CS 70

ON COMPUTATION OF FLOW PATTERNS OF COMPRESSIBLE FLUIDS IN THE TRANSONIC REGION

BY

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1. INTRODUCTION

The first task in devising a numerical procedure for solving a given problem is that of finding a constructive mathematical solution to the problem. But even after such a solution is found there is much to be done. Mathematical solutions normally involve infinite processes such as integration and differentiation as well as infinitely precise arithmetic and functions defined in arbitrarily involved ways. Numerical procedures suitable for a computer can involve only finite processes, fixed or at least bounded length arithmetic and rational functions. Thus one must find efficient methods which yield approximate solutions.

Of interest here are the initial and boundary value problems for compressible fluid flow. Constructive solutions to these problems can be found in [B]. As presented there, solution of the boundary value problem is limited to the subsonic region, and is given symbolically as a linear combination of orthogonal functions. A numerical continuation of this (subsonic) solution into the supersonic region can be done by using the (subsonic) solution and its derivative to set up an initial value problem. The solution to the initial value problem may then be valid in (some part of) the supersonic region. Whether this continuation will lead to a closed, meaningful flow is an open question. In this paper, we deal with the numerical solution of the initial value problem., We are currently working on the rest of the procedure described above.

2. THE INITIAL VALUE PROBLEM

The partial differential equation describing the flow of a compressible fluid is nonlinear when considered in the physical plane (x,y-plane). However, when transformed into the so called hodograph plane $(H,\theta\text{-plane})$, this equation becomes a linear one, namely

(2.1) $$\frac{\partial^2\psi}{\partial H^2} + \ell(H)\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad \ell(H) = \frac{1 - M^2}{\rho^2}$$

where

(2.2) $$H = H(v) = \int_{v_1}^{v} \frac{\rho}{v}\, dv$$

(2.3) $$\rho = \{1 - \tfrac{1}{2}(k - 1)(\frac{v}{a_0})^2\}^{1/(k-1)}$$

(2.4) $$M = v/\{a_0^2 - \tfrac{1}{2}(k-1)v^2\}^{\frac{1}{2}}$$

and θ is the angle which the velocity vector forms with the positive direction of the x-axis, v is the speed, $\psi(H,\theta)$ is the stream function, M is the Mach number, ρ is the density, v_1 is the speed when M-= 1 (i.e., the speed on the sonic line), k is a constant depending on the fluid and a_0 is a conveniently chosen constant.

We shall describe a numerical procedure for solving the initial value problem in which the stream function, $\psi(H_0,\theta) = f(\theta)$, and its derivative, $\left.\frac{\partial\psi(H,\theta)}{\partial H}\right|_{H=H_0} = g^{(1)}(\theta)$, are specified on an arbitrary line, $H = H_0$. The basis for this procedure is provided by the following:

Theorem 2.1. (See [B , p. 895]). Let α and β satisfy $\alpha < \beta < H(a_0\sqrt{2/(k-1)})$. Suppose that, for $|\theta| \le \theta_1$ and a given. $H_0 \in [\alpha,\beta]$ we have

$$(2.5) \qquad \psi(H_0,\theta) = \sum_{n=0}^{\infty} C_n\theta^n \equiv f(\theta) \ , \quad \left.\frac{\partial\psi(H,\theta)}{\partial H}\right|_{H=H_0} = \sum_{n=0}^{\infty} nD_n\theta^{n-1} \equiv g^{(1)}(\theta)$$

where the series $\Sigma C_n\theta^n$ and $\Sigma D_n\theta^n$ converge uniformly and absolutely for $|\theta| \le \theta_1$. Suppose that $|\ell(H)| \le c^2$, $0 < c < \infty$ for $H \in [\alpha,\beta]$. Let us define functions $s_m(H,H_0)$ by $s_0(H,H_0) = 1$, $s_1(H,H_0) = H-H_0$, and for $m = 2,3,\ldots$

$$(2.6) \qquad s_m(H,H_0) = \int_{H_0}^{H}\int_{H_0}^{H_1}\ell(H_2)\int_{H_0}^{H_2}\int_{H_0}^{H_3}\ell(H_4)\ \ldots\ dH_m dH_{m-1}\ldots dH_1 \ .$$

Then, for H and θ satisfying $|\theta| + c\,|H-H_0| \le \theta_1$ and $H \in [\alpha,\beta]$,

$$(2.7) \qquad \psi(H,\theta) = \sum_{j=0}^{\infty}(-1)^j\{s_{2j}(H,H_0)f^{(2j)}(\theta) + s_{2j+1}(H,H_0)g^{(2j+1)}(\theta)\}$$

is the (analytic) solution of (2.1) satisfying (2.5). Here $f^{(1)} \equiv \frac{df}{d\theta}$, $f^{(2)} = \frac{d^2f}{d\theta^2}$, etc.

