CS31

ON THE APPROXIMATION OF WEAK SOLUTIONS OF LINEAR PARABOLIC EQUATIONS BY A CLASS OF MULTISTEP DIFFERENCE METHODS

BY

PIERRE ARNAUD RAVIART

TECHNICAL REPORT CS31

DECEMBER 6, 1965

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

ON THE APPROXIMATION OF WEAK SOLUTIONS OF LINEAR PARABOLIC EQUATIONS BY A CLASS OF MULTISTEP DIFFERENCE METHODS

BY

Pierre Arnaud Raviart

We consider evolution equations of the form

(1) $$\frac{du(t)}{dt} + A(t)u(t) = f(t) \ , \quad 0 \leq t \leq T \ , \ f \text{ given} \ ,$$

with the initial condition

(2) $$u(o) = u_o \ , \ u_o \text{ given} \ ,$$

where each $A(t)$ is an unbounded linear operator in a Hilbert space H, which is in practice an elliptic partial differential operator subject to appropriate boundary conditions.

Let V_h be a Hilbert space which depends on the parameter h. Let k be the time-step such that $m = \frac{T}{k}$ is an integer. We approximate the solution u of (1), (2) by the solution $u_{h,k}$ ($u_{h,k} = \{u_{h,k}(rk) \in V_h, \ r = 0,1,\ldots,m-1\}$) of the multistep difference -scheme

(3) $$\frac{u_{h\ k}(rk) - u_{h\ k}((r-1)k)}{k} + \sum_{\ell=0}^{p} \gamma_\ell \ A_h((r-\ell)k) \ u_{h,k}((r-\ell)k)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell \ f_{h,\ k}((r-\ell)k) \ , \qquad r = p,\ldots,m-1$$

(4) $$u_{h,k}(o),\ldots,u_{h,k}((p-1)k) \quad \text{given} \ ,$$

where each $A_h(rk)$ is a linear continuous operator from V_h into V_h, $f_{h,k}(rk)$ $(r = 0,1,\ldots,m-1)$ are given, and $\gamma_\ell (\ell=0,\ldots,p)$ are given complex numbers,

Our paper is mainly concerned by the study of the stability of the approximation. The methods used here are very closely related to those developed in Raviart [7] and we shall refer to [7] frequently, In §1, 2, we define the continuous and approximate problems in precise terms, In §4, we find sufficient conditions for $u_{h,k}$ to satisfy some a priori estimates. The definition of the stability is given in §5 and we use the a priori estimates for proving a general stability theorem. In 56 we prove that the stability conditions may be weakened when A(t) is a self-adjoint operator (or when only the principal part of A(t) is self-adjoint), We give in §7 a weak convergence theorem. $8 is concerned by regularity properties., We apply our abstract analysis to a class of parabolic partial differential equations with variable coefficients in §9.

Strong convergence theorems can be obtained as in Raviart [7] (via compactness arguments) or as in Aubin [1]. We do not study here the discretization error (see [1]).

For the study of the stability of multistep difference methods in the case of the Cauchy problem for parabolic differential operators, we refer to Kreiss [3], Widlund [8].

1. The continuous problem.

We are given two separable Hilbert spaces V and H such that $V \subset H$, the inclusion mapping of V into H 'is continuous, and V is dense in H.

If X is a Banach space with norm $\| \|_X$, we denote by $L^p(0,T;X)$ the space of (classes of) functions f which are L^p over [0,T] with values in X, provided with the usual norm $(1 \leq p < \infty)$:

$$\left(\int_0^T \|f(t)\|_X^p \, dt \right)^{1/p}$$

and the usual modification in case $p = \infty$

For every $t \in [0,T]$, we are given a continuous sesquilinear form on $V \times V$:

$$u, v \rightarrow a(t;u,v) , \qquad (u,v \in V) .$$

We assume that:

i) $t \rightarrow a(t;u,v)$ is measurable $(u,v \in V)$,

ii) there exists a constant K such that

(1.1) $$|a(t;u,v)| \leq K \|u\|_V \|v\|_V \qquad (u,v \in V, \quad t \in [0,T]) .$$

iii) there exist constants λ, α $(\alpha > 0)$ such that

(1.2) $$\operatorname{Re} a(t;v,v) + \lambda\|v\|_H^2 \geq \alpha\|v\|_V^2 \qquad (v \in V, \; t \in [0,T]) .$$

Then we have the following result (cf. Lions [4])

Theorem 1.1:

Given:

(1.3) $$f \in L^2(0,T;H) ,$$

(1.4) $$u_0 \in H .$$

There exists a unique function u satisfying

(1.5) $$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

(1.6) $$\int_0^T \{a(t; u(t), \varphi(t)) - (u(t), \varphi'(t))_H\} dt = \int_0^T (f(t), \varphi(t))_H \, dt + (u_0, \varphi(0))_H ,$$

for every function φ satisfying

(1.7) $$\varphi \in L^2(0,T;V), \ \varphi' \in L^2(0,T;H), \ \varphi(T) = 0.$$

Remarks:

i) The derivatives are taken in the sense of distributions.

ii) We may assume that $\lambda = 0$ in Hypothesis (1.2) (Replace $u(t)$ by $u(t) \exp(\chi t), \chi$ a real number chosen sufficiently large).

iii) We define V' to be the antidual of V. Since $v \to a(t;u,v)$ is a continuous conjugate linear form on V, we may write:

$$a(t;u,v) = \langle A(t)u,v \rangle \qquad \text{for all } v \in V,$$

where $A(t) \in \mathcal{L}(V;V')$. Then $u' \in L^2(0,T;V')$ so that u is equal a.e. to a continuous function from $[0,T]$ to H (cf. Lions [5]) and equation (1.6) may be replaced by

(1.8) $$u'(t) + A(t)\, u(t) = f(t), \qquad \text{for a.e. } t \in [0,T] ,$$

(1.9) $$u(0) = u_0 .$$

In the following, we shall assume that the function $t \to a(t;u,v)$ is once continuously differentiable for every $u, v \in V$. Then, because of the uniform boundedness principle, there exists a constant L such that

$$(1.10) \qquad \left|\frac{d}{dt} a(t;u,v)\right| \le L \|u\|_V \|v\|_V \qquad (u,v \in V, \quad t \in [0,T]).$$

For the study of the approximation of the solution u of equation (1.6) when the function $t \to a(t;u,v)$ is only measurable, we refer to Raviart [7].

2. The approached problem.

Let $\{V_h\}$ be a family of Hilbert spaces where the parameter $h = (h_1,\dots,h_n)$ is a strictly positive vector of R^n such that

$$|h| = h_1 + \dots + h_n \le h_0 \quad ,$$

$h_0 > 0$ being a fixed number. We provide each Hilbert space V_h with two scalar products denoted by $(\ ,\ )_h$ and $((\ ,\ ))_h$ respectively. We assume that the corresponding norms $|\ |_h$ and $\|\ \|_h$ are equivalent and verify

$$(2.1) \qquad c(h)\ |u_h|_h \le \|u_h\|_h \le C(h)\ |u_h|_h \qquad (u_h \in V_h) \quad ,$$

where $C(h)$ may tend towards $+\infty$ when h tends towards 0.

Let O_h be an operator belonging to $\mathcal{L}(H;V_h)$ with

$$(2.2) \qquad |O_h|_h = \sup_{u \in H} \frac{|O_h u|_h}{\|u\|_H} < C_1 \ ,$$

where C_1 is a constant independent of h.

For every $t \in [0,T]$, we are given a family of continuous sequilinear forms on $V_h \times V_h$:

$$u_h, v_h \to a_h(t; u_h, v_h) \qquad (u_h, v_h \in V_h) .$$

We assume that:

(i) $t \to a_h(t; u_h, v_h)$ is once continuously differentiable $(u_h, v_h \in V_h)$;

(ii) there exist constants M, P independent of h such that

$$(2.3) \qquad | a_h(t; u_h, v_h) | \leq M \|u_h\|_h \|v_h\|_h ,$$

$$(2.4) \qquad | \frac{d}{dt} a_h(t; u_n, v_h) | \leq P \|u_h\|_h \|v_h\|_h ;$$

(iii)

$$(2.5) \qquad \text{Re } a_h(t; v_h, v_h) \geq \alpha \|v_h\|_h^2 \qquad (v_h \in V_h) ,$$

where α is the constant involved in inequality (1.2).

From (2.1), (2.3) and (2.4) we deduce that

$$(2.6) \qquad |a_h(t; u_h, v_h)| \leq M(h) |u_h|_h |v_h|_h ,$$

$$(2.7) \qquad |a_h(t; u_h, v_h)| \leq N(h) \|u_h\|_h |v_h|_h ,$$

$$(2.8) \qquad | \frac{d}{dt} a_h(t; u_h, v_h)| \leq P(h) |u_h|_h |v_h|_h ,$$

Where M(h), N(h) and P(h) may tend towards $+\infty$ when $h \to 0$. Moreover M(h), $(N(h))^2$, P(h) have the same order of magnitude when $h \to 0$. We denote by $\mu(h)$ and $\nu(h)$ respectively the principal parts of M(h) and N(h) when $h \to 0$.

Let us introduce now a set of consecutive integers $\{0,1,\dots,r,\dots,m\}$ and the "time-step" $k = \frac{T}{m}$. We define $E_k(r_1 k, r_2 k; V_h)$ to be the space of sequences $u_{h,k}$ of the form

$$u_{h,k} = \{u_{h,k}(rk),\ r = r_1,\dots,r_2 - 1\}$$

where each $u_{h,k}(rk)$ belongs to V_h. We provide $E_k(0,T;V_h)$ with the two following sequences of equivalent norms:

$$|\ |_{h,k,p} \quad \text{and} \quad \|\ \|_{h,k,p} \qquad (1 \le p \le +\infty)$$

defined by

$$|u_{h,k}|_{h,k,p} = \left(k \sum_{r=0}^{m-1} |u_{h,k}(rk)|_h^p\right)^{1/p} \qquad (1 \le p < +\infty),$$

$$\|u_{h,k}\|_{h,k,p} = \left(k \sum_{r=0}^{m-1} \|u_{h,k}(rk)\|_h^p\right)^{1/p} \qquad (1 \le p < +\infty),$$

$$|u_{h,k}|_{h,k,\infty} = \sup_{0 \le r \le m-1} |u_{h,k}(rk)|_h ,$$

$$\|u_{h,k}\|_{h,k,\infty} = \sup_{0 \le r \le m-1} \|u_{h,k}(rk)\|_h .$$

If $u_{h,k} \in E_k(r_1 k, r_2 k; V_h)$, we may define

$$\overline{\nabla}_k u_{h,k} = \{\overline{\nabla}_k u_{h,k}(rk)\ r = r_1+1,\dots,r_2 - 1\} \in E_k((r_1 + 1)k, r_2 k, V_h)$$

by

(2.9) $$\overline{\nabla}_k u_{h,k}(rk) = \frac{1}{k}\,[u_{h,k}(rk) - u_{h,k}((r-1)k)] \quad .$$

For every $t \in [sk, T]$ $(s \geq 0)$, we define for all $u_h, v_h \in V_h$:

(2.10) $$\{\overline{\nabla}_{sk}\, a_h\}\,(t;u_h,v_h) = \frac{1}{sk}\,[a_h(t;u_h,v_h) - a_h(t-sk;u_h,v_h)] \quad .$$

We introduce now $f_{h,k} \in E_h(0,T;V_h)$ by

(2.11) $$f_{h,k}(rk) = \frac{1}{k}\int_{rk}^{(r+1)k} 0_h f(t)\, dt.$$

The following inequality is easily verified:

(2.12) $$|f_{h,k}|_{h,k,2} \leq C_1\left(\int_0^T \|f(t)\|_H^2 \, dt\right)^{\frac{1}{2}} \quad .$$

Let us consider now the "approached" problem

<u>Problem I</u>:

<u>Find</u> $u_{h,k} \in E_k(0,T;V_h)$ <u>satisfying</u>

(2.13) $$(\overline{\nabla}_k u_{h,k}(rk), v_h)_h + \sum_{\ell=0}^{p} \gamma_\ell \, a_h((r-\ell)k;\, u_{h,k}((r-\ell)k), v_h)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell \, (f_{h,k}((r-\ell)k),\, v_h)_h, \qquad r = p,\ldots, m-1,$$

for all $v_h \in V_h$,

(2.14) $u_{h,k}(0)$, $u_{h,k}(k), \ldots, u_{h,k}((p-1)k)$ <u>given</u> <u>in</u> V_h $\gamma_0, \gamma_1, \ldots, \gamma_p$ <u>are chosen complex numbers</u>.

