CS109A Final Examination
Wednesday, Dec. 12, 8:30–11:30 am

Directions

The exam is *open book*; any written materials may be used. Answer all 16 questions on the exam paper itself. The total number of points is 300. You may make additional comments at the bottoms of pages or at the end if you feel the need to clarify or justify your answers. Do not forget to **sign the pledge** below.

I acknowledge and accept the honor code.

____

Print your name here: ____

1		5		9		13	
2		6		10		14	
3		7		11		15	
4		8		12		16	

Problem 1: (30 points) We are going to discuss the existence or nonexistence of certain kinds of relations over the particular domain $\{a, b, c\}$. Remember that for a relation R to be over this domain, both the first and second components of each pair in R must be either a, b, or c, but it is *not necessary* that each of these three elements appear as a first component and/or second component.

To facilitate your answers, we provide a space for you to draw relations. Each space has "nodes" labeled a, b, and c. To express the idea that xRy, draw an arrow from the node for x to the node for y, i.e., $x \to y$.

For each of the combinations of properties (or their negation) listed below, either draw a relation that has all those properties, or write "none exists."

a) Antisymmetric and transitive, but not reflexive.

$a \qquad b \qquad c$

b) Reflexive and antisymmetric, but not transitive.

$a \qquad b \qquad c$

c) Reflexive and transitive, but not antisymmetric.

a b c

d) Antisymmetric and symmetric.

a b c

e) Reflexive, transitive, and a function.

a b c

f) Antisymmetric and a one-to-one function.

a b c

Problem 2: (20 points) Suppose we represent sets by sorted lists of elements. An element may appear only once on a list. Write an ML function `diff(S,T)` that computes the difference of two sets S and T represented in this way and returns the result as a sorted list with no duplicates.

Problem 3: (10 points) Below we see a sequence of steps used to prove the statement "if x is in $S-(T \cup R)$, then x is in $(S-T)-R$." For each, you need to provide a reason, which is either:

a) Given.
b) Definition of "difference."
c) Definition of "union."

Note: The problem is scored by deducting 2 points for each unanswered part and 4 points for each wrong answer, with a minimum of 0.

(1) x is in $S - (T \cup R)$._______________
(2) x is in S._______________
(3) x is not in $T \cup R$._______________
(4) x is not in T._______________
(5) x is in $S - T$._______________
(6) x is not in R._______________
(7) x is in $(S - T) - R$._______________

Problem 4: (15 points) Now, write your own proof of the converse of Exercise (3), that is: "if x is in $(S - T) - R$, then x is in $S - (T \cup R)$.

Problem 5: (20 points) Consider the fragment of Pascal code

```
(1) sum := 0;
(2) for i := 1 to n do
(3)     sum := sum +A[i];
```

We assume that there is an integer array A with elements 1 to n, available to this code. We wish to prove that at the end of this code, sum is the sum of the elements in $A[1]$ through $A[n]$. To that end, let statement $S(j)$ be "if we reach the for-loop test $i > n$, with i having the value j, and j is between 1 and $n + 1$, then sum equals $A[1] + A[2] + \cdots + A[j - 1]$."

a) What value of j is the basis?_______________

b) Prove the basis.

c) What is the inductive step?

d) Prove the inductive step.

Problem 6: (20 points) Suppose that we represent a binary tree by left-child and right-child pointers, and we wish to prove the following statement $S(T)$ by *structural induction*. "If T is a nonempty binary tree with n nodes, then there are $n+1$ `NIL` pointers among all its nodes." *Warning*: You will receive zero credit if you give a proof that is *not* a structural induction.

a) What is the basis case?

b) Prove the basis case here.

c) State the inductive step here.

d) Prove the inductive step here.

Problem 7: (20 points) Consider the four functions

f_1: $2n^2 + 10$.
f_2: n^3.
f_3: n^2 if n is odd, and n^3 if n is even.
f_4: n^2 if n is a prime and n^3 if n is composite.

For each of the following statements, answer either "true" or "false." If "false," you need do nothing further. If "true," you must provide a value of c and a value of n_0 that proves the big-oh relationship. In what follows, it may help to remember that every prime except 2 is odd.

a) $f_1(n)$ is $O(f_2(n))$.____

b) $f_1(n)$ is $O(f_3(n))$.____

c) $f_3(n)$ is $O(f_4(n))$.____

d) $f_4(n)$ is $O(f_3(n))$.____

Problem 8: (20 points) Consider the following fragment of Pascal code, which happens to initialize the n by n array A to a particularly messy matrix and then turns it into an upper triangular matrix (it will actually fail for reasonably sized n, because the problem is numerically unstable, but that's irrelevant, here).

```
begin
    for i := 1 to n do
        for j := 1 to n do
            A[i,j] := i+j;
    for i := 1 to n-1 do
        for j := i+1 to n do
            for k := i to n do
                A[j,k] := A[j,k] - A[i,k]*A[j,i]/A[i,i]
end
```

a) In the space below, draw a tree grouping the statements as in Fig. 3.13 on p. 106 of the notes.

b) For each statement in your answer to (a) show, next to its node, a smooth and tight big-oh bound on the running time of that statement. All your expressions can be functions of the value of n only.

