CS109B Quiz
Friday, May 3, 2:15–3:05 PM

Directions

Answer all 5 questions. The total number of points is 100. In each question, except (1), where a complete proof is requested, justify your answers with the appropriate formulas, principles, and/or explanations. Please be brief. Do not forget to **sign the pledge**.

Problem 1: (20 points) If T is a binary tree, let $I(T)$ be the number of interior nodes in T, that is, the number of nodes that have a left-child, a right-child, or both. Let $R(T)$ be the number of nodes that have a right-sibling. A node has a right sibling if it is a left-child of its parent, and that parent also has a right-child. Finally, let $N(T)$ be the number of nodes of T. Give an inductive proof that $I(T) + R(T) = N(T) - 1$. Your induction could be on the height of T or on the number of nodes of T, for example. State clearly what your induction parameter is, what the basis is, and what the inductive step is.

Problem 2: (15 points)

a) How many ways are there to assign nonnegative integers to x, y, and z, so that $x + y + z = 12$, and $x = y$?

b) How many ways are there to assign nonnegative integers to x, y, and z, so that $x + y + z = 12$, and $x > y$?

Problem 3: (30 points) Suppose we have a graph consisting of n nodes, named $1, 2, \ldots, n$, with edges $\{i, i+1\}$ for all $i = 1, 2, \ldots, n-1$. That is, the graph looks like

$$1 - 2 - 3 - 4 - \cdots - n$$

We wish to partition the nodes into two sets A and B, with node 1 in A. In parts (a), (b), and (c), there is no constraint on the number of nodes in the sets A and B (but node 1 is in A, so that set cannot be empty).

a) In how many ways (as a function of n) can we assign the nodes to the two sets? (Remember that here, as in all parts, node 1 is in set A.)

b) Suppose we require that there be exactly one edge that connects a node in A with a node in B; all other edges are either between a pair of nodes in A or a pair of nodes in B. How many ways to assign nodes to the sets are there?

c) Suppose we require that there be exactly 3 edges that connect nodes in A with nodes in B; all other edges are either between a pair of nodes in A or a pair of nodes in B. How many ways to assign nodes to the sets are there?

In the next three parts, we assume n is even and require that there be exactly $n/2$ nodes in each of the sets A and B.

d) Repeat part (a), on the assumption that A and B are each required to have exactly $n/2$ nodes.

e) Repeat part (b), on the assumption that A and B are each required to have exactly $n/2$ nodes.

f) Repeat part (c), on the assumption that A and B are each required to have exactly $n/2$ nodes.

Problem 4: (20 points) Suppose we wish to form a string of 10 beads, each bead red, blue, or green. How many strings are there such that there is at least one color that is used for exactly three beads? Note that in a string, the order of beads is important.

Problem 5: (15 points) How many anagrams are there for the word `axolotl`?