Theorem 2.1:

Assume that γ_0 is a real number ≥ 0. Then Problem I has a unique solution.

Proof:

Note that the theorem is trivial for $\gamma_0 = 0$. Let us assume $\gamma_0 > 0$. Then equation (2.13) may be written

$$\hat{a}_{h,k}(rk;u_{h,k}(rk), v_h) = (F_{h,k}(rk), v_h)_h \tag{2.15}$$

where $\hat{a}_{h,k}(rk;u_h,v_h)$ is a continuous sequilinear form on $V_h \times V_h$ which verifies for all $v_h \in V_h$

$$|\hat{a}_{h,k}(rk;v_h,v_h)| \geq \text{Re } a_{h,k}(rk;v_h,v_h) \geq |v_h|_h^2 + k\gamma_0\,\alpha\,\|v_h\|_h^2 \geq |v_h|_h^2 \,. \tag{2.16}$$

and $F_{h,k}(rk)$ is an element of V_h which does not depend on $u_{h,k}(rk)$. Then (2.16) is a sufficient condition for equation (2.15) to have a unique solution.

In the following, we shall always assume that γ_0 is > 0.

Remarks:

(i) We could slightly generalize by choosing other complex constants than $\gamma_0,\ldots,\gamma_p$ in the left-hand side of equation (2.13).

(ii) Let $A_h(t) \in \mathcal{L}(V_h;V_h)$ be the operator defined by

$$(A_h(t)u_h, v_h)_h = a_h(t;u_h,v_h) \qquad (u_h,v_h \in V_h) \,.$$

Then equation (2.13) may be written on the equivalent form

$$(2.17)\qquad \overline{\nabla}_k u_{h,k}(rk) + \sum_{\ell=0}^{p} \gamma_\ell A_h((r-\ell)k)u_{h,k}((r-\ell)k)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell f_{h,k}((r-\ell)k), \qquad r = p,\ldots,m-1.$$

3. Some Lemmas.

Lemma 3.1:

We denote by E a linear space and by $b(.,.)$ a sequilinear hermitian form on $E \times E$. Let φ be a mapping from Zk into E. Then

$$(3.1)\qquad \overline{\nabla}_k\, b(\varphi(rk), \varphi(rk)) = 2\mathrm{Re}\, b(\varphi(rk), \overline{\nabla}_k \varphi(rk)) - k\, b(\overline{\nabla}_k\varphi(rk), \overline{\nabla}_k\varphi(rk))$$

$$= 2\mathrm{Re}\, b(\varphi((r-1)k), \overline{\nabla}_k\varphi(rk)) + k\, b(\overline{\nabla}_k\varphi(rk), \overline{\nabla}_k\varphi(rk))$$

Lemma 3.2:

. Let φ be any scalar function defined on Zk, Then

$$(3.2)\qquad k \sum_{s=r_1+1}^{r_2} \overline{\nabla}_k\, \varphi(sk) = \varphi(r_2 k) - \varphi(r_1 k)\ .$$

Lemma 3.3:

Let φ be any real function defined on $\{r_1 k, (r_1+1)k, \ldots, r_2 k\}$. We assume that the following inequality holds for all integer r $(r_1 \le r \le r_2)$

$$(3.3)\qquad \varphi(rk) + \alpha(rk) \le C + \frac{k}{k_0} \sum_{s=r_1}^{r-1} \varphi(sk)\ , \qquad \alpha(rk) \ge 0,$$

where $C \ge 0$ and $k_0 > 0$ are given constants. Then

$$(3.4)\qquad \varphi(rk) + \alpha(rk) < C(1 + \frac{k}{k_0})^{r-r_1} \le C \exp\left(\frac{(r-r_1)k}{k_0}\right)\ .$$

If we replace for all r inequality (3.3) by

$$(3.5)\qquad \varphi(rk) + \alpha(rk) \leq C + \frac{k}{k_0} \sum_{s=r_1+1}^{r} \varphi(sk), \qquad \alpha(rk) \geq 0,$$

Then, for, $k < k_0$,

$$(3.6)\qquad \varphi(rk) + \alpha(rk) \leq C\left(1 - \frac{k}{k_0}\right)^{-r+r_1} \leq C \exp\left(\frac{(r-r_1)k}{k_0}\right).$$

All these lemmas are obvious.

4. A priori estimates for the solution of Problem I.

Before establishing an energy inequality for the solution $u_{h,k}$ of Problem I, we shall put equation (2.13) into a more convenient form.

a) Another form for equation (2.13)

Equation (2.17) may be replaced by

$$(4.1)\qquad \overline{\nabla}_k u_{h,k}(rk) + \left(\sum_{\ell=0}^{p} \gamma_\ell\right) A_h(rk)\, u_{h,k}(rk)$$

$$+ k \sum_{\ell=1}^{p} \beta_\ell\, \overline{\nabla}_k \{A_h((r-\ell+1)k) u_{h,k}((r-\ell+1)k\}$$

where $\qquad \beta_\ell = - \sum_{\ell'=\ell}^{p} \gamma_{\ell'}, \qquad \ell = 1, \ldots p .$

Assume that the operator $A_h(t)$ is consistent with the operator $A(t)$ in a certain sense which will be precised later in §7. Then the operator $\sum_{\ell=1}^{p} \gamma_\ell A_h(t-\ell k)$ is consistent with A(t) if and only if

(4.2) $$\sum_{\ell=0}^{p} \gamma_\ell = 1 .$$

In the following, we shall assume that equality (4.2) is always verified. Then equation (4.1) may be written:

(4.3) $$\bar{\nabla}_k u_{h,k}(rk) + A_h(rk)\ [u_{h,k}(rk) + k \sum_{\ell=1}^{p} \beta_\ell\ \bar{\nabla}_k\ u_{h,k}((r-\ell+1)k)]$$

$$+ \sum_{\ell=2}^{p} \beta_\ell\ \{A_h((r-\ell+1)k) - A_h(rk)\}\ u_{h,k}((r-\ell+1)k)$$

$$+ \sum_{\ell=1}^{p} \beta_\ell\ \{A_h(rk) - A_h((r-\ell)k\}\ u_{h,k}((r-\ell)k)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\ f_{h,k}((r-\ell)k) .$$

We define

(4.4) $$v_{h,k}(rk) = u_{h,k}(rk) + k \sum_{\ell=1}^{p} \beta_\ell\ \bar{\nabla}_k\ u_{h,k}((r-\ell+1)k)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\ u_{h,k}((r-\ell)k) , \qquad r = p, \ldots, m-1 .$$

Hence another form for equation (2.13) is given by

$$(4.5) \qquad (\bar{\bar{\nabla}}_k\, u_{h,k}(rk),\, v_h)_h + a_h(rk;\, v_{h,k}(rk),\, v_h)$$

$$- k \sum_{\ell=2}^{p} (i-1)\, \beta_\ell\, \{\nabla_{(\ell-1)k}\, a_h\}\, (rk;\, u_{h,k}((r-\ell+1)k),\, v_h)$$

$$+ k \sum_{\ell=1}^{p} \ell\beta_\ell\, \{\bar{\bar{\nabla}}_{\ell k}\, a_h\}\, (rk;\, u_{h,k}((r-\ell)k,\, v_h)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\, (f_{h,k}((r-\ell)k),\, v_h)_h, \qquad r = p,\dots,m-1 \quad .$$

b) <u>The energy-inequality</u>.

In the following, $D_1, D_2, \dots$ will be positive constants independent of h and k.

We replace v_h in equation (4.5) by $v_{h,k}(rk)$. We obtain

$$(4.6) \qquad (\bar{\bar{\nabla}}_k\, u_{h,k}(rk),\, u_{h,k}(rk))_h + k \sum_{\ell=1}^{p} \beta_\ell\, (\bar{\bar{\nabla}}_k u_{h,k}(rk),\, \bar{\bar{\nabla}}_k u_{h,k}((r-\ell+1)k))_h$$

$$+ a_h(rk;\, v_{h,k}(rk),\, v_{h,k}(rk))$$

$$- k \sum_{\ell=2}^{p} (\ell-1)\, \beta_\ell\, \{\bar{\bar{\nabla}}_{(\ell-1)k}\}(rk;\, u_{h,k}((r-\ell+1)k),\, v_{h,k}(rk))$$

$$+ k \sum_{\ell=1}^{p} \ell\beta_\ell\, \{\bar{\bar{\nabla}}_{\ell k}\, a_h\}\, (rk;\, u_{h,k}((r-\ell)k),\, v_{h,k}(rk))$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\, (f_{h,k}((r-\ell)k),\, v_{h,k}(rk))_h, \qquad r = p, \dots, m-1 \,.$$

Taking real parts and applying lemma 3.1 gives:

$$(4.7)\qquad \bar{\nabla}_k |u_{h,k}(rk)|_h^2 + k|\bar{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+ 2k\ \mathrm{Re}\{\sum_{\ell=1}^{p} \beta_\ell (\bar{\nabla}_k u_{h,k}(rk), \bar{\nabla}_k u_{h,k}((r-\ell+1)k))_h\}$$

$$+ 2\mathrm{Re}\ a_h(rk; v_{h,k}(rk), v_{h,k}(rk))$$

$$= 2k\ \mathrm{Re}\{\sum_{\ell=2}^{p} (\ell-1)\beta_\ell \{\bar{\nabla}_{(\ell-1)k} a_h\}(rk; u_{h,k}((r-\ell+1)k), v_{h,k}(rk))\}$$

$$- 2k\ \mathrm{Re}\{\sum_{\ell=1}^{p} \ell\beta_\ell \{\bar{\nabla}_{\ell k} a_h\}(rk; u_{h,k}((r-\ell)k), v_{h,k}(rk))\}$$

$$+ 2\mathrm{Re}\{\sum_{\ell=0}^{p} \gamma_\ell (f_{h,k}((r-\ell)k), v_{h,k}(rk))_h\}, \qquad r = p,\ldots,m-1 .$$

Note that $\beta_1 = \gamma_0 - 1$ is a real number. Then it follows from the Cauchy-Schwarz inequality and Hypotheses (2.5), (2.8) that

$$(4.8)\qquad \bar{\nabla}_k |u_{h,k}(rk)|_h^2 + (1 + 2\beta_1)k\,|\bar{\nabla}_k u_{h,k}(rk)|_h^2$$

$$- 2k\sum_{\ell=2}^{p} |\beta_\ell|\,|\bar{\nabla}_k u_{h,k}(rk)|_h\,|\bar{\nabla}_k u_{h,k}((r-\ell+1)k)|_h + 2\alpha\,\|v_{h,k}(rk)\|_h^2$$

$$\leq 2k\ P(h) \sum_{\ell=2}^{p} (\ell-1)|\beta_\ell|\,|u_{h,k}((r-\ell+1)k)|_h\,|v_{h,k}(rk)|_h$$

$$+ 2k\ P(h) \sum_{\ell=1}^{p} \ell|\beta_\ell|\,|u_{h,k}((r-\ell)k)|_h\,|v_{h,k}(rk)|_h$$

$$+ 2\sum_{\ell=0}^{p} |\gamma_\ell|\,|f_{h,k}((r-\ell)k|_h\,|v_{h,k}(rk)|_h, \quad r = p,\ldots,m-1 .$$

We deduce from the inequality $2ab \leq a^2 + b^2$ (a,b real numbers):

$$(4.9) \qquad \bar{\bar{\nabla}}_k |u_{h,k}(rk)|_h^2 + (1 + 2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell|)\, k\, |\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h^2$$

$$- k \sum_{\ell=2}^{p} |\beta_\ell|\, |\bar{\bar{\nabla}}_k u_{h,k}((r-\ell+1)k)|_h^2 + 2\alpha \|v_{h,k}(rk)\|_h^2$$

$$\leq (D_1 k P(h) + D_2) \sum_{\ell=0}^{p} |u_{h,k}((r-\ell)k)|_h^2 + D_3 \sum_{\ell=0}^{p} |f_{h,k}((r-\ell)k|_h^2 ,$$

$$r = p, \ldots, m-1 .$$

Multiplying equation (4.9) by k and summing from r=p to r = s ($p \leq s \leq m-1$) gives:

$$(4.10) \qquad |u_{h,k}(sk)|_h^2 + (1+2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell|)\, k^2 \sum_{r=p}^{s} |\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h^2$$

$$+ 2\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2$$

$$\leq |u_{h,k}((p-1)k)|_h^2 + k^2 \sum_{\ell=2}^{p} |\beta_\ell| \sum_{r=p-\ell+1}^{p-1} |\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h^2$$

$$+ (D_1 k P(h) + D_2)\, k \sum_{\ell=1}^{p} \sum_{r=p-\ell}^{p-1} |u_{h,k}(rk)|_h^2$$

$$+ D_3 k \sum_{\ell=0}^{p} \sum_{r=p-\ell}^{s-\ell} |f_{h,k}(rk)|_h^2$$

$$+ (D_1 k P(h) + D_2)\, k \sum_{\ell=0}^{p} \sum_{r=p}^{s-\ell} |u_{h,k}(rk)|_h^2 .$$