Proof: It is easy to check that (2.7) satisfies (2.1) and (2.5). For a proof of (absolute and uniform) convergence see [B, p. 896]. (However, there is an incorrect specification of the domain of convergence in this reference. The domain stated there is $|\theta|+c|H-H_0| \le \theta_1$, whereas the domain of convergence actually established by his proof is

$$\{(H,\theta)|\ |\theta| + c|H-H_0| \le \theta_1 \text{ and } |H-H_0| \le H_1\} \ .$$

The constraint $|H-H_0| \le H_1$ corresponds to our constraint, $H \in [\alpha,\beta]$.)

The domain of convergence guaranteed by this theorem is pictured in Figure 2.1. If the initial conditions are specified as a Fourier series instead of a power series, then a theorem similar to this one can be proved. In that case, the domain of guaranteed convergence would be rectangular.

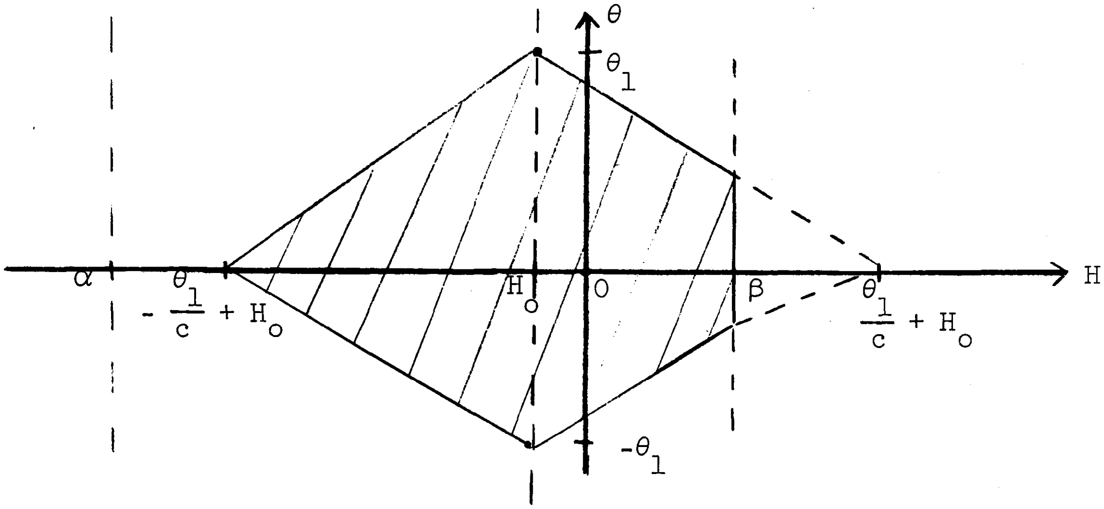


Figure 2.1

In numerical evaluation of the right hand side of (2.7) we have to approximate all functions in a convenient way and we must truncate the series. We shall denote approximation functions by adding a horizontal bracket $(\overline{\ \ })$ over the function. In this manner (2.7) becomes

$$(2.8) \qquad \overline{\psi}_n(H,H_o,\theta) = \sum_{j=0}^{n} (-1)^j \{\overline{s}_{2j}(H,H_o)\overline{f^{(2j)}}(\theta) + \overline{s}_{2j+1}(H,H_o)\overline{g^{(2j+1)}}(\theta)$$

where n is an (arbitrary) positive integer denoting the degree of truncation. (Notice the approximation, $\overline{\psi}_n$, to ψ depends on H_o, whereas ψ does not.) Since computers can only perform the basic operations + , - , X , 3 , we must use rational approximations, The following remarks about $f^{(2j)}$ will apply to $g^{(2j+1)}$ as well. In general, obtaining approximations $\overline{f^{(2j)}}$, for $j = 0,1,\ldots,n$, is not difficult. In fact, in the usual application of this of this procedure, $f^{(2j)}$ will be defined in terms of functions customarily available on computers, such as sine, cosine, etc., and it will be possible to calculate $f^{(2j)}$ to almost full machine accuracy. In such cases the fact that we are really calculating a $\overline{f^{(2j)}}$ is somewhat obscured by our ability to express it, in current programming languages, in precisely the form of its formal definition. For example, the Algol statement to calculate an approximation to $f(x) = \sin x$ is just "f(x) := sin(x)". However, when only $\overline{f}$, and not f, is' known, perhaps as the result of solving the boundary value problem alluded to 'earlier in this paper, a severe error is incurred. This is why we keep track of $f^{(2j)} - \overline{f^{(2j)}}$ in what follows.

The values of $\overline{f^{(2j)}}(\theta)$ may be derived from an approximation, $\overline{f}$. For example, if f is given as in (2.5), we can truncate that series to obtain $\overline{f}$. We can then use an iterative synthetic division scheme to evaluate $\frac{\overline{f}^{(2j)}}{(2j)!}$, for $j = 0,1,\ldots,n$. */ Of course the error of $\overline{f}^{(2j)}$ incurred by such a procedure increases as j grows. However, if (some norm of) the $f^{(2j)}$, considered as functions of j, does not increase too rapidly for $j \leq n$, then the absolute errors of $s_{2j}f^{(2j)}$ will not increase as j grows and remains $\leq n$. **/ This is

*/ Note that $\overline{f}^{(m)}$ denotes the m-th derivative of $\overline{f}$ and $\overline{f^{(m)}}$ denotes an approximation to $f^{(m)}$; $\overline{f}^{(m)}$ need not be a very good $\overline{f^{(m)}}$.

