cache Locality

- Suppose array A has column-major layout

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A[1,1] | A[2,1] | … | A[1,2] | A[2,2] | … | A[1,3] | … |

```
for i = 1, 100
  for j = 1, 200
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

- Loop nest has **poor** spatial cache locality.

# Loop Interchange

- Suppose array A has column-major layout

| A[1,1] | A[2,1] | … | A[1,2] | A[2,2] | … | A[1,3] | … |
|---|---|---|---|---|---|---|---|

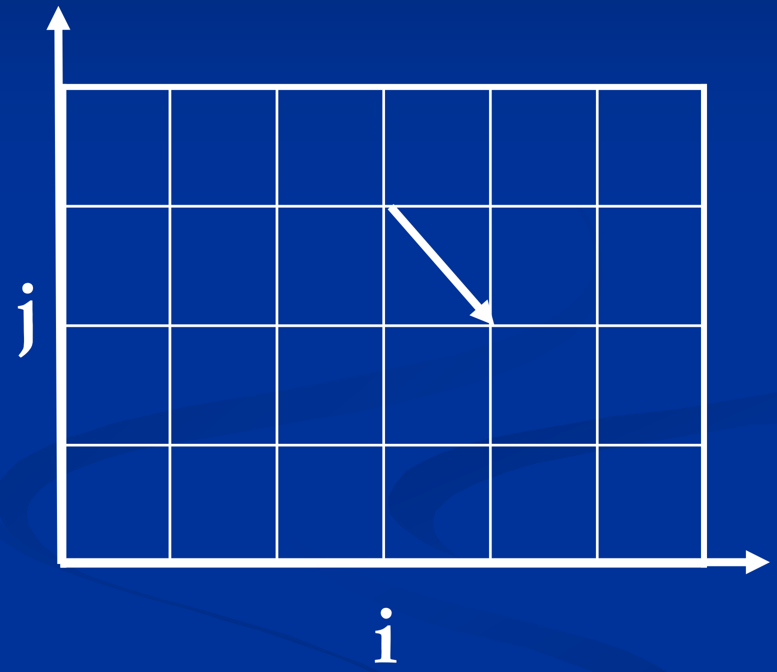
```
for i = 1, 100
  for j = 1, 200
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

```
for j = 1, 200
  for i = 1, 100
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

- New loop nest has **better** spatial cache locality.

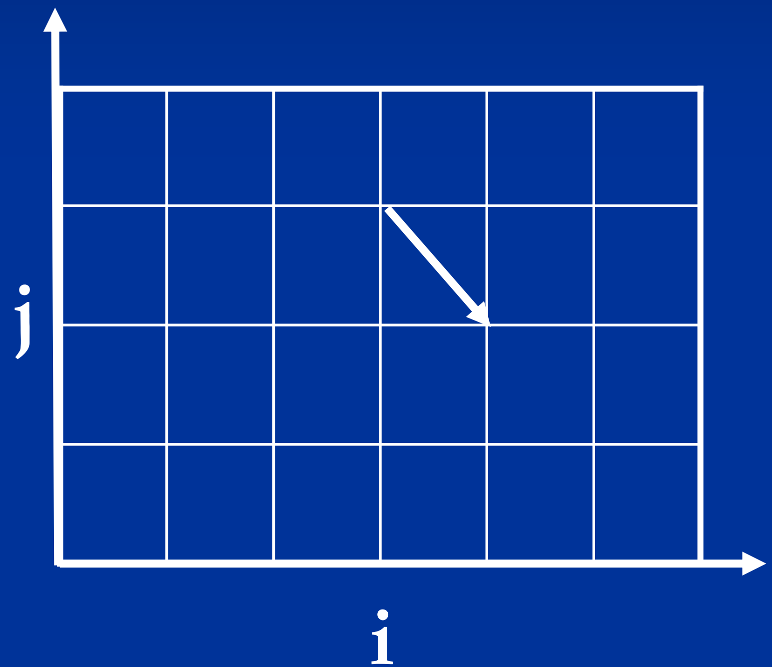
# Interchange Loops?