Let us assume that there exist two constants K_1, K_2 independent of h and k such that

$$(4.11) \qquad |u_{h,k}(rk)|_h \leq K_1 , \qquad r = 0,\ldots, p-1$$

$$(4.12) \qquad k^2 \sum_{r=1}^{p-1} |\overline{\nabla}_k u_{h,k}(rk)|_h^2 \leq K_2 .$$

We consider now two different cases according to the sign of

$$1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| .$$

(i) <u>1^{st} Case</u>: $\qquad 1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0 .$

Let us assume that

$$(4.13) \qquad k(\nu(h))^2 \leq \rho ,$$

where ρ is an arbitrary > 0 constant independent of h and k. Note that (4.13) is equivalent to

$(4.13)'$ $\qquad k\, P(h) \leq \rho'$, $\rho' > 0$ arbitrary constant independent of h and k. Then, because of (2.12), we obtain:

$$(4.14) \qquad |u_{h,k}(sk)|_h^2 + (1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell|) k^2 \sum_{r=p}^{s} |\overline{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+ 2\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2 \leq D_4 + D_5 k \sum_{r=p}^{s} |u_{h,k}(rk)|_h^2 ,$$

$$s = p,\ldots, m-1$$

By applying lemma 3.3, we find for $kD_5 < 1$ and for every s ($p \le s \le m-1$) the following energy inequality

$$(4.15) \qquad |u_{h,k}(sk)|_h^2 + 2\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2 \le D_4 \exp(D_5(s-p+1)k)$$

$$\le D_4 \exp(D_5(T - pk))$$

(ii) 2nd Case: $\quad 1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| < 0$.

We give an estimate for $|\overline{\nabla}_k u_{h,k}(rk)|_h^2$, r =p,..., m-1, which is obtained as follows: We replace v_h in equation (4.5) by $\overline{\nabla}_k u_{h,k}(rk)$. Applying inequalities (2.7) and (2.8) gives:

$$(4.16) \qquad |\overline{\nabla}_k u_{h,k}(rk)|_h \le N(h) \|v_{h,k}(rk)\|_h$$

$$+ k\, P(h) \sum_{\ell=2}^{p} (\ell - 1)|\beta_\ell|\, |u_{h,k}((r - \ell + 1)k)|_h$$

$$+ k\, P(h) \sum_{\ell=1}^{p} \ell |\beta_\ell|\, |u_{h,k}((r-\ell)k)|_h + \sum_{\ell=0}^{p} |\gamma_\ell|\, |f_{h,k}((r-\ell)k)|_h .$$

Then

$$(4.17) \qquad |\overline{\nabla}_k u_{h,k}(rk)|_h^2 \le (N(h))^2 (1 + \delta) \|v_{h,k}(rk)\|_h^2$$

$$+ \left(D_6 + \frac{D_7}{\delta}\right) k^2 (P(h))^2 \sum_{\ell=1}^{p} |u_{h,k}((r-\ell)k)|_h^2$$

$$+ \left(D_8 + \frac{D_9}{\delta}\right) \sum_{\ell=0}^{p} |f_{h,k}((r-\ell)k)|_h^2 ,$$

where $\delta > 0$ may be chosen as small as we please. Assume that Hypothesis (4.13) is verified. Hence we deduce from (4.10) and 4.17):

$$(4.18) \qquad |u_{h,k}(sk)|_h^2$$

$$+ \left(2\alpha - \left(2\sum_{\ell=2}^{p}|\beta_\ell| - 2\beta_1 - 1\right) k(N(h))^2 (1+\delta)\right) k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2$$

$$\leq D_{10} + D_{11}\, k \sum_{r=p}^{s} |u_{h,k}(rk)|_h^2 \ .$$

Let us assume now that

$$(4.19) \qquad k(N(h))^2 \leq \frac{2\alpha}{2\sum_{\ell=2}^{p}|\beta_\ell| - 2\beta_1 - 1}(1-\delta) \ .$$

Then

$$(4.20) \qquad |u_{h,k}(sk)|_h^2 + 2\alpha\delta^2 \ k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2$$

$$\leq D_{10} + D_{11} \ k \sum_{r=p}^{s} |u_{h,k}(rk)|_h^2 \ .$$

By applying lemma 3.3, we find for $k\ D_{11} < 1$ and for every s ($p \leq s \leq m-1$) the energy inequality

$$(4.21) \qquad |u_{h,k}(sk)|_h^2 + 2\alpha\delta^2 \ k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2$$

$$\leq D_{10} \exp(D_{11}(s-p+1)k) \leq D_{10} \exp(D_{11}(T-pk)) \ .$$

Note that (4.19) implies (4.13). When k is small enough, we may replace condition (4.19) by

$$(4.22) \qquad k(\nu(h))^2 \leq \frac{2\alpha}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta'$$

where $\delta' > 0$ is arbitrarily small.

Thus, under some appropriate assumptions, we have obtained an energy inequality for any value of $1 + 2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell|$. Let us assume that there exists a positive constant K_3 independent of h and k such that

$$(4.23) \qquad k \sum_{r=0}^{p-1} \|u_{h,k}(rk)\|_h^2 \leq K_3 .$$

We define $v_{h,k} \in E_k(0,T;V_h)$ by

$$(4.24) \qquad v_{h,k} = \{u_{h,k}(rk),\ r = 0,\ldots, p-1,\ \ v_{h,k}(rk),\ r = p,\ldots, m-1\} .$$

Then we have proved the following result:

Theorem 4.1:

Assume Hypotheses

$$(4.11) \qquad |u_{h,k}(rk)|_h \leq K_1 , \qquad r = 0,\ldots, p-1 ,$$

$$(4.12) \qquad k^2 \sum_{r=1}^{p-1} |\bar{\nabla}_k u_{h,k}(rk)|_h^2 \leq K_2 ,$$

$$(4.23) \qquad k \sum_{r=0}^{p-1} \|u_{h,k}(rk)\|_h^2 \leq K_3 .$$

There exist positive constants E_1, E_2 independent of h and k such that

(4.25) $$|u_{h,k}|_{h,k;\infty} \leq E_1 \quad ,$$

(4.26) $$\|v_{h,k}\|_{h,k;2} \leq E_2 \quad ,$$

in the two following cases

(i) 1st Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0,$

(4.13) $$k(\nu(h))^2 \leq \rho \ .$$

(ii) 2nd Case: $1 + 2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell| < 0 ,$

(4.2) $$k(\nu(h))^2 \leq \frac{2\alpha}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta' \quad , \quad \delta'>0 \text{ arbitrarily}$$

small, k small enough,

Remarks:

(i) It is easy to determine $\{u_{h,k}(rk),\ r=0,\dots,p-1\}$; verifying (4.11), (4.12) and (4.23) by two-level difference schemes: see Raviart [7].

(ii) If the operator $A_h(t)$ is independent of t, P(h) = 0 and Hypothesis (4.13) may be suppressed.

5. The Stability Theorem.

Let X be a Banach space with norm $\| \ \|_X$. For every $\{h,k\}$ ($|h| \leq h_0$, $k \leq k_0$) let $P_{h,k}$ be an operator of $\mathcal{L}(E_k(0,T;V_h); L^{\infty}(0,T;X))$ such that $P_{h,k}\,u_{h,k}(t)$ is defined in X for all $t \in [0,T]$ and all $u_{h,k} \in E_k(0,T;V_h)$: $P_{h,k}$ is called a prolongation operator.

Definition 5.1:

Let $u_{h,k}$ be the solution of Problems I. $P_{h,k}\,u_{h,k}$ is said to be $L^{\infty}(0,T;X)$-stable if there exists a constant $C \geq 0$ independent of h and k such that:

$$\|P_{h,k}\,u_{h,k}(t)\|_X \leq C \quad , \text{ for all } t \in [0,T],$$
$$\text{for all } \{h,k\} \ (|h| \leq h_0, k \leq k_0)$$

Let F be a Hilbert space such that H is a closed subspace of F. Let π denote the projection operator from F onto H. For every $\{h,k\}$, let $P_{h,k}$ be a prolongation operator of $\mathcal{L}(E_k(0,T;V_h);L^{\infty}(0,T;F))$. Then $Q_{h,k} = \pi \circ P_{h,k}$ is a prolongation operator of $\mathcal{L}(E_k(0,T;V_h); L^{\infty}(0,T;H))$. We assume that

(5.1) $$\|P_{h,k}u_{h,k}\|_{L^2(0,T;F)} \leq C_2 \ \|u_{h,k}\|_{h,k,2} \, ,$$

(5.2) $$\sup_{t\in[0,T]} \|Q_{h,k}\ u_{h,k}(t)\|_H \leq C_3 \ |u_{h,k}|_{h,k,\infty} \, ,$$

for all $u_{h,k} \in E_k(0,T;V_h)$, where C_2 and C_3 are positive constants independent of h and k.

Then the following result can be deduced from theorem 4.1.

Theorem 5.1:

Assume Hypotheses (4.11), (4.12), (4.23), (5.1) (5.2). Let $u_{h,k}$ be the solution of Problem I. Then $Q_{h,k}u_{h,k}$ is $L^{\infty}(0,T;H)$-stable and $P_{h,k} \cdot v_{h,k}$ remains in a bounded set of $L^2(0,T;F)$ in the two following cases:

(i) 1st Case: $$1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell| \geq 0 ,$$

(4.13) $k(\nu(h))^2 \leq \rho$ (No restriction if $A_h(t)$ is independent of t),

(ii) 2nd Case: $$1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta^\ell| < 0 ,$$

(4.22) $$k(\nu(h))^2 < \frac{2\alpha}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta', \quad \delta' > 0$$ arbitrarily small, k small enough.

6. The Hermitian Case:

Let us assume now that for every $t \in [0,T]$ the sesquiiinear form $a(t;u,v)$ is hermitian (i.e. $a(t;u,v) = \overline{a(t;v,u)}$). So we choose the family of sequilinear forms $a_h(t;u_h,v_h)$ such that

(6.1) $$a_h(t;u_h,v_h) = \overline{a_h(t;v_h,u_h)} \qquad (u_h,v_h \in V_h,\ t \in [0,T])$$

Then it is possible to weaken condition (4.22) in case

$$1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell| < 0$$

First we give a new estimate for $k \sum_{r=p}^{s} |\bar{\nabla}_k u_{h,k}(rk)|_h^2$, $p \leq s \leq m-1$. We replace v_h in equation (4.5) by $\bar{\nabla}_k u_{h,k}(rk)$:

$$(6.2)\qquad |\bar{\nabla}_k u_{h,k}(rk)|_h^2 + a_h(rk; u_{h,k}(rk), \bar{\nabla}_k u_{h,k}(rk))$$

$$+ \; k \sum_{\ell=1}^{p} \beta_\ell \; a_h(rk; \; \nabla_k u_{h,k}((r-\ell+1)k), \; \bar{\nabla}_k u_{h,k}(rk))$$

$$- \; k \sum_{\ell=2}^{p} (\ell-1)\beta_\ell \; \{\bar{\nabla}_{(\ell-1)k} \, a_h\} \; (rk; u_{h,k}((r-\ell+1)k), \; \bar{\nabla}_k u_{h,k}(rk))$$

$$+ \; k \sum_{\ell=1}^{p} \ell\beta_\ell \; \{\bar{\nabla}_{\ell k} \, a_h\} \; (rk; \; u_{h,k}((r-\ell)k), \; \bar{\nabla}_k u_{h,k}(rk))$$

$$= \sum_{\ell=0}^{p} \gamma_\ell \; (f_{h,k}((r-\ell)k), \; \bar{\nabla}_k u_{h,k}(rk))_h \;, \quad r=p,\ldots,m-1 \,.$$

From (2.1) and (6.1), it follows that

$$(6.3)\qquad 2\mathrm{Re} \;\; a_h(rk; \; u_{h,k}(rk), \; \bar{\nabla}_k u_{h,k}(rk) \,)$$

$$= \bar{\nabla}_k a_h(rk; \; u_{h,k}(rk), \; u_{h,k}(rk))$$

$$+ \; k \;\; a_h(rk; \; \bar{\nabla}_k u_{h,k}(rk), \; \bar{\nabla}_k u_{h,k}(rk))$$

$$- \; \{\bar{\nabla}_k a_h\} \; (rk; \; u_{h,k}((r-1)k), \; u_{h,k}((r-1)k)) \;.$$

