CS109A Solutions for Class Quiz #1

February 8, 1995

Problem 1:

a) $n = 2$ is the basis.
b) When $n = 2$, $n! = 2$ and $n^n = 4$. Thus, $2! < 2^2$, which is $S(2)$.
c) $S(n+1)$ is $(n+1)! < (n+1)^{n+1}$.
d) We may assume $S(n)$, i.e., $n! < n^n$. Multiply both sides by $n+1$. Then $(n+1)! < (n+1)n^n$. Surely $n^n \leq (n+1)^n$, so $(n+1)! < (n+1)^{n+1}$; the latter is $S(n+1)$.

Problem 2:

a) $S(n)$ is "If L_1 is of length n, then $longer(L_1, L_2)$ returns value `true` if and only if L_2 is shorter than length n."
b) The basis must be $n = 0$.
c) If the first argument is `nil` (has length 0), then the pattern of rule (1) matches, and `longer` returns `false`. That is correct, since no matter what L_2 is, it cannot be shorter than `nil`.
d) Assume $S(n)$ and let the first argument have length $n+1$. Since $n \geq 0$, the first argument cannot be `nil`, and line (1) cannot match. If line (2) matches, then the second argument is `nil` and `longer` correctly returns `true`. Otherwise, both lists are nonempty, and the pattern of line (3) calls `longer` on the tails. By the inductive hypothesis, since `xs` is of length n, this call correctly tells whether or not `xs` is longer than `ys`. But `x::xs` is longer than `y::ys` if and only if `xs` is longer than `ys`. Thus, the call to `longer` with first argument `x::xs` of length $n+1$ correctly tells whether the first argument is longer.

Problem 3:

```
val longer = fn : 'a list * 'b list -> bool
```

Problem 4:

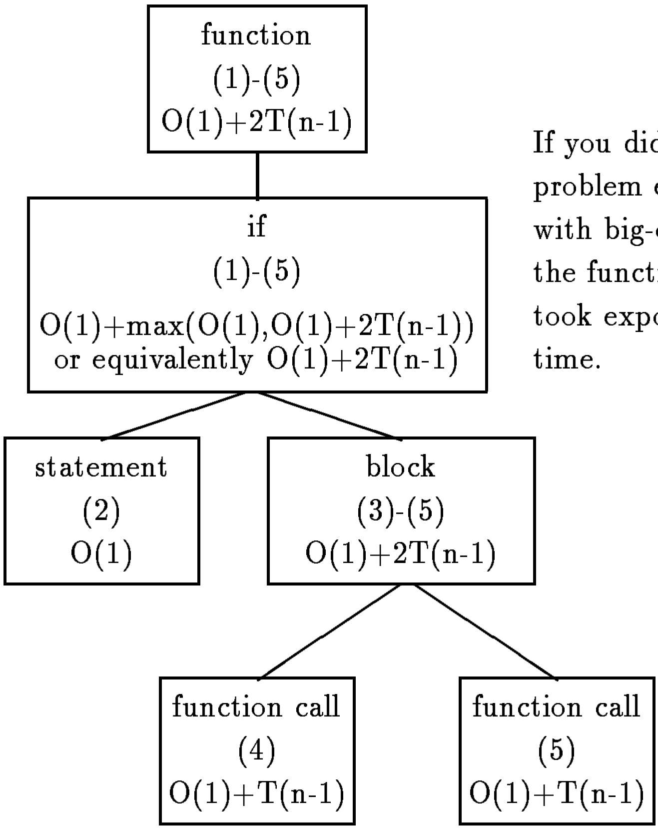
```
fun flip [] = []
 |  flip [x] = [x]
 |  flip (x::y::r) = y::x::(flip r);
```

Problem 5:

a) The answer was 15/4 or 4 if you gave an integer. One way to derive this answer is to start from the definition of big-oh $n^2 + 3n + 5 \leq cn^2$ and divide through by n^2 to get $1 + 3/n + 5/n^2 \leq c$, which is equivalent since $n \neq 0$. Notice that the left-hand side decreases as n increases, so the largest value it can attain is when n is as small as possible. Since $n_0 = 2$, we plug in 2 and get that c must be greater or equal to 15/4.
b) The answer was $(3 + \sqrt{29})/2$ (about 4.19) or 5 if you gave an integer. Again, working from the definition of big-oh, we start with $n^2 + 3n + 5 \leq 2n^2$, which is equivalent to $0 \leq n^2 - 3n - 5$. Recalling our grade-school algebra, we know that $n^2 - 3n - 5$ is a parabola that opens upward, so anything to the right of the larger of the two roots is positive. Plugging into the quadratic formula $(-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)})/2$ and choosing the larger root gives the desired answer.

Problem 6:

a)



If you did this problem entirely with big-ohs, the function calls took exponential time.

b)

$$
\begin{aligned}
T(0) &= c_1 \quad \text{for some constant } c_1 \text{ (Basis)} \\
T(n) &= c_2 + 2T(n-1) \quad \text{for some constant } c_2 \text{ and } n > 0
\end{aligned}
$$

Problem 7:

a) $T(0) = 2$; $T(1) = 2$; $T(2) = 4$; $T(3) = 12$; $T(4) = 48$.

b) $T(n) = n(n-1)T(n-2)$.

c) $T(n) = n(n-1)(n-2)T(n-3)$.

d) $T(n) = \frac{n!}{(n-i)!}T(n-i)$.

e) $i = n$.

f) $T(n) = 2n!$.