```
for i = 2, 100
  for j = 1, 200
    A[i, j] = A[i-1, j+1]+3
  end_for
end_for
```




- e.g. dependence from (3,3) to (4,2)

# Dependence Vectors




- Distance vector (1,-1) = (4,2)-(3,3)
- Direction vector (+, -) from the signs of distance vector
- Loop interchange is not legal if there exists dependence (+, -)

# Agenda

- Introduction
- *Loop Transformations*
- Affine Transform Theory

# Loop Fusion

```
for i = 1, 1000
  A[i] = B[i]  + 3
end_for

for j = 1, 1000
  C[j] = A[j]  + 5
end_for
```

```
for i = 1, 1000
  A[i] = B[i]  + 3
  C[i] = A[i]  + 5
end_for
```

- Better reuse between A[i] and A[i]

# Loop Distribution

```
for i = 1, 1000
  A[i] = A[i-1]  + 3
  C[i] = B[i]  + 5
end_for
```

```
for i = 1, 1000
  A[i] = A[i-1]  + 3
end_for

for i = 1, 1000
  C[i] = B[i]  + 5
end_for
```

- 2nd loop is parallel

# Register Blocking

```
for j = 1, 2*m
  for i = 1, 2*n
    A[i, j] = A[i-1, j]
+  A[i-1, j-1]
  end_for
end_for
```

```
for j = 1, 2*m, 2
  for i = 1, 2*n, 2
    A[i, j] = A[i-1,j] + A[i-1,j-1]
    A[i, j+1] = A[i-1,j+1] + A[i-1,j]
    A[i+1, j] = A[i, j] + A[i, j-1]
    A[i+1, j+1] = A[i, j+1] + A[i, j]
  end_for
end_for
```

- Better reuse between A[i,j] and A[i,j]

# Virtual Register Allocation

```
for j = 1, 2*M, 2
  for i = 1, 2*N, 2
    r1 = A[i-1,j]
    r2 = r1 + A[i-1,j-1]
    A[i, j] = r2
    r3 = A[i-1,j+1] + r1
    A[i, j+1] = r3
    A[i+1, j] = r2 + A[i, j-1]
    A[i+1, j+1] = r3 + r2
  end_for
end_for
```

- Memory operations reduced to register load/store
- 8MN loads to 4MN loads

# Scalar Replacement

```
for i = 2, N+1
       = A[i-1]+1
  A[i] =
end_for
```

```
t1 = A[1]
for i = 2, N+1
       = t1 + 1
  t1 =
  A[i] = t1
end_for
```

- Eliminate loads and stores for array references

# Unroll-and-Jam

```
for j = 1, 2*M
  for i = 1, N
    A[i, j] = A[i-1, j]
+  A[i-1, j-1]
  end_for
end_for
```

```
for j = 1, 2*M, 2
 for i = 1, N
  A[i, j]=A[i-1,j]+A[i-1,j-1]

  A[i, j+1]=A[i-1,j+1]+A[i-1,j]
 end_for
end_for
```

- Expose more opportunity for scalar replacement

# Large Arrays

- Suppose arrays A and B have row-major layout

```
for i = 1, 1000
  for j = 1, 1000
    A[i, j] = A[i, j]  +  B[j, i]
  end_for
end_for
```

- B has poor cache locality.
- Loop interchange will not help.

# Loop Blocking

```
for v = 1, 1000, 20
  for u = 1, 1000, 20
    for j = v, v+19
      for i = u, u+19
        A[i, j] = A[i, j]  +  B[j, i]
      end_for
    end_for
  end_for
end_for
```

- Access to small blocks of the arrays has good cache locality.

# Loop Unrolling for ILP

```
for i = 1, 10
  a[i] = b[i];
  *p = ...
end_for
```

```
for I = 1, 10, 2
  a[i] = b[i];
  *p = …
  a[i+1] = b[i+1];
  *p = …
end_for
```

- Large scheduling regions. Fewer dynamic branches
- Increased code size

# Agenda

- Introduction
- Loop Transformations
- *Affine Transform Theory*

# Objective

- Unify a large class of program transformations.
- Example:

```
float Z[100];
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Iteration Space

- A d-deep loop nest has d index variables, and is modeled by a d-dimensional space. The space of iterations is bounded by the lower and upper bounds of the loop indices.
- Iteration space i = 0,1, …9

```
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Matrix Formulation

- The iterations in a d-deep loop nest can be represented mathematically as
- $$\{\vec{i} \in Z^d \mid B\vec{i} + b \geq 0\}$$
  - Z is the set of integers
  - B is a d x d integer matrix
  - b is an integer vector of length d, and
  - 0 is a vector of d 0's.

# Example

```
for i = 0, 5
  for j = i, 7
    Z[j,i] = 0;
```

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- E.g. the $3^{rd}$ row $-i+j \geq 0$ is from the lower bound $j \geq i$ for loop j.

# Symbolic Constants