Taking real parts in equation (6.2) and applying identity (6.3) gives:

$$(6.4) \qquad |\nabla_k u_{h,k}(rk)|_h^2 + \frac{1}{2} \overline{\nabla}_k a_h (rk; u_{h,k}(rk), \ u_{h,k}(rk))$$

$$+ \frac{k}{2}(1+2\beta_1)\, a_h(rk; \overline{\nabla}_k u_{h,k}(rk), \ \overline{\nabla}_k u_{h,k}(rk))$$

$$+ \ k \operatorname{Re} \{\sum_{\ell=2}^{p} \beta_\ell \ a_h(rk; \ \overline{\nabla}_k u_{h,k}((r-\ell+1)k), \ \overline{\nabla}_k u_{h,k}(rk))\}$$

$$- \ \frac{1}{2} \ \{\overline{\nabla}_k a_h\} (rk; \ u_{h,k}((r-1)k), \ u_{h,k}((r-1)k))$$

$$\leq k \ P(h) \sum_{\ell=2}^{p} (\ell-1) \ |\beta_\ell| \ |u_{h,k}((r-\ell+1)k)|_h \ |\overline{\nabla}_k u_{h,k}(rk)|_h$$

$$+ \ k \ P(h) \sum_{\ell=1}^{p} \ell |\beta_\ell| \ |u_{h,k}((r-\ell)k)|_h \ |\overline{\nabla}_k u_{h,k}(rk)|_h$$

$$+ \sum_{\ell=0}^{p} |\gamma_\ell| \ |f_{h,k}((r-\ell)k)|_h \ |\overline{\nabla}_k u_{h,k}(rk)|_h \ .$$

Hypotheses (2.5) and (6.1) imply that

$$(6.5) \qquad |a_h(t;u_h,v_h)| \leq (a_h(t;u_h,u_h))^{\frac{1}{2}} \ (a_h(t;v_h,v_h))^{\frac{1}{2}} \ , \quad (u_h,v_h \in V_h) \ .$$

Hence

$$(6.6) \qquad 2|a_h(t;u_h,v_h)| \ \leq \ a_h(t;u_h,u_h) + a_h(t;v_h,v_h) \ .$$

By using (6.6), (6.4) becomes:

$$(6.7)\qquad 2|\overline{\nabla}_k u_{h,k}(rk)|_h^2 + \overline{\nabla}_k a_h(rk;\ u_{h,k}(rk),\ u_{h,k}(rk)$$

$$+\ k(1+2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell|)\ a_h(rk;\ \overline{\nabla}_k u_{h,k}(rk),\ \overline{\nabla}_k u_{h,k}(rk))$$

$$-\ k \sum_{\ell=2}^{p} |\beta_\ell|\ a_h(rk;\ \overline{\nabla}_k u_{h,k}((r-\ell+1)k),\ \overline{\nabla}_k u_{h,k}((r-\ell+1)k))$$

$$-\ \{\overline{\nabla}_k a_h\}\ (rk;\ u_{h,k}((r-1)k),\ u_{h,k}((r-1)k))$$

$$\leq\ \epsilon\ |\overline{\nabla}_k u_{h,k}(rk)|_h^2 + \frac{D_{12}}{\epsilon}\ k^2(P(h))^2 \sum_{\ell=1}^{p} |u_{h,k}((r-\ell)k)|_h^2$$

$$+\ \frac{D_{13}}{\epsilon} \sum_{\ell=0}^{p} |f_{h,k}((r-\ell)k)|_h^2\ , \qquad r = p, \ldots, m-1\ .$$

where $\epsilon > 0$ may be chosen as small as we please.

Using (2.6), multiplying (6.7) by k and summing from $r = p$ to $r = s$ $(p \leq s \leq m-1)$ gives:

$$(6.8)\qquad (2-kM(h)\ (2 \sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1) - \epsilon)\ k \sum_{r=p}^{s} |\overline{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+\ a_h(sk;\ u_{h,k}(sk),\ u_{h,k}(sk))$$

$$-\ k \sum_{r=p}^{s} \{\overline{\nabla}_k a_h\}\ (rk; u_{h,k}((r-1)k),\ u_{h,k}((r-1)k)$$

$$\leq a_h((p-1)k;\ u_{h,k}((p-1)k),\ u_{h,k}((p-1)k)$$

$$+ D_{14}\ kM(h)k \sum_{r=1}^{p-1} | \overline{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+ \frac{D_{12}}{\epsilon} k^2 (P(h))^2\ k \sum_{\ell=1}^{p} \sum_{r=p}^{s} |u_{h,k}((r-\ell)k)|_h^2$$

$$+ \frac{D_{13}}{\epsilon} k \sum_{\ell=0}^{p} \sum_{r=p}^{s} |f_{h,k}((r-\ell)k)|_h^2 \quad .$$

Let us assume that

$$(6.9) \qquad k\,M(h) < \frac{2}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} (1 - \delta) \quad ,$$

where $\delta > 0$ is arbitrarily small

We can choose $\epsilon = \delta$. Then equation (6.8) becomes

$$(6.10) \qquad k \sum_{r=p}^{s} | \overline{\nabla}_k u_{h,k}(rk)|_h^2 + \frac{1}{\delta} a_h(sk; u_{h,k}(sk),\ u_{h,k}(sk))$$

$$\leq \frac{1}{\delta} a_h((p-1)k;\ u_{h,k}((p-1)k),\ u_{h,k}((p-1)k)$$

$$+ D_{15}\ k \sum_{r=1}^{p-1} | \overline{\nabla}_k u_{h,k}(rk) |_h^2$$

$$+ \frac{1}{\delta}\ k \sum_{r=p}^{s} \{\overline{\nabla}_k a_h\} (rk; u_{h,k}((r-1)k),\ u_{h,k}((r-1)k))$$

$$+ D_{16}\ k \sum_{r=0}^{s-1} |u_{h,k}(rk)|_h^2 + D_{17}\ k \sum_{r=0}^{s} |f_{h,k}(rk)|_h^2 \quad ,$$

where D_{15}, D_{16}, D_{17} depend on δ.

We deduce now from (4.10) and 6.10):

$$(6.11) \qquad |u_{h,k}(sk)|_h^2 + 2\,\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2$$

$$\leq (2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1)\, k \Big[\frac{1}{\delta}\, a_h((p-1)k; u_{h,k}((p-1)k), u_{h,k}((p-1)k))$$

$$+ D_{15}\, k\sum_{r=1}^{p-1} |\overline{\nabla}_k u_{h,k}(rk)|_h^2 + \frac{1}{\delta}\, k \sum_{r=p}^{s} \{\overline{\nabla}_k a_h\}(rk; u_{h,k}((r-1)k), u_{h,k}((r-1)k))$$

$$+ D_{16}\, k\sum_{r=0}^{s-1} |u_{h,k}(rk)|_h^2 + D_{17}\, k \sum_{r=0}^{s} |f_{h,k}(rk)|_h^2\Big]$$

$$+ |u_{h,k}((p-1)k)|_h^2 + k^2 \sum_{\ell=2}^{p} |\beta_\ell| \sum_{r=p-\ell+1}^{p-1} |\overline{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+ (D_1\, kP(h) + D_2)\, k \sum_{\ell=1}^{p} \sum_{r=p-\ell}^{p-1} |u_{h,k}(rk)|_h^2$$

$$+ D_3 k \sum_{\ell=0}^{p} \sum_{r=p-\ell}^{s-\ell} |f_{h,k}(rk)|_h^2$$

$$+ (D_1\, kP(h) + D_2)\, k \sum_{\ell=0}^{p} \sum_{r=p}^{s-\ell} |u_{h,k}(rk)|_h^2 \;.$$

But

$$a_h((p-1)k;\, u_{h,k}((p-1)k),\; u_{h,k}((p-1)k)) \leq M(h)\, |u_{h,k}((p-1)k)|_h^2 ,$$

$$|\{\overline{\nabla}_k a_h\}(rk; u_{h,k}((r-1)k),\, u_{h,k}((r-1)k))| \leq P(h)|u_{h,k}((r-1)k)|_h^2 \;.$$

Thus because of Hypotheses (4.11), (4.12), (6.9), we may write:

$$(6.12)\qquad |u_{h,k}(sk)|_h^2 + 2\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2 \le D_{18} + D_{19}\, k \sum_{r=p}^{s} |u_{h,k}(rk)|_h^2 , \qquad p \le s \le m-1 .$$

Hence for $D_{19}\, k < 1$

$$(6.13)\qquad |u_{h,k}(sk)|_h^2 + 2\alpha k \sum_{r=p}^{s} \|v_{h,k}(rk)\|_h^2 \le D_{18}\, \exp(D_{19}(s-p+1)k)$$

$$\le D_{18}\, \exp(D_{19}(T-pk)).$$

Note that we may replace condition (6.9) by

$$(6.14)\qquad k\ \mu(h) \le \frac{2}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta' ,$$

where 6' > 0 is arbitrarily small and k is small enough. Then we have proved the following result

Theorem 6.1:

Assume Hypotheses of Theorem 5.1 and, in addition, Hypothesis (6.1). Assume that $1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell| < 0$. Then a sufficient condition for $Q_{h\gamma k}u_{h,k}$ to be $L^\infty(0,T;H)$ - stable and $P_{h,k}v_{h,k}$ to remain in a bounded set of $L^2(0,T;F)$ is given by

$$(6.14)\qquad k\ \mu(h) \le \frac{2}{2\sum_{\ell=2} |\beta_\ell| - 2\beta_1 - 1} - \delta', \qquad \delta' > 0 \text{ arbitrarily small, } k \text{ small enough}.$$

Remark:

We could slightly generalize by replacing $a(t;u,v)$ by $a(t,u,v) + a^1(t;u,v)$ where $a^1(t;u,v)$ is a continuous sesquilinear form on $V \times H$ and the function $t \to a^1(t;u,v)$ is once continuously differentiable. It is easy to see that the results given above remain valid in this case (cf. RAVIART [7]).

7. A weak convergence theorem.

We examine now the convergence of the solution $u_{h,k}$ of Problem I towards the solution of equation (1.6) when h and k tend towards 0. We shall only prove a weak convergence theorem.

Let ϖ be an operator of $\mathcal{L}(V;F)$. Let $\mathcal{V}$ denote a dense subspace of V and let r_h be a linear mapping from $\mathcal{V}$ into V_h. Under the assumptions of Theorem 5.1 or Theorem 6.1, we may extract a subsequence $\{u_{h_1,k_1}\}$ from $\{u_{h,k}\}$ such that

(7.1) $$P_{h_1,k_1} v_{h_1,k_1} \to U \quad \text{weakly in } L^2(0,T;F) ,$$

(7.2) $$Q_{h_1,k_1} u_{h_1,k_1} \to u \quad \text{weakly in } L^\infty(0,T;H) ,$$

when h and k tend towards 0.

Now clearly $P_{h_1,k_1} \to U$ weakly in $L^2(0,T;F)$ and $u = \pi . U$. Then we assume that:

(7.3) $$u \in L^2(0,T;V) , \quad U = \varpi u .$$

Let $\psi(t)$ be a scalar function once continuously differentiable in $[0,T]$ with $\psi(T) = 0$. We assume the following consistency Hypotheses:

$$(7.4)\quad \begin{cases} \text{If} \quad P_{h,k}u_{h,k} \to \varpi u \quad \text{weakly in} \quad L^2(0,T;F), \quad \text{then} \\ k \sum_{r=r_1}^{m-r_2} a_h(rk\ ;\ u_{h,k}(rk),\ \psi((r+r_2-1)k)r_h w) \to \int_0^T a(t;u(t),\psi(t)w)dt \end{cases}$$

where r_1 and r_2 are positive arbitrary integers independent of k; ((7.4) means that $A_h(t)$ is consistent with A(t). Then a necessary and sufficient condition for $\sum_{\ell=0}^{p} \gamma_\ell A_h(t-\ell k)$ to be consistent with A(t) is $\sum_{\ell=0}^{p} \gamma_\ell = 1$)

$$(7.5)\quad \begin{cases} \text{If} \quad Q_{h,k}u_{h,k} \to u \quad \text{weakly in} \quad L^\infty(0,T;H), \quad \text{then} \\ k \sum_{r=r_1}^{m-r_2} (u_{h,k}(rk),\ \bar{\nabla}_k \psi((r+1)k)r_h w)_h \to \int_0^T (u(t),\psi'(t)w)_H dt \ ; \end{cases}$$

$$(7.6)\qquad k \sum_{r=r_1}^{m-r_2} (f_{h,k}((r+r_2-1)k),\ \psi(rk)r_h w)_h \to \int_0^T (f(t),\psi(t)w)_H dt \ ;$$

$$(7.7)\qquad (u_{h,k}((p-1)k),\ \psi(pk)r_h w)_h \to (u_o,\psi(o)w)_H \ ,$$

for all $w \in \mathcal{V}$.

