On Exact and Approximate Cut Covers of Graphs

Rajeev Motwani *
Computer Science Department
Stanford University
Stanford, CA 94305

Joseph (Seffi) Naor
Computer Science Department
Technion
Haifa 32000, Israel

Abstract

We consider the minimum cut cover problem for a simple, undirected graphs $G(V, E)$: find a minimum cardinality family of cuts $\mathcal{C}$ in G such that each edge $e \in E$ belongs to at least one cut $C \in \mathcal{C}$. The cardinality of the minimum cut cover of G is denoted by $c(G)$. The motivation for this problem comes from testing of electronic component boards.

Loulou has shown that the cardinality of a minimum cut cover in the complete graph is precisely $\lceil \log n \rceil$. However, determining the minimum cut cover of an arbitrary graph was posed as an open problem by Loulou. In this note we settle this open problem by showing that the cut cover problem is closely related to the graph coloring problem, thereby also obtaining a simple proof of Loulou's main result. We show that the problem is NP-complete in general, and moreover, the approximation version of this problem still remains NP-complete. Some other observations are made, all of which follow as a consequence of the close connection to graph coloring.

*Supported by an IBM Faculty Development Award, an OTL grant, and NSF Young Investigator Award CCR-9357849, with matching funds from IBM, Schlumberger Foundation, Shell Foundation, and Xerox Corporation.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. A cut C in G is a partition of V into two disjoint subsets, V_r and V_ℓ, such that $V = V_r \cup V_\ell$. Equivalently, C is the set of edges connecting V_r and V_ℓ. In the sequel, we will consistently use n to refer to the number of vertices of the graph under consideration, and all logarithms will be to base 2.

Loulou [Lou92] introduced the minimum cut cover problem: find a minimum cardinality family of cuts $\mathcal{C}$ such that each edge $e \in E$ belongs to at least one cut $C \in \mathcal{C}$. The cardinality of the minimum cut cover of G is denoted by $c(G)$. The motivation for this problem comes from testing of electronic component boards. (The interested reader is referred to Loulou [Lou92] for more details.)

Loulou showed that the cardinality of a minimum cut cover in the complete graph is precisely $\lceil \log n \rceil$. However, determining the minimum cut cover of an arbitrary graph was posed as an open problem by Loulou. In this note we settle this open problem by showing that the cut cover problem is closely related to the graph coloring problem, thereby also obtaining a simple proof of Loulou's main result. We show that the problem is NP-complete in general, and moreover, the approximation version of this problem still remains NP-complete. Some other observations are made, all of which follow as a consequence of the close connection to graph coloring.

An interesting open question is to study the complexity of this problem when generalized to the case of weighted graphs. For example, we could define a cost function over the cuts in the graph, and the problem would then be to find a minimum cost collection of cuts that covers the graph. Although the hardness results given here still apply using a unit cost function, it is possible that the approximation version of this problem is significantly harder.

2 The Complexity of Minimum Cut Cover

We start by proving two lemmas relating the cut cover number $c(G)$ with the chromatic number $\chi(G)$.

Lemma 2.1 *For any graph G, $\chi(G) \leq 2^{c(G)}$.*

Proof: Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a set of cuts in G. We construct from $\mathcal{C}$ a labeling of the vertices of G as follows. Let V_r^i and V_ℓ^i be the partition of V induced by cut C_i. We now associate with each vertex $v \in V$ a label $l(v)$ from $\{0,1\}^k$. The i'th position in $l(v)$ is equal to 0 (1) if vertex v belongs to V_ℓ^i (V_r^i). We claim that $\mathcal{C}$ is a cut cover of G if and only if, for each edge $e = (v, w)$, $l(v) \neq l(w)$. This follows by observing that if $l(v) \neq l(w)$, then there exists a position, say i, in which they differ. But, then, cut C_i must cover edge e.

We can now interpret the labels of the vertices as colors, and it is easy to see that this is a legal vertex coloring of the graph using at most 2^k colors. □

Lemma 2.2 *For any graph G, $c(G) \leq \lceil \log \chi(G) \rceil$.*

Proof: Consider any k-coloring of G, and assume that the color labels are $\lceil \log k \rceil$-bit strings. This defines a cut cover of size $\lceil \log k \rceil$, when we interpret the i'th bit in a color as the indicator for the i'th cut. □

By these lemmas we have that $c(G) = \lceil \log \chi(G) \rceil$. The following theorem settles the open problem posed by Loulou.

Theorem 2.1 *The problem of deciding for a graph G and integer k whether $c(G) \leq k$ is NP-complete.*

Proof: Recall that for any fixed $k \geq 3$, it is NP-hard to decide if a graph is k-colorable [GJ79]. In particular, it is NP-hard to decide whether a graph is 4-colorable. But note that a graph is 4-colorable if and only if it has a cut cover of cardinality at most 2. From this it follows that determining the cardinality of the minimum cut cover is an NP-complete problem. □

Corollary 2.1 *An optimal cut cover of a planar graph can be computed in polynomial time, and its cardinality is equal to 2 unless the graph is bipartite.*

Proof: First note that we can check in polynomial time if a graph is bipartite, which happens if and only if $c(G) = 1$. The lemma follows by observing that a planar graph can be colored by 4 colors in polynomial time [SK86], and therefore non-bipartite planar graphs have chromatic number either 3 or 4. □

We remark that the NP-hardness result also applies for a variety of special types of graphs, as enumerated by Garey and Johnson [GJ79]. For example, the problem remains NP-hard for intersection graphs of line segments in the plane, circle graphs, and circular-arc graphs. These special cases are of interest as it is likely that in applications, such as electronic component boards, the graphs may possess a special structure.