```
for i = 0, n
  Z[i] = 0;
```

$$\left\{ i \in Z \,\middle|\, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ n \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right.$$

- E.g. the 1st row $-i+n \ge 0$ is from the upper bound $i \le n$.

# Data Space

- An n-dimensional array is modeled by an n-dimensional space. The space is bounded by the array bounds.
- Data space a = 0,1, …99

```
float Z[100]
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Processor Space

- Initially assume unbounded number of virtual processors (vp1, vp2, …) organized in a multi-dimensional space.
  - (iteration 1, vp1), (iteration 2, vp2),…
- After parallelization, map to physical processors (p1, p2).
  - (vp1, p1), (vp2, p2), (vp3, p1), (vp4, p2),…

# Affine Array Index Function

- Each array access in the code specifies a mapping from an iteration in the iteration space to an array element in the data space
- Both i+10 and i are affine.

```
float Z[100]
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Array Affine Access

- The bounds of the loop are expressed as affine expressions of the surrounding loop variables and symbolic constants, and
- The index for each dimension of the array is also an affine expression of surrounding loop variables and symbolic constants

# Matrix Formulation

- Array access maps a vector i within the bounds to array element location Fi+f.

$$\{\vec{i} \in Z^d \mid B\vec{i} + b \geq 0\}$$

- E.g. access X[i-1] in loop nest i,j

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} - 1$$

# Affine Partitioning

- An affine function to assign iterations in an iteration space to processors in the processor space.
- E.g. iteration i to processor 10-i.

```
float Z[100]
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Data Access Region

- An affine function to assign iterations in an iteration space to processors in the processor space.
- Region for Z[i+10] is $\{a \mid 10 \le a \le 20\}$.

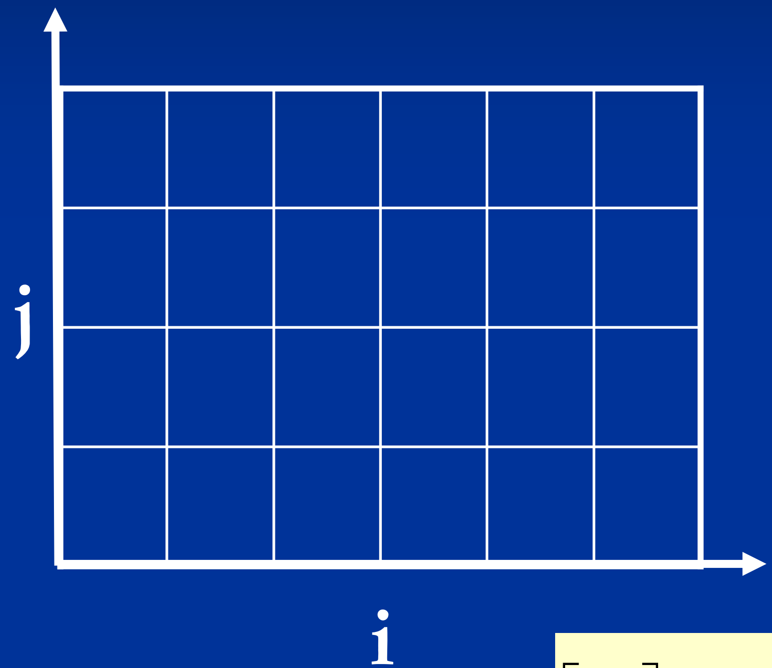
```
float Z[100]
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

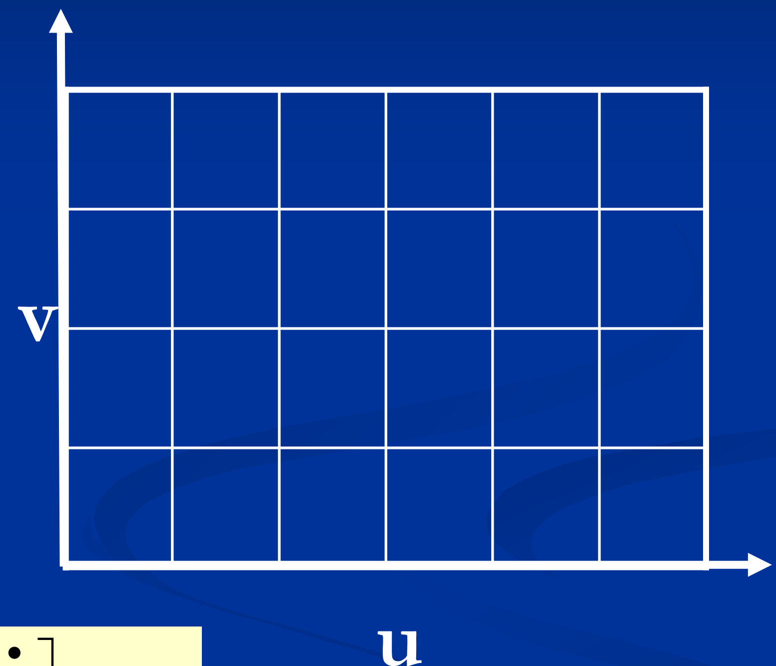
# Data Dependences

- Solution to linear constraints as shown in the last lecture.
  - There exist $i_r$, $i_w$, such that
  - $0 \le i_r, i_w \le 9$,
  - $i_w + 10 = i_r$