Theorem 7.1:

Assume Hypotheses (7.1),...,(7.7). Then $P_{h,k}u_{h,k} \to \varpi u$ weakly in $L^2(0,T;F)$, $Q_{h,k}u_{h,k} \to u$ weakly in $L^\infty(0,T;H)$, where $u_{h,k}$ denotes the solution of Problem I and u denotes the solution of equation (1.6).

Proof:

We deduce from (2.13):

$$(7.8)\qquad k\sum_{r=p}^{m-1}(\overline{\nabla}_k u_{h,k}(rk),\ \psi(rk)\ r_h w)_h\ +$$

$$+\ k\sum_{r=p}^{m-1}\sum_{\ell=0}^{p}\gamma_\ell\ a_h((r-\ell)k\ ;\ u_{h,k}((r-\ell)k),\ \psi(rk)r_h w)$$

$$=k\sum_{r=p}^{m-1}\sum_{\ell=0}^{p}\gamma_\ell\ (f_{h,k}((r-\ell)k),\psi(rk)r_h w)_h\ .$$

It is easy to see that

$$(7.9)\qquad k\sum_{r=p}^{m-1}(\overline{\nabla}_k u_{h,k}(rk),\psi(rk)r_h w)_h\ =\ -k\sum_{r=p}^{m-1}(u_{h,k}(rk),\ \overline{\nabla}_k\psi((r+1)k)r_h w)_h$$

$$-\ (u_{h,k}((p-1)k),\ \psi(pk)r_h w)_h\ .$$

Then (7.8) may be written

$$(7.10)\qquad -\ k\sum_{r=p}^{m-1}(u_{h,k}(rk),\overline{\nabla}_k\psi((r+1)k)r_h w)_h$$

$$+\ k\sum_{\ell=0}^{p}\gamma_\ell\sum_{r=p-\ell}^{m-\ell-1}a_h(rk;u_{h,k}(rk),\ \psi((r+\ell)k)r_h w)$$

$$=k\sum_{\ell=0}^{p}\gamma_\ell\sum_{r=p-\ell}^{m-\ell-1}(f_{h,k}(rk),\psi((r+\ell)k)r_h w)_h\ +\ (u_{h,k}((p-1)k),\psi(pk)r_h w)_h.$$

If h and k $\to 0$, $P_{h_1,k_1} u_{h_1,k_1} \to \varpi u$ weakly in $L^2(0,T;F)$ and $Q_{h_1,k_1} u_{h_1,k_1} \to u$ weakly in $L^\infty(0,T,H)$. So u satisfies

$$(7.11) \quad \int_0^T \{-(u(t),\psi'(t)w)_H + a(t\,;u(t),\psi(t)w)\}dt = \int_0^T (f(t),\psi(t)w)_H dt$$

$$+ (u_0,w)_H \ \psi(0) \ ,$$

for all $w \in \mathcal{V}$. It follows from the density of $\mathcal{V}$ in V that (7.11) is true for all $w \in V$. Moreover the space of functions $\Sigma\, \psi \otimes w$ (Σ denotes a finite sum), $\psi \in C^1(0,T)$ with $\psi(T) = 0$, $w \in V$, is dense in the space of functions φ satisfying (1.7) provided with the norm $\left(\int_0^T (\|\varphi(t)\|_V^2 + \|\varphi'(t)\|_H^2\, dt\right)^{1/2}$ (cf. Lions [4]). Then u satisfies (1.06). We deduce from the uniqueness of the solution u of (1.6) that $P_{h,k}u_{h,k} \to \varpi u$ weakly in $L^2(0,T;F)$ and $Q_{h,k}u_{h,k} \to u$ weakly in $L^\infty(0,T;H)$.

8. Regularity theorems:

a) The Hermitian Case:

For every $t \in [0,T]$, we are given a continuous sesquilinear form $a(t;u,v)$ on V X V with the following hypotheses:

(i) $t \to a(t;u,v)$ is once continuously differentiable in $[0,T](u,v \in V)$;

(ii) $a(t;u,v) = \overline{a(t;v,u)}$ $(u,v \in V)$;

(iii) $a(t;v,v) + \lambda\|v\|_H^2 \geq \alpha\|v\|_V^2 \quad \alpha > 0,\ v \in V$.

Then we have the following regularity theorem (LIONS [4])

Theorem 8.1:

Given:

(8.1) $$f \in L^2(0,T;H) ,$$

(8.2) $$u_0 \in V .$$

There exists a unique function u satisfying

(8.3) $$u \in L^\infty(0,T;V) ,$$

(8.4) $$u' \in L^2(0,T;H) ,$$

(8.5) $$u'(t) + A(t)u(t) = f(t) , \quad \text{for} \quad \text{a.e. } t \in [0,T] ,$$

(8.6) $$u(0) = u_0 .$$

Remark:

For a slight generalization see the remark following Theorem 6.1.

For every $t \in [0,T]$, we are given a family of continuous sesquilinear forms $a_h(t;u_h,v_h)$ on $V_h \times V_h$ as in §6 (i.e. satisfying Hypotheses (i), (ii), (iii) of §2 and Hypothesis (6.1)). Let $u_{h,k} \in E_k(0,T;V_h)$ be the solution of Problem I and let us assume now that there exist two positive constants K_4, K_5 independent of h and k such that

(8.7) $$|u_{h,k}(rk)|_h, \ \|u_{h,k}(rk)\|_h \leq K_4, \quad r = 0,\ldots\ldots,p-1 .$$

(8.8) $$k \sum_{r=1}^{p-1} | \bar{\nabla}_k u_{h,k}(rk) |_h^2 \leq K_5 .$$

Theorem 8.2:

Assume Hypotheses (i), (ii), (iii) of §2 and Hypotheses (6.1), (8.7), (8.8). There exist two positive constants E_3, E_4 independent of h and k such that

(8.9) $$\|u_{h,k}\|_{h,k,\infty} , \; |u_{h,k}|_{h,k,\infty} \leq E_3 ,$$

(8.10) $$k \sum_{r=1}^{m-1} |\bar{\nabla}_k u_{h,k}(rk)|_h^2 \leq E_4 ,$$

in the two following cases

(i) 1st Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0$,

(4.13) $k(\nu(h))^2 \leq \rho$ (No restriction if $A_h(t)$ is independent of t),

(ii) 2nd Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| < 0$,

(6.14) $k\mu(h) \leq \dfrac{2}{2 \sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta'$, $\delta' > 0$ arbitrarily small,

k small enough .

Proof:

(i) 1st Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0$:

Theorem 4.1 gives:

(4.25) $$|u_{h,k}|_{h,k,\infty} \le E_2 .$$

Inequation (6.4) is true and we deduce for $p \le s \le m - 1$:

(8.11)
$$k \sum_{r=p}^{s} |\overline{\nabla}_k u_{h,k}(rk)|_h^2 + \frac{1}{2} a_h(sk ; u_{h,k}(sk), u_{h,k}(sk))$$

$$+ \frac{k}{2}(1 + 2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell|)k \sum_{r=p}^{s} a_h(rk ; \overline{\nabla}_k u_{h,k}(rk), \overline{\nabla}_k u_{h,k}(rk))$$

$$- \frac{k}{2} \sum_{\ell=2}^{p} |\beta_\ell| k \sum_{r=p}^{s} a_h(rk ; \overline{\nabla}_k u_{h,k}((r-\ell+1)k), \overline{\nabla}_k u_{h,k}((r-\ell+1)k))$$

$$\le \frac{1}{2} a_h((p-1)k ; u_{h,k}((p-1)k), u_{h,k}((p-1k)) + \frac{P}{2} k \sum_{r=p}^{s} \|u_{h,k}((r-1)k)\|_h^2$$

$$+ k\, P(h) \sum_{\ell=2}^{p} (\ell-1) |\beta_\ell| k \sum_{r=p}^{s} |u_{h,k}((r-\ell+1)k)|_h |\overline{\nabla}_k u_{h,k}(rk)|_h$$

$$+ k\, P(h) \sum_{\ell=1}^{p} \ell |\beta_\ell| k \sum_{r=p}^{s} |u_{h,k}((r-\ell)k)|_h |\overline{\nabla}_k u_{h,k}(rk)|_h$$

$$+ \sum_{\ell=0}^{p} |\gamma_\ell| k \sum_{r=p}^{s} |f_{h,k}((r-\ell)k)|_h |\overline{\nabla}_k u_{h,k}(rk)|_h .$$

But

$$a_h(rk ; \overline{\nabla}_k u_{h,k}((r-\ell+1)k), \overline{\nabla}_k u_{h,k}((r-\ell+1)k))$$

$$= a_h((r-\ell+1)k ; \overline{\nabla}_k u_{h,k}((r-\ell+1)k), \overline{\nabla}_k u_{h,k}((r-\ell+1)k))$$

$$+ (\ell-1)k \{ \overline{\nabla}_{(\ell-1)k} a_h \} (rk ; \overline{\nabla}_k u_{h,k}((r-\ell+1)k), \overline{\nabla}_k u_{h,k}((r-\ell+1)k)),$$

and

$$|\{\overline{\overline{\nabla}}_{(\ell-1)k}a_h\}(rk\ ;\overline{\overline{\nabla}}_k u_{h,k}((r-\ell+1)k),\overline{\overline{\nabla}}_k u_{h,k}((r-\ell+1)k))|$$

$$\leq P(h)\ |\overline{\overline{\nabla}}_k u_{h,k}((r-\ell+1)k)|_h^2\ .$$

Hence we deduce from (8.11), Hypotheses (4.13), (8.7), (8.8) and (4.25) that for $p \leq s \leq m-1$ and k small enough.

(8.12)
$$k\sum_{r=p}^{s}|\overline{\overline{\nabla}}_k u_{h,k}(rk)|_h^2 + \alpha\|u_{h,k}(sk)\|_h^2$$

$$\leq D_1 + D_2 k\sum_{r=p}^{s-1}\|u_{h,k}(rk)\|_h^2\ .$$

Then applying lemma 3.3 gives the first part of the theorem.

(ii) 2nd Case: $1 + 2\beta_1 - 2\sum_{\ell=2}^{p}|\beta_\ell| < 0$:

The second part 'of the theorem can be easily deduced from inequality (6.10).

. Remark:

It is easy to determine $\{u_{h,k}(rk),\ r = 0,\dots,p-1\}$ verifying (8.7) and (8.8) when $u_0 \in V$. See RAVIART [7].

Let us assume now that (cf. §5):

(8.13)
$$\sup_{t\in[0,T]}\|P_{h,k}u_{h,k}(t)\|_F \leq C_4\|u_{h,k}\|_{h,\ k,\ \infty}\ ,$$

(8.14)
$$\|\frac{d}{dt}Q_{h,k}u_{h,k}\|^2_{L^2(0,T;H)} \leq C_5 k\sum_{r=1}^{m-1}|\overline{\overline{\nabla}}_k u_{h,k}(rk)|_h^2\ ,$$

for all $u_{h,k} \in E_k(0,T;V_h)$, where C_4 and C_5 are positive constants independent of h and k.

Theorem 8.3:

Let $u_{h,k}$ be the solution of Problem I. Under the assumptions of Theorem 8.2 and, in addition, Hypothese P_h(8.13), $({}_k u_h 4)$, k, is $L^\infty(0,T;F)$-stable and $\frac{d}{dt} Q_{h,k} u_{h,k}$ remains in a bounded set of $L^2(0,T;H)$ in the two following cases:

(i) 1st Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0$,

(4.13) $k(\nu(h))^2 \leq \rho$ (No restriction if $A_h(t)$ is independent of t)

(ii) 2nd Case: $1 + 28, - 2 \sum_{\ell=2}^{p} |\beta_\ell| < 0$,

(6.14) $N\text{-}44 \leq \frac{2}{2 \sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta'$; $8' > 0$ arbitrarily small, k small enough .

If, in addition, the assumptions of Theorem 7.1 are verified, then

$$P_{h,k} u_{h,k} \to \varpi u \text{ weakly in } L^\infty(0,T;F) ,$$

$$\frac{d}{dt} Q_{h,k} u_{h,k} \to u' = \frac{du}{dt} \text{ weakly in } L^2(0,T;H) ,$$

in cases (i), (ii).

b) A general regularity theorem.