Problem 9: (20 points) Consider the following ML function `foo`.

```
fun foo [] = []
| foo [x] = [x]
| foo (x::y::tail) = [x]@foo(y::tail)@foo(tail)
```

This function takes a list of length n and returns a list of some other length $L(n)$.

a) What is `foo([1,2,3,4])`?

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b) Write a recurrence relation for $L(n)$ below. You need not solve the recurrence. Do not forget the basis case(s).

Problem 10: (20 points) Consider the recurrence

BASIS: $T(0) = 1$.

INDUCTION: $T(n) = 3T(n-1) - 2$ for integers $n > 0$.

Note that since there are no symbolic constants, this is not a "big-oh" problem; we can solve for $T(n)$ exactly.

Also note that the answers below should be simplified. For example, replace arithmetic or geometric sums by their equivalents, and combine terms in algebraic expressions whenever you can.

a) To begin, what are the initial values of $T(n)$?

$T(0) =$________; $T(1) =$________; $T(2) =$________; $T(3) =$________;

b) Expand the inductive rule so $T(n)$ is expressed in terms of $T(n-2)$.

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c) Express $T(n)$ in terms of $T(n-3)$.

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d) What is the general pattern? That is, express $T(n)$ in terms of $T(n-i)$.

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e) For what value of i can we eliminate $T(n-i)$ from the expression? $i =$________

f) Use your answer to (e) to express $T(n)$ as a function of n alone (i.e., without any terms involving the function T).

$T(n) =$________________.

Problem 11: (10 points) Suppose the Oxford English Dictionary contains 1,000,000 words, and that among these words there are 5,000,000 different prefixes. Let us store these words in a trie with branching factor 26.

a) How many nodes are there in the trie?________________

b) How many `NIL` pointers are there in the trie?________________

Problem 12: (10 points) If a trie has height 4 and branching factor 10, what is the least number of nodes it can have?________________ What is the largest number of nodes it can have?________________

Problem 13: (20 points) There are a number of data structures besides a heap that we could use to implement the ADT PRIORITY QUEUE.

I. A linked list, with the elements sorted, *largest* first (i.e., in the reverse of the usual smallest-first order).

II. A binary search tree.

For each of these structures, suppose that n is the number of elements currently in the priority queue. In the table below, give tight worst-case and average-case big-oh bounds on the running time of the two priority-queue operations *insert* and *deleteMax*.

Data Structure	Worst-Case *insert*	Average *insert*	Worst-Case *deleteMax*	Average *deleteMax*
Sorted Linked List				
Binary Search Tree				

Problem 14: (20 points) Define relation R on the domain of pairs of real numbers as follows. $(a, b)R(c, d)$ if either $a = c$ or $b = d$ (or both).

a) Describe the set of points (x, y) such that $(3, 4)R(x, y)$. That is, explain what they *look like* in the plane; don't just substitute $a = 3$ and so on, in the definition of R.

b) Is R reflexive? Prove or give a counterexample to show it is not.

c) Is R symmetric? Prove or give a counterexample to show it is not.

b) Is R transitive? Prove or give a counterexample to show it is not.

Problem 15: (30 points) *Note*: It is suggested that you work this out on scratch paper first. Consider the recurrence

BASIS: $T(1) = d$.

INDUCTION: $T(n + 1) = T(n) + cn^2$ for integers $n \geq 1$.

Suppose we wish to prove, by induction on n, an upper bound statement of the form $S(n)$:

"$T(n) \leq an^3 + bn^2$." We do so by establishing some inequalities among a, b, c, and d that allow both the basis and the inductive step to be carried out.

a) What inequality is sufficient to prove the basis, $S(1)$?

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b) For the inductive step, we need to assume that $S(n)$ holds. That lets us establish an upper bound on $T(n+1)$, by using the inductive part of the definition of T. What is this upper bound on $T(n+1)$?

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c) This upper bound on $T(n+1)$ must, in turn, be less than or equal to the bound on $T(n+1)$ that is asserted by statement $S(n+1)$. Write the resulting inequality here.

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d) Simplify your answer to (c). It should be written in the form $fn^2 + gn + h \geq 0$, where f, g, and h are functions of a, b, and c only.

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e) One way to make sure the inequality (d) is satisfied is to require that f, g, and h all be greater than or equal to 0. What are the resulting inequalities involving a, b, and c?

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f) Assuming c and d are *positive* constants, is it always possible to find values of a and b that satisfy your inequalities from parts (a) and (e)?____________ If so, give formulas for a and b in terms of c and d that satisfy these inequalities.

Problem 16: (15 points) Quickies:

a) If the tail of the tail of list L is empty, how long is L?____________

b) If a binary tree has one leaf and three nodes that are not leaves, what is the height of the tree?____________

c) If $R \cup (T \cap S) = \emptyset$, what is R?____________