It is interesting to note that for graphs for which the maximum degree Δ is bounded by a constant, there exists a cut cover which has cardinality bounded by $\lceil \log(\Delta+1) \rceil$. This follows from the fact that such graphs can be colored by at most $\Delta + 1$ colors.

3 NP-hardness of Approximations

We now make use of the recent results on approximate graph coloring to obtain corresponding results for the cut cover problem. First, we focus on approximations with an additive error. Making use of the (weak) approximation algorithm for graph coloring due to Halldorsson [Hal90] that achieves a ratio of $O(n(\log\log n)^2/(\log n)^3)$, we obtain the following theorem.

Theorem 3.1 *There exists a polynomial time algorithm which can find an approximate cut cover of size* $c(g) + \log n - 3\log\log n$.

It is known that this algorithm cannot be improved significantly. In particular, based on the results of Arora, Lund, Motwani, Sudan and Szegedy [ALMSS92], it was shown by Lund and Yannakakis [LY93] that there exists a constant $\epsilon > 0$ such that no polynomial time algorithm can approximate the chromatic number of a graph to within a ratio of n^ϵ, unless P = NP. As a consequence, we obtain the following theorem which pins down the approximability of the minimum cut cover problem rather precisely.

Theorem 3.2 *There exists a constant* $\epsilon > 0$ *such that no polynomial time algorithm can approximate the minimum cut cover of a graph to within* $c(G) + \epsilon \log n$*, unless P = NP.*

Consider now the problem of approximations with a multiplicative error. It is easy to see that we can approximate the minimum cardinality cut cover within a ratio of $\log n$, since there always exists a cut cover of this size. The following theorem provides a complementary result with respect to *absolute* approximation ratios (see [GJ79] for a definition).

Theorem 3.3 *Unless P = NP, there is no polynomial time algorithm for approximating the minimum cardinality cut cover within an absolute ratio of* 1.5.

Proof: Recently, Khanna, Linial and Safra [KLS93] showed that it is NP-hard to distinguish between graphs of chromatic number 3 from those of chromatic number 5. Suppose there were a polynomial time algorithm for approximating the cut cover within a ratio strictly smaller than 1.5. Noting that $\lceil \log 3 \rceil = 2$ and $\lceil \log 5 \rceil = 3$, we can see that it is then possible to distinguish between graphs of chromatic number 3 versus 5. □

A weaker result can be proved for the case *asymptotic* approximation ratio, in particular that there is no polynomial time approximation scheme (PTAS) for this problem (see [GJ79] for a definition).

Theorem 3.4 *Unless P = NP, there is no polynomial time (asymptotic) approximation scheme (PTAS) for the minimum cardinality cut cover problem. In other words, there exists an* $\epsilon > 0$ *such that no polynomial time algorithm can approximate the cut cover problem with an asymptotic ratio of* $1 + \epsilon$ *unless P = NP.*

Proof: Notice that an algorithm with asymptotic approximation ratio equal to $1+\epsilon$ can approximate the cut cover problem within an additive error of $\epsilon \log n$, since any graph has a cut cover of cardinality $\lceil \log n \rceil$. Any PTAS gives such an algotihm for all fixed $\epsilon > 0$, and the result now follows from Theorem 3.2. □

It remains an interesting open problem to determine the approximability of cut cover more precisely. However, this looks pretty hard since any result along these lines would imply new results for approximate graph coloring. For example, if we could prove that there is a polynomial time approximation algorithm for cut cover achieving a ratio $r < \alpha \log n$, it would imply that we could color a 3-chromatic graph using $O(n^{2\alpha})$ colors. The best known such result for 3-colorable graphs is that they can be colored with roughly $O(n^{3/8})$ colors [Bl90]. Similar consequences would follow from any hardness results for cut cover approximations.

References

[ALMSS92] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. "Proof verification and intractability of approximation problems." In *Proceedings of the 33rd IEEE Symposium on Foundations of Computer Science* (1992), pp. 14–23.

[Bl90] A. Blum. "Some Tools for Approximate 3-Coloring." In *Proceedings of the 33rd IEEE Symposium on Foundations of Computer Science* (1990), pp. 554–562.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman (1979).

[Hal90] M. M. Halldórsson. "A still better performance guarantee for approximate graph coloring." Technical Report 90-44, DIMACS (1990).

[KLS93] S. Khanna, N. Linial, and S. Safra. "On the Hardness of Approximating the Chromatic Number." In *Proceedings 2nd Israeli Symposium on Theory and Computing Systems* (1992), pp. 250–260.

[Lou92] R. Loulou. "Minimal cut cover of a graph with an application to the testing of electronic boards." *Operations Research Letters*, vol. 12 (1992), pp. 301–305.

[LY93] C. Lund and M. Yannakakis. "On the Hardness of Approximating Minimization Problems." In *Proceedings of the 25th ACM Symposium on Theory of Computing* (1993), pp. 286–293.

[SK86] T. L. Saaty and P. C. Kainen, *The four-color problem: assaults and conquest.* Dover Publications, Inc,, New York (1986).