For every $t \in [0,T]$, we are given a continuous sesquilinear form $a(t;u,v)$ on $V \times V$ with the following hypotheses:

(i) $t \to a(t;u,v)$ is once continuously differentiable in $[0,T]$ $(u,v \in V)$;

(ii) there exist constants λ, α such that

$$\operatorname{Re} a(t;v,v) + \lambda\|v\|_H^2 \geq \alpha\|v\|_V^2 \ , \ \alpha > 0 \ , \ v \in V \ .$$

Let $A(t) \in \mathcal{L}(V;V')$ be defined by

$$a(t;u,v) = \langle A(t)u,v \rangle \quad (u,v \in V) \ .$$

We denote by $D(A(t))$ the set of all u in V such that $A(t)u \in H$.

We provide $D(A(t))$ with the norm

$$\|u\|_{D(A(t))} = \left(\|u\|_H^2 + \|A(t)u\|_H^2 \right)^{1/2} .$$

Theorem 8.4:

Let f, u_o be given satisfying

(8.15) $$f \in L^2(0,T;H), \ f' \in L^2(0,T;H) \ ,$$

(8.16) $$u_o \in D(A(0)) \ .$$

There exists a unique function u which verifies:

(8.17) $$u \in L^\infty(0,T;V) \ ,$$

(8.18) $$u' \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

(8.19) $$u'(t) + A(t)u(t) = f(t) \ , \text{ for a.e. } t \in [0,T] \ ,$$

(8.20) $$u(o) = u_o \ .$$

For every $t \in [0,T]$, we are given a family of continuous sesquilinear forms $a_h(t;u_h,v_h)$ on $V_h \times V_h$ as in §2 (i.e., satisfying

Hypotheses (i), (ii), (iii) of §2). Let $u_{h,k} \in E_k(0,T;V_h)$ denote the solution of Problem I and let us assume that there exist five positive constants $K_6,\dots,K_{10}$ independent of h and k such that:

$$|u_{h,k}(rk)|_h,\ \|u_{h,k}(rk)\|_h \leq K_6, \qquad r = 0,\dots,p, \tag{8.21}$$

$$|A_h(rk)u_{h,k}(rk)|_h \leq K_7, \qquad r = 0,\dots,p-1, \tag{8.22}$$

$$|\bar{\nabla}_k u_{h,k}(rk)|_h \leq K_8, \qquad r = 1,\dots,p-1, \tag{8.23}$$

$$k \sum_{r=1}^{p-1} \|\bar{\nabla}_k u_{h,k}(rk)\|_h^2 \leq K_9, \tag{8.24}$$

$$k^2 \sum_{r=2}^{p-1} |\bar{\nabla}_k^2 u_{h,k}(rk)|_h^2 \leq K_{10}. \tag{8.25}$$

Moreover let us assume that:

(8.26) if $\|v_{h,k}\|_{h,k,2}$ is bounded by a constant independent of h and k ($v_{h,k}$ defined by (4.4) and (4.24)), then $\|u_{h,k}\|_{h,k,2}$ has the same property.[1]

Theorem 8.5:

Assume Hypotheses (i), (ii), (iii) of §2 and Hypotheses (8.21),...,(8.26). There exist four positive constants $E_5,\dots,E_8$ independent of h and k such that

---

[1] Hypothesis (8.26) is trivially verified when Theorem 7.1 may be applied and $\|w_{h,k}\|_{h,k,2} \leq C_2'\|P_{h,k}w_{h,k}\|_{L^2(0,T;F)}$, for all $w_{h,k} \in E_k(0,T;V_h)$, where C_2' is a constant independent of h and k.

(8.27) $$|u_{h,k}|_{h,k,\infty}\ ,\ \|u_{h,k}\|_{h,k,2}\ ,\ \|v_{h,k}\|_{h,k,\infty} \leq E_5 \quad ,$$

(8.28) $$|A_h(\cdot)u_{h,k}(\cdot)|_{h,k,\infty} \leq E_6 \ ,$$

(8.29) $$|\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h \leq E_7 \ , \qquad r = 1,\ldots\ldots,m-1 \ ,$$

(8.30) $$k \sum_{r=1}^{m-1} \|\bar{\bar{\nabla}}_k v_{h,k}(rk)\|_h^2 \leq E_8 \ ,$$

in the two following cases:

(i) 1st Case: $$1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell| \geq 0 \quad ,$$

(4.13) $$k(\nu(h))^2 \leq \rho \quad ,$$

(ii) 2nd Case: $$1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell| < 0 \quad ,$$

(4.22) $$k(\nu(h))^2 \leq \frac{2\alpha}{2\sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta', \quad \delta' > 0$$ arbitrarily small, k small enough.

Proof:

First, we may apply Theorem 4.1 and, in cases (i), (ii), we obtain:

(8.31) $$|u_{h,k}|_{h,k,\infty} \leq E_1 \ ,$$

(8.32) $$\|v_{h,k}\|_{h,k,2} \leq E_2 \ .$$

We deduce from (8.26) and (8.32):

(8.33) $$\|u_{h,k}\|_{h,k,2} \leq E_2' \quad .$$

Then a "discrete" differentiation of equation (2.13) with respect to t gives:

$$(8.34)\qquad (\bar{\bar{\nabla}}_k^2 u_{h,k}(rk), v_h)_h + \sum_{\ell=0}^{p} \gamma_\ell\, a_h((r-\ell)k\ ;\ \bar{\bar{\nabla}}_k u_{h,k}((r-\ell)k), v_h)$$

$$+ \sum_{\ell=0}^{p} \gamma_\ell\, \{\bar{\bar{\nabla}}_k a_h\}\ ((r-\ell)k;\ u_{h,k}((r-\ell+1)k), v_h)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\, (\bar{\bar{\nabla}}_k f_{h,k}((r-\ell)k), v_h)_h\ , \qquad r = p+1, \ldots\ldots, m-1\ .$$

We put (8.34) into a more convenient form as in §4.b.

$$(8.35)\qquad (\bar{\bar{\nabla}}_k^2\, u_{h,k}(rk), v_h)_h + a_h(rk\ ;\ \bar{\bar{\nabla}}_k v_{h,k}(rk), v_h)$$

$$- k \sum_{\ell=2}^{p} (\ell-1)\, \beta_\ell\, \{\bar{\bar{\nabla}}_{(\ell-1)k} a_h\}\ (rk\ ;\ \bar{\bar{\nabla}}_k u_{h,k}((r-\ell+1)k), v_h)$$

$$+ k \sum_{\ell=1}^{p} \ell\beta_\ell\, \{\bar{\bar{\nabla}}_{\ell k} a_h\}\ (rk\ ;\ \bar{\bar{\nabla}}_k u_{h,k}((r-\ell)k), v_h)$$

$$+ \sum_{\ell=0}^{p} \gamma_\ell\, \{\bar{\bar{\nabla}}_k a_h\}\ ((r-\ell)k\ ;\ u_{h,k}((r-\ell-1)k), v_h)$$

$$= \sum_{\ell=0}^{p} \gamma_\ell\, (\bar{\bar{\nabla}}_k f_{h,k}((r-\ell)k), v_h)_h \qquad r = p+1, \ldots\ldots, m-1\ .$$

We replace v_h in equation (8.35) by $\bar{\bar{\nabla}}_k v_{h,k}(rk)$. We obtain:

$$(8.36) \quad \bar{\nabla}_k |\bar{\nabla}_k u_{h,k}(rk)|_h^2 + (1 + 2\beta_1 - \sum_{\ell=2}^{p} |\beta_\ell|)\, k |\bar{\nabla}_k^2 u_{h,k}(rk)|_h^2$$

$$- k \sum_{\ell=2}^{p} |\beta_\ell|\, |\bar{\nabla}_k^2 u_{h,k}((r-\ell+1)k)|_h^2 + 2\alpha \|\bar{\nabla}_k v_{h,k}(rk)\|_h^2$$

$$\leq (D_3 kP(h) + D_4) \sum_{\ell=0}^{p} |\bar{\nabla}_k u_{h,k}((r-\ell)k)|_h^2 + \epsilon \|\bar{\nabla}_k v_{h,k}(rk)\|_h^2$$

$$+ D_5(\epsilon) \sum_{\ell=0}^{p} \|u_{h,k}((r-\ell-1)k)\|_h^2 + D_6 \sum_{\ell=0}^{p} |\bar{\nabla}_k f_{h,k}((r-\ell)k)|_h^2 ,$$

$r = p+1,\ldots\ldots,m-1$, where $\epsilon > 0$ may be chosen as small as we please, Multiplying equation (8.36) by k and summing from $r = p+1$ to $r = s$ ($p+1 \leq s \leq m-1$) gives:

$$(8.37) \quad |\bar{\nabla}_k u_{h,k}(sk)|_h^2 + (1 + 2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell|)k^2 \sum_{r=p+1}^{s} |\bar{\nabla}_k^2 u_{h,k}(rk)|_h^2$$

$$+ (2\alpha-\epsilon)\, k \sum_{r=p+1}^{s} \|\bar{\nabla}_k v_{h,k}(rk)\|_h^2$$

$$\leq |\bar{\nabla}_k u_{h,k}(pk)|_h^2 + k^2 \sum_{\ell=2}^{p} |\beta_\ell| \sum_{r=p-\ell+2} |\bar{\nabla}_k^2 u_{h,k}(rk)|_h^2$$

$$+ (D_3 kP(h) + D_4)k \sum_{\ell=1}^{p} \sum_{r=p-\ell+1}^{p} |\bar{\nabla}_k u_{h,k}(rk)|_h^2$$

$$+ D_6\, k \sum_{\ell=0}^{p} \sum_{r=p+1}^{s} |\bar{\nabla}_k f_{h,k}((r-\ell)k)|_h^2 + D_5(\epsilon)k \sum_{\ell=0}^{p} \sum_{r=p+1}^{s}$$

$$\|u_{h,k}((r-\ell-1)k)\|_h^2$$

$$+ (D_3 kP(h) + D_4)k \sum_{\ell=0}^{p} \sum_{r=p+1}^{s-\ell} |\bar{\nabla}_k u_{h,k}(rk)|_h^2 .$$

It is very easy to prove (cf. RAVIART [7]):

$$(8.38) \qquad k \sum_{r=1}^{m-1} |\bar{\bar{\nabla}}_k f_{h,k}(rk)|_h^2 \leq C_1^2 \int_0^T \|f'(t)\|_H^2 \, dt \ ,$$

$$(8.39) \qquad |f_{h,k}|_{h;\infty}^2 \leq 2C_1^2 \left(\frac{1}{T} \int_0^T \|f(t)\|_H^2 \, dt + T \int_0^T \|f'(t)\|_H^2 \, dt\right) .$$

Then, it follows from the inequalities (4.13), (8.21), (8.23), (8.25) (8.33) and (8.38) that

$$(8.40) \qquad |\bar{\bar{\nabla}}_k u_{h,k}(sk)|_h^2 + (1 + 2\beta_1 - 2 \sum_{\ell=1}^{p} |\beta_\ell|) k^2 \sum_{r=p+1}^{s} |\bar{\bar{\nabla}}_k^2 u_{h,k}(rk)|_h^2$$

$$+ (2\alpha - \epsilon) k \sum_{r=p+1}^{s} \| \bar{\bar{\nabla}}_k v_{h,k}(rk) \|_h^2$$

$$\leq D_7 \ |\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h^2 + k^2 \sum_{\ell=2}^{p} |\beta_\ell| \, |\bar{\bar{\nabla}}_k^2 u_{h,k}(pk)|_h^2 + D_8$$

$$+ D_9 k \sum_{r=p+1}^{s} |\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h^2 \ , \qquad s = p+1, \ldots\ldots, m-1 \ .$$

Now,

$$k^2 |\bar{\bar{\nabla}}_k^2 u_{h,k}(pk)|_h^2 \leq 2 |\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h^2 + 2 |\bar{\bar{\nabla}}_k u_{h,k}((p-1)k)|_h^2 \ .$$

