CS109B Notes for Lecture 5/31/95

Predicates

Essentially Boolean-valued functions with arguments of arbitrary type.

- But predicates are *uninterpreted*; a predicate named *less*, for example, need not give $less(3,4)$ the value `TRUE`.
- In the deeper realms of logic, one forces a predicate like *less* to be what one wants by asserting expressions about it that can only be satisfied by a predicate that behaves as you intend.
- But back here in the real world, that is too hard. Thus we use extra-logical means to explain and use the "meaning" of a symbol.
  - □ E.g., we said p stands for "T is a MWST" and spoke informally about what that meant, while still using formal logic for matters like the contrapositive law.

Example: We might assert a logical expression like

$$less(X,Y) \text{ AND } less(Y,Z) \rightarrow less(X,Z)$$

i.e., the transitive law for predicate *less*.

- That narrows down somewhat what *less* can be, but it still could be "greater than," "equals," or any transitive relation.

Logical Expressions: The Predicate Logic Case

Basis: An *atomic formula* is a logical expression. These are predicate symbols applied to *arguments*, which are either variables or constants.

- Convention: predicate names and constants begin with a lower-case letter, while variables begin with an upper-case letter.
- Numbers and (quoted) character strings are also constants.

Example: Here are some atomic formulas: $p(X, Y)$, $q(0, X, a)$, p.

- The second has first and third arguments constant.
- p is a zero-ary predicate; it is essentially the same as a propositional variable, since its value is either TRUE or FALSE, independent of any arguments.

Induction: Logical expressions can be built from smaller logical expressions by

1. The usual logical connectives: AND, $\rightarrow$, etc.
2. The *quantifier* $\forall$ ("for all"). It is used in expressions like $(\forall X)p(X, X)$, i.e., "for all X, $p(X, X)$ is true.
   - □ That might be the case if, say, p were the predicate $\geq$, i.e., "for all X, $X \geq X$."
3. The quantifier $\exists$ ("there exists"). It is used in expressions like $(\exists Y)\big(p(X, Y)\texttt{AND}p(Y, Z)\big)$, i.e., "there exists a value of Y such that both $p(X, Y)$ and $p(Y, Z)$ are true.
   - □ That might make sense if, e.g., p were the predicate $<$, and $X \neq Z$.

Class Problems

Suppose that $lt(X, Y)$ is the predicate that is true iff $X < Y$ and $ne(X, Y)$ is true iff $X \neq Y$. Write logical expressions for the following:

1. "For all X other than 0, there is some Y such that $0 < Y < X$."
2. "There is some X such that for all Y and Z, X is equal to neither Y nor Z."

Are these expressions true or false?

Bound/Free Variables

Think of a quantified expression $(\forall X)E$ or $(\exists X)E$ as a "declaration" of X that applies to the expression E.

- Uses of X within E are said to be *bound* to that quantification of X.

- But another quantification of X within E supercedes the outer quantification.

  - ☐ Analogous to a local definition of `x` within a C or ML function superceding a global or external declaration of `x`.

- A use of a variable that has no associated quantification within an expression E is said to be *free* in E.

  - ☐ I.e., a free variable is like an external variable in C.

Example: Consider:

$$(\forall X)\Big((\exists Y)\big((\forall X)p(X,Y) \texttt{ AND } q(X,Y)\big)\Big)$$

- Convention: quantifiers have highest precedence and so bind only the shortest well-formed expression that follows them.

  - ☐ Thus, the innermost quantified expression is just $(\forall X)p(X,Y)$.

  - ☐ Note: X is bound (to the $(\forall X)$ in this subexpression; Y is free.

- Here is the same expression with bindings of variables to quantifiers indicated by subscripts.

  - ☐ You may think of the subscripted variables as distinct variables. As with local variables in C, you can rename them at will, as long as you don't accidently use a name that has another declaration at that point.

$$(\forall X_1)\Big((\exists Y_2)\big((\forall X_3)p(X_3,Y_2) \texttt{ AND } q(X_1,Y_2)\big)\Big)$$

What does this expression "mean"? Roughly:

- $(\forall X)p(X,Y)$ is true for a given value of Y if no matter what value X has, $p(X,Y)$ is true.

  - ☐ Call this condition $S(Y)$.

- □ We don't know what p means, so we don't know whether $S(Y)$ is true, but for a given p we could decide whether $S(Y)$ is true.

- $(\exists Y)\big((\forall X)p(X,Y)$ AND $q(X,Y)\big)$ is true for any given X if there is some Y such that

  1. $S(Y)$ is true, and
  2. $q(X,Y)$ is true.

  □ Call this statement $T(X)$. Again, we don't know how to tell whether $T(X)$ is true, but given p and q we could decide.

- The entire statement says that $T(X)$ is true for every X.