CS109B Notes for Lecture 6/2/95

Why Interpretations?

- Predicate logic provides an easy way to represent what a logical expression "means": its truth table.
- "Interpretations" are the things that correspond to rows of the truth table (= truth assignments).
  - □ While the number of truth assignments for a propositional expression is finite, the number of interpretations is infinite, making our job harder and less intuitive.

Meaning of a Propositional Logic Expression

We want to get to an understanding of what a predicate logic expression "means." To do so, we need first to review what a propositional logic expression means.

- E is a function from truth-assignments to {`TRUE`, `FALSE`}. In ML terms, its type is `(vars->bool)->bool`, where `vars` is the set of propositional variables in E.
  - □ Thus a truth assignment is a function from the variables to truth-values; you give it a variable like p and it tells you whether it thinks p is true or false.
  - □ Also thus, you give E a truth assignment as input, and it gives you back `TRUE` or `FALSE`.
- A tautology is an expression that always gives you back `TRUE`.

Meaning of Predicates

Propositional variables are simple; they can only take the value `TRUE` or `FALSE`. Predicates are more complex:

- They each have 1 or more arguments, and the type of arguments is arbitrary.
- The value of a predicate is a function from assignment of values for each of its arguments to {TRUE, FALSE}.
- In ML terms, the type of a predicate is

  ```
  (int->'a)->bool
  ```

  That is, the "input" to a predicate is a function that gives for each integer (from 1 up to the number of arguments) a value in some *domain* type `'a`. The output is `TRUE` or `FALSE`.

Interpretations

Like a truth assignment, an *interpretation* assigns a value to each symbol of an expression E that needs to be defined externally.

- These are the predicates and any free variables in E.
- The type of an interpretation, in ML terms, is

  ```
  (free->'a) * (preds->((int->'a)->bool))
  ```

  where `free` is the set of free variables and `preds` the set of predicates in E.

Meaning of a Predicate-Logic Expression

An expression E is a function from interpretations to {TRUE, FALSE}; i.e., its ML type is:

```
((free->'a) * (preds->((int->'a)->bool))) -> bool
```

Tautologies of Predicate Logic

An expression whose value is a function with range value `TRUE` for every argument is a *tautology*.

Computing the Meaning

We compute the meaning of an expression E for a given interpretation I by a structural induction on the expression tree for E.

- But a variable bound in E may be free in some subexpression of E.
  - □ Thus, the interpretation or interpretations applied to a subexpression may extend E with assignments of a value to some free variables.
- Propositional logic operators applied in the obvious way.
- $(\exists X)F$ evaluates to true if there is some value v in the domain of I such that F is `TRUE` under interpretation J, where:
  - □ $J = I$ extended to assign the free variable X the value v.
- $(\forall X)F$ evaluates to `TRUE` if the above is true for *every* v in the domain of I.

Example: Let E be $(\forall X)(\exists Y)p(X, Y)$.

- Let I have integers as domain, and let p be the function that is true iff its second argument is larger than its first argument.

1. Whole expression: We ask whether subexpression $(\exists Y)p(X, Y)$ is true for every interpretation I_j, where:
   - □ I_j is I extended to assign integer j to free variable X.
2. Subexpression $(\exists Y)p(X, Y)$: We ask whether there is some integer k such that $p(X, Y)$ is true under the interpretation I_{jk} that assigns j and k, respectively, to free variables X and Y.

2$'$. For any j we find that $k = j + 1$ makes $p(j, k)$ true, so the answer to (2) is "yes" for any j.

1$'$. Now we have our answer to (1): indeed $(\exists Y)p(X, Y)$ is true under every interpretation I_j.

- Thus, E is true under interpretation I.
- Note: this does not mean E is a tautology. *It isn't.* It is just true under this interpretation.

- There are many interpretations under which E is false, e.g. let the domain be the complex numbers and let $p(X, Y)$ be true if the magnitude of Y is less than the magnitude of X.
  - □ $(\exists Y)p(X, Y)$ is false for $X = 0 + 0i$.

Class Problem

Consider expression

$$(\exists X)p(X) \rightarrow (\forall X)p(X)$$

a) What does it intuitively say?

b) Give one interpretation for which it is true, one for which it is false.

  - □ Hint: if you are having trouble thinking about integer or real domains, try a finite domain.