An estimate for $|\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h$ is obtained as follows, Equation (2.13) gives for $r = p$:

$$|\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h^2 \leq - \gamma_0 \ \mathrm{Re} \ a_h(pk \ ; u_{h,k}(pk), \bar{\bar{\nabla}}_k u_{h,k}(pk))$$

$$+ [\sum_{\ell=1}^{p} |\gamma_\ell| \ |A_h((p-\ell)k) u_{h,h}((p-\ell)k)|_h + \sum_{\ell=0}^{p} |\gamma_\ell| \, |f_{h,k}((p-\ell)k)|_h]$$

$$|\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h \ .$$

But

$$a_h(pk;u_{h,k}(pk), \bar{\bar{\nabla}}_k u_{h,k}(pk) = a_h(pk;u_{h,k}((p-1)k), \bar{\bar{\nabla}}_k u_{h,k}(pk))$$

$$+ k\ a_h(pk; \bar{\bar{\nabla}}_k u_{h,k}(pk), \bar{\bar{\nabla}}_k u_{h,k}(pk)).$$

and

$$-\text{Re}\ a_h(pk;u_{h,k}(pk), \bar{\bar{\nabla}}_k u_{h,k}(pk) \le |A_h((p-1)k\ u_{h,k}((p-1)k)|_h |\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h$$

$$+ k\ P(h)\ \|u_{h,k}((p-1)k\|_h |\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h\ .$$

Hence,

(8.41) $$|\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h \le \gamma_o\ |A_h((p-1)k)\, u_{h,k}((p-1)k)|_h$$

$$+ \sum_{\ell=1}^{p} |\gamma_\ell|\ |A_h((p-\ell)k)\ u_{h,k}((p-\ell)k)|_h + \sum_{\ell=0}^{p} |\gamma_\ell|\ |f_{h,k}((p-\ell)k)|_h$$

$$+ k\ P(h)\ \|u_{h,k}((p-1)k)\|_h\ .$$

Then the inequalities (8.21),(8.22), (8.39) imply that $|\bar{\bar{\nabla}}_k u_{h,k}(pk)|_h$ is-bounded independently of h and k. So (8.40) becomes:

(8.42) $$|\bar{\bar{\nabla}}_k u_{h,k}(sk)|_h^2 + (1+2\beta_1 - 2\sum_{\ell=2}^{p} |\beta_\ell|)k^2 \sum_{r=p+1}^{s} |\bar{\bar{\nabla}}_k^2\ u_{h,k}(rk)|_h^2$$

$$+ (2\alpha-\epsilon)k \sum_{r=p+1}^{s} \|\bar{\bar{\nabla}}_k v_{h,k}(rk)\|_h^2 \le D_{10} + D_g\ k \sum_{r=p+1}^{s} |\bar{\bar{\nabla}}_k u_{h,k}(rk)|_h^2,$$

$s = p+1,\ldots,m-1$.

(i) 1^{st} Case:, $\quad 1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \geq 0$.

By applying lemma 3.3, we find for $kD_9 < 1$ and for $p + 1 \leq s \leq m - 1$

$$(8.43) \qquad |\overline{\nabla}_k u_{h,k}(sk)|_h^2 + (2\alpha - \epsilon)\, k \sum_{r=p+1}^{s} \|\overline{\nabla}_k v_{h,k}(rk)\|_h^2$$

$$\leq D_{10} \exp (D_9 (T-((p+1)k))) .$$

But

$$\alpha \|\overline{\nabla}_k v_{h,k}(pk)\|_h^2 \leq \mathrm{Re}\ a_h(pk; \overline{\nabla}_k v_{h,k}(pk), \overline{\nabla}_k v_{h,k}(pk) \leq M(h)\ |\overline{\nabla}_k v_{h,k}(pk)|_h^2$$

$$\leq D_{11}\ M(h) \left(|\overline{\nabla}_k u_{h,k}(pk)|_h^2 + \sum_{r=1}^{p-1} |\overline{\nabla}_k u_{h,k}(rk)|_h^2 \right) .$$

Thus, because of Hypothesis (4.13), $k\|\overline{\nabla}_k v_{h,k}(pk)\|_h^2$ is bounded independently of h and k.

Then we find

$$(8.44) \qquad |\overline{\nabla}_k u_{h,k}(rk)|_h^2 \leq D_{12} \qquad r = 1,\ldots,m-1 .$$

$$(8.45) \qquad k \sum_{r=1}^{m-1} \|\overline{\nabla}_k v_{h,k}(rk)\|_h^2 \leq D_{13} .$$

(ii) 2^{nd} Case: $\quad 1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| < 0$.

We deduce from (2.4) that

$$(8.46) \qquad \left|\frac{d}{dt} a_h(t; u_h, v_h)\right| \leq Q(h)\ \|u_h\|_h\ |v_h|_h \qquad (u_h, v_h \in V_h) ,$$

where $Q(h)$ has the same order of magnitude as $N(h)$ when $h \to 0$. To obtain an estimate for $|\overline{\nabla}_k^2 u_{h,k}(rk)|_h^2$, $r = p+1,\ldots,m-1$, we replace v_h in equation (8.35) by $\overline{\nabla}_k^2 u_{h,k}(rk)$ and apply inequalities (2.7), (2.8), (8.46):

$$|\overline{\nabla}_k^2 u_{h,k}(rk)|_h \leq N(h) \; \|\overline{\nabla}_k v_{h,k}(rk)\|_h + k\ P(h) \sum_{\ell=2}^{p} (\ell-1)|\beta_\ell|\overline{\nabla}_k u_{h,k}((r-\ell+1)k)|_h$$

$$+ k\ P(h) \sum_{\ell=1}^{p} \ell|\beta_\ell||\overline{\nabla}_k u_{h,k}((r-\ell)k)|_h + Q(h) \sum_{\ell=0}^{p} |\gamma_\ell| \; \|u_{h,k}((r-\ell-1)k)\|_h$$

$$+ \sum_{\ell=0}^{p} |\gamma_\ell||\overline{\nabla}_k f_{h,k}((r-\ell)k)|_h \quad .$$

Using Hypothesis (4.22) and by the same device as in §4.b,(ii), this gives (8.44) and (8.45).

Now, we have:

$$v_{h,k}(rk) = u_{h,k}((p-1)k) + k \sum_{s=p}^{r} \overline{\nabla}_k v_{h,k}(sk) \quad .$$

This identity implies that

$$\|v_{h,k}\|_{h,k,\varphi} \leq \; E_5$$

'because of the inequalities (8.32),(8.45) and (8.21).

It remains to show the inequality (8.28). This is a trivial consequence of equation (2.17) and inequalities (8.22), (8.39), (8.44). This completes the proof of the theorem, Remark:

It is easy to determine $\{u_{h,k}(rk),\ r = 0,\dots,p-1\}$ verifying (8.21),. . ., (8.25) by two level difference schemes when $u_o \in D(A(0))$: We choose $u_{h,k}(0) = Q_h u_o$ with

$$|Q_h' u_o|_h,\ \|Q_h' u_o\|_h,\ |A_h(0)\, Q_h' u_o|_h \le C_1 \|u_o\|_{D(A(0))},$$

and apply Theorem 8.5, with $p = 1$.

Let us assume now that

(8.47) $$\sup_{t\in[0,T]} \|P_{h,k} u_{h,k}(t)\|_F \le C_6 \|u_{h,k}\|_{h,k,\infty},$$

(8.48) $$\left\|\frac{d}{dt} P_{h,k} u_{h,k}\right\|_{L^2(0,T;F)} \le C_7 \left(k \sum_{r=1}^{m-1} \|\bar{\nabla}_k u_{h,k}(rk)\|_h^2\right)^{1/2},$$

(8.49) $$\sup_{t\in[0,T]} \left\|\frac{d}{dt} Q_{h,k} u_{h,k}(t)\right\|_H \le C_8 \sup_{r=1,\dots,m-1} |\bar{\nabla}_k u_{h,k}(rk)|_h,$$

for all $u_{h,k} \in E_k(0,T;V_h)$, where C_6, C_7, C_8 are positive constants independent of h and k.

Theorem 8.6:

Let $u_{h,k}$ be the solution of Problem. Under the assumptions of Theorem 8.5 and, in addition, Hypotheses (8.47), (8.48), (8.49), $\dot{Q}_{h,k} u_{h,k}$ is $L^\infty(0,T;H)$-stable, $P_{h,k} v_{h,k}$ is $L^\infty(0,T;F)$-stable, $\frac{d}{dt} Q_{h,k} u_{h,k}$ is $L^\infty(0,T;H)$-stable, $\frac{d}{dt} P_{h,k} v_{h,k}$ remains in a bounded set of $L^2(0,T;F)$ in the two following cases:

(i) 1st Case: $$1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| \ge 0,$$

(4.13) $$k(\nu(h))^2 \le \rho,$$

(ii) 2nd Case: $1 + 2\beta_1 - 2 \sum_{\ell=2}^{p} |\beta_\ell| < 0$,

(4.22) $$k(\nu(h))^2 \leq \frac{2\alpha}{2 \sum_{\ell=2}^{p} |\beta_\ell| - 2\beta_1 - 1} - \delta', \quad \delta' > 0 \text{ arbitrarily small,}$$

k small enough.

If, in addition, the assumptions of Theorem 7.1 are verified, then, in cases P_h, $\{u_{h,k}\}$ is $L^\infty(0,T;F)$-stable and

$$P_{h,k}u_{h,k} \to \varpi u \quad \text{weakly in } L^\infty(0,T;F) ,$$

$$\frac{d}{dt} Q_{h,k}u_{h,k} \to u' = \frac{du}{dt} \quad \text{weakly in } L^\infty(0,T;H) ,$$

$$\frac{d}{dt} P_{h,k}u_{h,k} \to \varpi u' = \varpi \frac{du}{dt} \quad \text{weakly in } L^2(0,T;F)$$

Proof:

The first part of the theorem is trivial. We deduce, as in Theorem 7.1, that $P_{h,k}u_{h,k} \to \varpi u$ weakly in $L^\infty(0,T;F)$,

$$\frac{d}{dt} \varphi_{h,k}u_{h,k} \to u' \quad \text{weakly in } L^\infty(0,T;H) .$$

$$\frac{d}{dt} P_{h,k}u_{h,k} \to \varpi u' \quad \text{weakly in } L^2(0,T;F) .$$

Then $P_{h,k}u_{h,k}$ remains in a bounded set of $L^\infty(0,T;F)$ and $\frac{d}{dt} P_{h,k}u_{h,k}$ remains in a bounded set of $L^2(0,T;F)$. But it is easy to see that

(8.50) $$\sup_{t \in [0,T]} \|P_{h,k}u_{h,k}(t)\|_F^2 \leq D_{14} \left[\int_0^T \{\|P_{h,k}u_{h,k}(t)\|_F^2 + \|\frac{d}{dt} P_{h,k}u_{h,k}(t)\|_F^2\} dt\right]$$

Thus, $P_{h,k}u_{h,k}$ is $L^\infty(0,T;F)$-stable.

9. Applications to parabolic partial differential equations.

We shall study here a simple example. For other examples see Lions [4]. Let Ω be a bounded set in $\mathcal{R}^n$. We choose

$$H = L^2(\Omega),$$

$$V = H^1(\Omega) = \{u \mid u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1,\dots,n\} ,$$

$$a(t;u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t) \frac{\partial u(x)}{\partial x_j} \overline{\frac{\partial v(x)}{\partial x_i}} dx$$
$$+ \sum_{i=1}^{n} \int_\Omega a_i(x, t) \frac{\partial u(x)}{\partial x_i} \bar{v}(x) dx$$
$$+ \int_\Omega a_o(x,t) u(x) \overline{v(x)} dx ,$$

where $a_{ij}, a_i, a_o \in L^\infty(\Omega \times (0,T))$.