```
float Z[100]
for i = 0, 9
  Z[i+10] = Z[i];
end_for
```

# Affine Transform




$$\begin{bmatrix} u \\ v \end{bmatrix} = B \begin{bmatrix} i \\ j \end{bmatrix} + b$$

# Locality Optimization

```
for i = 1, 100
  for j = 1, 200
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

```
for u = 1, 200
  for v = 1, 100
    A[v,u] = A[v,u]+ 3
  end_for
end_for
```

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

# Old Iteration Space

```
for i = 1, 100
  for j = 1, 200
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} -1 \\ 100 \\ -1 \\ 200 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -1 \\ 100 \\ -1 \\ 200 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# New Iteration Space

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -1 \\ 100 \\ -1 \\ 200 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -1 \\ 100 \\ -1 \\ 200 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
for u = 1, 200
  for v = 1, 100
    A[v,u] = A[v,u]+ 3
  end_for
end_for
```

# Old Array Accesses

```
for i = 1, 100
  for j = 1, 200
    A[i, j] = A[i, j]  + 3
  end_for
end_for
```

$$A\left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}i \\ j\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}i \\ j\end{bmatrix}\right]$$

$$\begin{bmatrix}u \\ v\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}i \\ j\end{bmatrix}$$

$$A\left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}u \\ v\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}u \\ v\end{bmatrix}\right]$$

# New Array Accesses

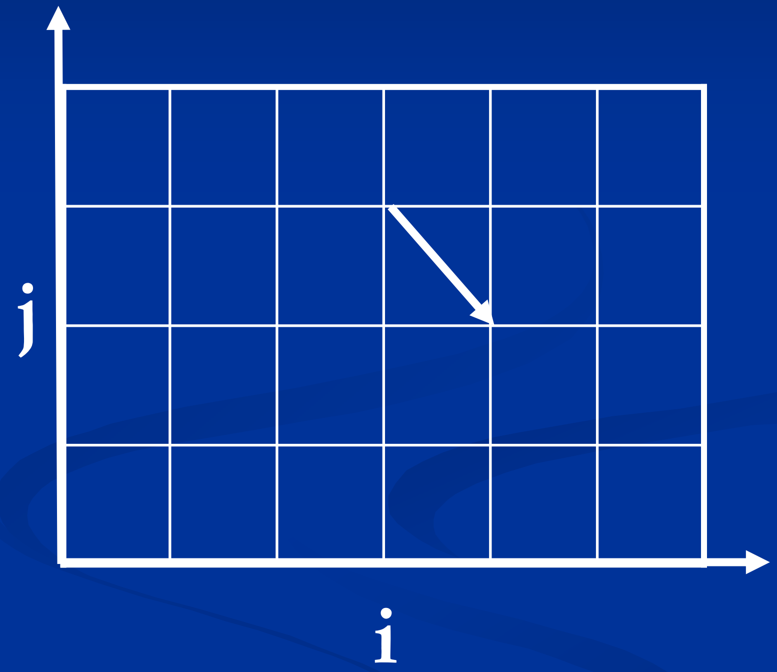
$$A\left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\right]$$

```
for u = 1, 200
  for v = 1, 100
    A[v,u] = A[v,u]+ 3
  end_for
end_for
```

$$A\left[\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix},\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}\right]$$

# Interchange Loops?

```
for i = 2, 1000
  for j = 1, 1000
    A[i, j] = A[i-1, j+1]+3
  end_for
end_for
```




- e.g. dependence vector $d_{old} = (1,-1)$

# Interchange Loops?

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

$$d_{new} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} d_{old} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- A transformation is legal, if the new dependence is lexicographically positive, i.e. the leading non-zero in the dependence vector is positive.

# Summary

- Locality Optimizations
- Loop Transformations
- Affine Transform Theory