CS109A Notes for Lecture 2/27/95

Sets

- Defined by *membership* relation $\in$.
- *Atoms* may not have members, but may be members of a set.
    - □ Sets may also be members of sets.
- Membership is "once only." An element cannot be a member of a set more than once.
    - □ Unlike lists, where (a) elements are ordered, and (b) elements may repeat.
    - □ Distinguished from *multisets* or *bags*, where elements are unordered but may appear more than once.

Why Sets?

- Model underlying most representation of information.
    - □ Structured sets represent record structures, tables.
    - □ Relational model used in database systems derives both its properties and its operations (querying, updating) from operations on sets.
- Probability space is a simple kind of set.
- Set theory underlies much (all?) of mathematics, explicating concepts such as real numbers, infinities.

Representing Sets

1. By *extension*: list the elements, surrounded by curly brackets.

Example: $\{1, 2, \{3, 4\}\}$ = set with three elements: the atoms 1 and 2 and the set $\{3, 4\}$.

2. By *abstraction*: describe the elements belonging to the set.

Example: $\{x \mid 3 \leq x \leq 10 \text{ and } x \text{ is an integer}\}$ $= \{3, 4, \ldots, 10\}$.

- Read: "the set of x such that $3 \leq x \leq 10 \cdots$."

Algebra of Sets

Principal operators on a pair of sets S and T:

1. Union: set of elements in S or T or both.
2. Intersection: set of elements in both.
3. Difference: $S - T$ = elements in S, not T.

- To an extent, these share algebraic laws with $+$, $\times$, and $-$, respectively.

Equality of Sets

- Two sets are equal if they have exactly the same members.
- Two expressions involving sets are *equivalent* ($\equiv$) if they produce the same value regardless of what values we assign to the set-variables in the expressions.

Algebraic Laws

These are observations about pairs of expressions that are equivalent.

- Commutative laws of union, intersection. The order of operands may be reversed.
- Associative laws of union, intersection. Operations may be grouped in any order.
  - □ Similar law for union and difference: $S - (T \cup R) \equiv (S - T) - R$
- Distributive laws: like $x(y + z) = xy + xz$ for arithmetic. But there are 3 different laws for sets:
  - □ $S \cap (T \cup R) \equiv (S \cap T) \cup (S \cap R)$
  - □ $S \cup (T \cap R) \equiv (S \cup T) \cap (S \cup R)$
  - □ $(S \cup T) - R \equiv (S - R) \cup (T - R)$

- The *empty set* $\emptyset$ has important properties:
  - $\emptyset$ is the identity for union: $S \cup \emptyset = S$.
  - It is also the *annihilator* for intersection $S \cap \emptyset = \emptyset$.
  - There is no identity for intersection or annihilator for union, because "set containing everything" does not exist.
  - $S - S = \emptyset$.
  - $\emptyset - S = \emptyset$.
- *Idempotence* laws: Union and intersection are idempotent, e.g., $S \cup S = S$.

Proving Equivalences

Three approaches:

1. Manipulating known equivalences.
2. Classifying elements by sets of which they are members.
   - Venn diagrams and truth-tables (in logic, Ch. 12, FCS) are instances of this approach.
3. Proving containments in both directions, using definitions of operators.

Manipulating Equivalences

Equivalence is preserved by:

- Substituting an expression for *all* occurrences of some variable in an equivalence.
- Replacing a subexpression in an equivalence by a known equivalent expression.
- Use of transitivity of equivalence: If $E \equiv F$ and $F \equiv G$ then $E \equiv G$.
- Use of commutativity of equivalence: If $E \equiv F$ then $F \equiv E$.

Example: Let's show $(S \cup T) \cap S \equiv S$.

$S \cup (T \cap R) \equiv (S \cup T) \cap (S \cup R)$	Dist. law
$S \cup (T \cap \emptyset) \equiv (S \cup T) \cap (S \cup \emptyset)$	$R \Rightarrow \emptyset$
$S \cup \emptyset \equiv (S \cup T) \cap S$	Ident., Annih.
$S \equiv (S \cup T) \cap S$	Ident.
$(S \cup T) \cap S \equiv S$	Comm of $\equiv$

Enumerating Cases

If there are n sets in an expression, we can divide elements into 2^n classes, depending on whether they are in/out of each set ("painting houses").

- A table decides whether an equivalence holds.

Example: $S \equiv (S \cup T) \cap S$.

S	T	$S \cup T$	RHS
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

Equivalence Through Containment

- $S \subseteq T$ (S is *contained* in T) means every element of S is an element of T.
- $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.
- We can prove the equivalence of two expressions by showing the result of each is contained in the other.

Example: $S \equiv (S \cup T) \cap S$.

$\Rightarrow$ If $x \in S$ then $x \in (S \cup T)$ (def. of "union"). Thus, $x \in \big((S \cup T) \cap S\big)$ (def. of "intersection").

$\Leftarrow$ If $x \in \big((S \cup T) \cap S\big)$, then $x \in S$ (def. of "intersection").