We assume that

$$\text{Re} \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_j\bar{\xi}_i \geq \alpha \sum_{i=1}^{n} |\xi_i|^2, \quad \alpha > 0, \quad \xi_i \in \mathcal{C}, \text{ a.e. in } \Omega \times (0,T).$$

Then we may apply Theorem 1.1. There exists a unique function u which satisfies:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{i,j}(x,t) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} a_i(x,t) \frac{\partial u}{\partial x_i} + a_o(x,t)u = f ,$$

a. e. in $\Omega \times]0,T[$, with the initial condition

$$u(x,0) = u_o(x) ,$$

and the (formal) boundary condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t) \cos (n,x_i) \frac{\partial u}{\partial x_j} = 0 \text{ for } x \in \Gamma = \partial\Omega, \ t \in]0,T[.$$

where n denotes the exterior normal to I' in x. This boundary condition makes sense when Γ is smooth enough, see Lions-Magenes [6]. We examine now the approximation of the solution u.

a) The spaces V_h:

Let $\mathcal{R}_h$ denote the set of points $M \in \mathcal{R}^n$ such that

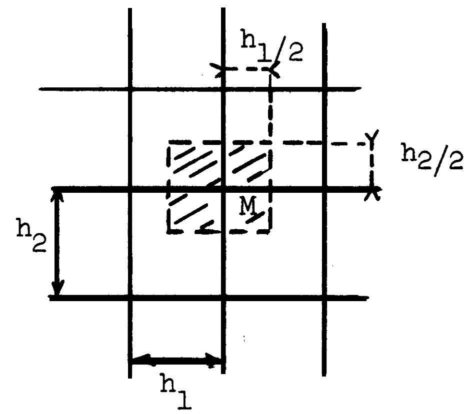
$$M = (e_1h_1,\ldots,e_nh_n)$$

where the e_i's are integers. Let $\sigma_h(M,o)$ be the set of points $x \in \mathcal{R}^n$ such that

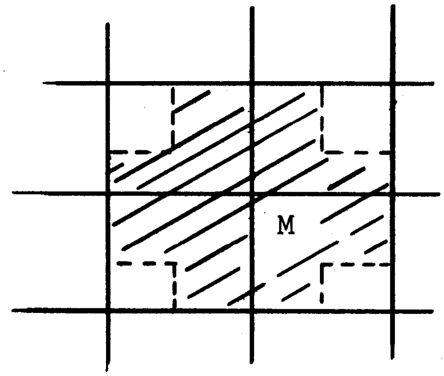
$$x_i(M) - \frac{h_i}{2} < x_i < x_i(M) + \frac{h_i}{2} .$$

$W_{h,M}$ denotes the characteristic function of $\sigma_h(M,o)$. Let $\sigma_h(M,1)$ be defined by

$$\sigma_h(M,1) = \bigcup_{i=1}^{n} \sigma_h(M \pm \frac{h_i}{2}\ 0)$$



$\sigma_h(M,0)$



$\sigma_h(M,1)$

We define: $\Omega_h = \{M \mid M \in \mathcal{R}_h, \ \sigma_h(M,1) \cap \Omega \neq \emptyset\}$,

$$R(h) = \bigcup_{M \in \Omega_h} \sigma_h(M,o) \quad .$$

Then V_h is the space of functions u_h of the form

$$u_h = \sum_{M \in \Omega_h} u_h(M)\, W_{h,M} \; , \qquad u_h(M) \in \mathcal{C} \; .$$

If u_h belongs to V_h, we may define $\delta_i u_h$ by

$$\delta_i u_h(x) = \frac{1}{h_i} \left[u_h\left(x + \frac{h_i}{2}\right) - u_h\left(x - \frac{h_i}{2}\right)\right] \quad \text{a.e. in } \Omega \; ,$$

We set:

$$(u_h, v_h)_h = \int_{\Omega(h)} u_h(x)\, \overline{v_h(x)}\, dx \; ,$$

$$((u_h, v_h))_h = \int_\Omega u_h(x)\, \overline{v_h(x)}\, dx + \sum_{i=1}^{n} \int_\Omega \delta_i u_h(x)\, \delta_i \overline{v_h(x)}\, dx \; .$$

We choose:

$$Q_h u = \frac{1}{h_1 \cdots h_n} \sum_{M \in \Omega_h} \left(\int_{\sigma_h(M,o)} \tilde{u} \, dx \right) W_{h,M}, \quad \text{for all } u \in L^2(\Omega),$$

where $\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{elsewhere.} \end{cases}$

Let $F = [L^2(\Omega)]^{n+1}$. If $U = (u, u_1, \ldots, u_n)$ belongs to $[L^2(\Omega)]^{n+1}$, we set: $u = \pi U \in L^2(\Omega)$.

If u belongs to $V = H^1(\Omega)$, we set:

$$\varpi u = \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \in [L^2(\Omega)]^{n+1} \quad .$$

Now, we define the prolongation operator $P_{h,k}$ (cf. §5). If $u_{h,k}$ belongs to $E_k(0,T;V_h)$, we may define:

$$u_{h,k}(t) = u_{h,k}(rk) + (t-rk) \overline{\nabla}_k u_{h,k}((r+1)k) \ , \ rk \leq t \leq (r+1)k \ ,$$

$$u_{h,k}(t) = u_{h,k}((m-1)k) \qquad , \ T - k \leq t \leq T \ .$$

Then,

$$P_{h,k} u_{h,k}(t) = (u_{h,k}(t), \delta_1 u_{h,k}(t), \ldots, \delta_n u_{h,k}(t)) \in [L^2(\Omega)]^{n+1}.$$

The verification of our Hypotheses is trivial and left to the reader.,
For other examples of spaces V_h, see CÉA [2], RAVIART [7].

b) <u>The forms</u> $a_h(t; u_h, v_h)$:

Let us assume that each a_{ij} (resp. a_i, a_o) has one continuous derivative in t which is bounded in $\Omega \times (0,T)$ together with a_{ij} (resp. a_i, a_o) itself. We choose:

(9.1) $$a_h(t;u_h,v_h) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t)\ \delta_j u_h(x)\ \delta_i \overline{v_h(x)}\ dx$$

$$+ \sum_{i=1}^{n} \int_\Omega a_i(x,t)\ \delta_i u_h(x)\ \overline{v_h(x)}\ dx$$

$$+ \int_\Omega a_o(x,t)\ u_h(x)\ \overline{v_h(x)}\ dx \ .$$

We assume that Hypothesis (2.5) is verified (cf. Remark (ii) §1). Now, we compute N(h) and v(h):

$$|a_h(t;u_h,v_h)| \le 2\|a\|_{L^\infty(\Omega \times (0,T))} \Big(\sum_{i=1}^{n} \frac{1}{h_i^2}\Big)^{1/2} \Big(\sum_{i=1}^{n} \int_\Omega |\delta_i u_h(x)|^2 dx\Big)^{1/2} |v_h|_h$$

$$+ \Big(\sum_{i=1}^{n} \|a_i\|^2_{L^\infty(\Omega\times(0,T))}\Big)^{1/2} \Big(\sum_{i=1}^{n} \int_\Omega |\delta_i u_h(x)|^2 dx\Big)^{1/2} |v_h|_h$$

$$+ \|a_o\|_{L^\infty(\Omega\times(0,T))} \Big(\int_\Omega |u_h(x)|^2 dx\Big)^{1/2} |v_h|_h \ ,$$

where $a(x,t)$ is the euclidean norm of the matrix $(a_{ij}(x,t))_{i,j=1,\ldots,n}$. Then,

(9.2) $$N(h) \le \Big\{ \Big[2\|a\|_{L^\infty(\Omega\times(0,T))} \Big(\sum_{i=1}^{n} \frac{1}{h_i^2}\Big)^{1/2} + \Big(\sum_{i=1}^{n} \|a_i\|^2_{L^\infty(\Omega\times(0,T))}\Big)^{1/2} \Big]^2$$

$$+ \|a_o\|^2_{L^\infty(\Omega\times(0,T))} \Big\}^{1/2} ,$$

(9.3) $$\nu(h) \le 2 \, \|a\|_{L^\infty(\Omega\times(0,T))} \left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right)^{1/2} .$$

We compute $M(h)$ and $\mu(h)$:

$$|a_h(t;u_h,v_h)| \le 4 \, \|a\|_{L^\infty(\Omega\times(0,T))} \left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right) |u_h|_h \, |v_h|_h$$

$$+ 2 \left(\sum_{i=1}^{n} \|a_i\|^2_{L^\infty(\Omega\times(0,T))}\right)^{1/2} \left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right)^{1/2} |u_h|_h |v_h|_h$$

$$+ \|a_o\|_{L^\infty(\Omega\times(0,T))} \, |u_h|_h \, |v_h|_h ,$$

(9.4) $$M(h) \le 4\|a\|_{L^\infty(\Omega\times(0,T))} \left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right) + 2\left(\sum_{i=1}^{n} \|a_i\|^2_{L^\infty(\Omega\times(0,T))}\right)^{1/2} \left(\sum_{i=1}^{n} \frac{1}{h_i^2}\right)^{1/2}$$

$$+ \|a_o\|_{L^\infty(\Omega\times(0,T))}$$

(9.5) $$\mu(h) \le 4 \, \|a\|_{L^\infty(\Omega\times(),T))} \sum_{i-1}^{n} \frac{1}{h_i^2} "$$

When

(9.6) $$a_{ij}(x,t) = \overline{a_{ji}(x,t)} ,$$

the principal part of $a_h(t;u_h,v_h)$ is hermitian (see the remark following Theorem 6.1).

c) <u>The initial values</u> $u_{h,k}(rk)$, $r = 0,1,\dots,p-1$:

We define $u_{h,k}(0)$ by

9.7 $$u_{h,k}(0) = O'_h u_o$$

where O'_h is a linear continuous mapping from $L^2(\Omega)$ (resp. $H^1(\Omega)$, $D(A(0))$) into V_h such that

(9.8) $$|O'_h u_o|_h \le C\|u_o\|_{L^2(\Omega)}$$

(resp. (9.9) $|O'_h u_o|_h, \|O'_h u_o\|_h \le C\|u_o\|_{H^1(\Omega)}$,

(9.10) $$|O'_h u_o|_h, \|O'_h u_o\|_h, |A_h(0)O'_h u_o|_h \le C\|u_o\|_{D(A(0))})\ ,$$

where C is a positive constant independent of h. If $u_o \in L^2(\Omega)$, we set

(9.11) $$O'_h u_o = \frac{1}{h_1\cdots h_n}\sum_{M\in\Omega_h}\left(\int_{\sigma_h(M,o)}\tilde{u}_o\,dx\right)W_{h,M}\ ,$$

and (9.8) is true with $C = 1$,

If $u_o \in H^1(\Omega)$ and if the boundary Γ of Ω is smooth enough, there exists an operator $P \in \mathcal{L}(H^1(\Omega); H^1(\mathbb{R}^n))$ such that

$$Pu_o = u_o \text{ in } \Omega \qquad \text{(cf. Lions [4])}\ .$$

Then we set

(9.12) $$O'_h u_o = \frac{1}{h_1\cdots h_n}\sum_{M\in\Omega_h}\left(\int_{\sigma_h(M,o)}Pu_o\,dx\right)W_{h,M}\ ,$$

and (9.9) is true (cf. [7]).

Let us examine now the case $u_o \in D(A(0))$. Generally we do not know how to choose O_h such that (9.10) is true for all $u_o \in D(A(0))$. However when u_o belongs to an appropriate subspace W of $D(A(0))$, it is possible to find O_h such that

$$|O'_h u_o|_h, \ \|O'_h u_o\|_h, \ |A_h(0)\, a'_h u_o|_h \le \|u_o\|_W \ .$$

For example, let $H^2(\Omega)$ be the space of functions u such that

$$u, \ \frac{\partial u}{\partial x_i}, \ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega), \qquad i,j = 1,\ldots,n$$

We provide $H^2(\Omega)$ with the following norm

$$\|u\|_{H^2(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)} + \sum_{i,j=1}^{n} \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|^2_{L^2(\Omega)} \right)^{1/2} .$$

We define $H^2_o(\Omega)$ to be the closure in $H^2(\Omega)$ of the smooth functions with compact support in Ω. Then if $A(t) = -\Delta$, $H^2_o(\Omega) \subset D(-\Delta)$ and we can prove:

$$|O'_h u_o|_h, \ \|O'_h u_o\|_h, \ |\Delta_h \, O'_h u_o| \le C \|u_o\|_{H^2(\Omega)} ,$$

where O'_h is defined by (9.11) (see [7]).

Then we define $u_{h,k}(k),\ldots,u_{h,k}((p-1)k)$ by one step difference methods (cf. [7]). Now we can easily see that the consistency hypotheses are verified,

It is very simple to state the stability theorems and the convergence theorems corresponding to our example: this is left to the reader.

REFERENCES

[1] J. P. Aubin, Thesis (to appear).

[2] J. Céa, Approximation variationnelle des problèmes aux limites, Annales Inst. Fourier, (1965).

[3] H. O. Kreiss, Über die Differenzen approximation höher Genauigkeit bei Anfangswertproblem fur partielle Differentialgleichungen, Num. Math. (1959), 1, pp. 186-202.

[4] J. L. Lions, Equations différentielles opérationmelles et problèmes aux limites, Springer Verlag, Berlin (1961).

[5] J. L. Lions, CIME Course (May 1963) (Ed. Cremonese Roma).

[6] J. L. Lions, E. Magenes, Problèmes aux limites non homogènes (II), Annales Inst. Fourier (1961), 11, pp. 137-178.

[7] P. A. Raviart, Sur l'approximation de certaines équations d'évolution linéaires et non linéaires, Thesis (to appear).

[8] O. B. Widlund, On the stability of parabolic difference schemes, Math. of Comput. (1965), 19, 89, pp. 1-13.