**/ This is discussed more precisely in Section 5.

because $s_m \to 0$ rapidly as $m \to \infty$ since, as indicated in [B],

$$(2.9) \qquad |s_m(H,H_o)| \le \frac{\delta_m^{-1}}{m!} c^m |H-H_o|^m$$

where $\delta_m = c$ for m odd and $\delta_m = 1$ for m even, and c is the constant in Theorem 2.1.

The determination of $\bar{s}_m$ presents more challenging problems. Due to the nature of R(H) , an exact formula for s_m has not been found. The numerical procedure which evaluates $\bar{\psi}_n$ will be used to trace the streamlines $\psi(H,\theta) = \text{const.}$ Such curves, when transformed into the (x,y)-plane, describe the fluid flow. This means that many evaluations of $\bar{\psi}_n$ will be required (we use approximately 1500 per run), and so the $\bar{\ell}(H)$ in (2.6) must be chosen to yield an efficient scheme. In the next section we derive such an approximation to $\ell(H)$ and thus to $s_m(H,H_o)$ for the special case in which the fluid under consideration is air. In this case

$$(2.10) \qquad v_1 = \sqrt{5/6} \qquad\qquad k = 1.4$$

and we choose $a_o = 1$ (see (2.3) and (2.4)). The function $\ell(H)$ takes the form shown in Figure 2.2. It has a singularity at $p = .25125...$ and is asymptotic to unity as $H \to -\infty$. Its only zero is at $H = 0$. This information will prove most useful in the next section.

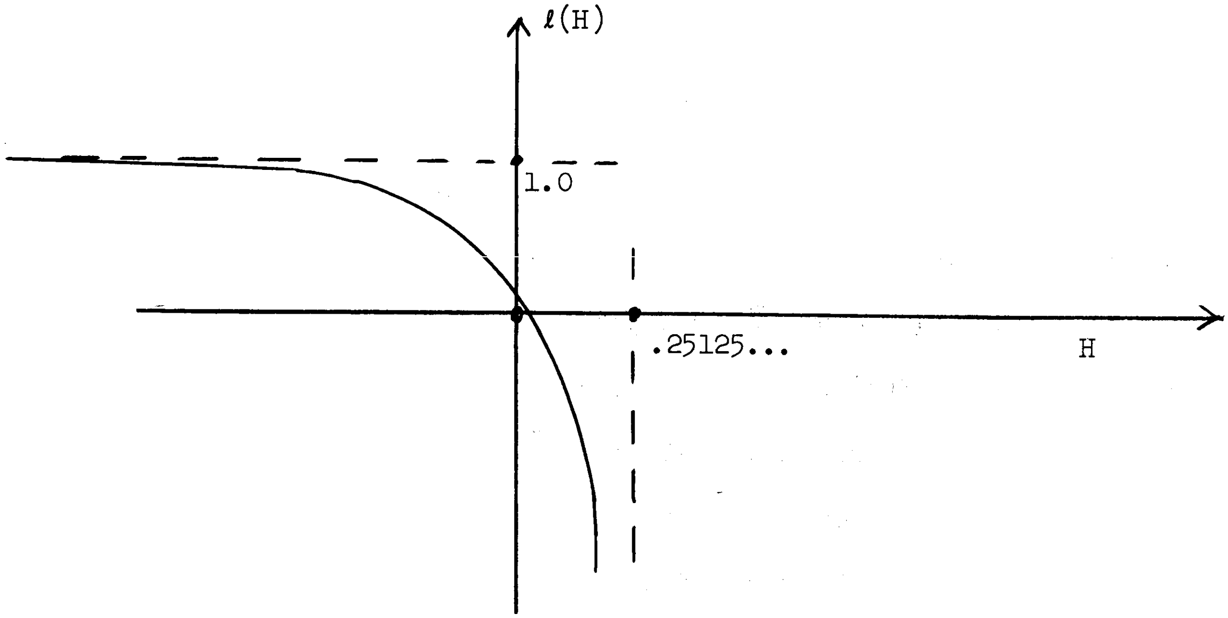


Figure 2.2

3. THE INTEGRALS $s_m(H,H_o)$ AND THEIR APPROXIMATION

The s_m of eqn. (2.6) satisfy the recurrence relation

(3.1) $$s_m(H,H_o) = \int_{H_o}^{H} \int_{H_o}^{H_1} \ell(H_2) s_{m-2}(H_2,H_o)\,dH_2\,dH_1 \quad \text{for } m \geq 0$$

with the starting values

(3.2) $$s_o(H,H_o) = 1 \qquad\qquad s_1(H,H_o) = H-H_o$$

where $H = H_o$ is the line on which the initial conditions of (2.5) are specified. We will consider H_o satisfying $H_o < .25125\ldots = p$, since as $H \rightarrow p$ the Mach number, $M(H)$, approaches infinity. A major problem in this implementation was the construction of an approximation, $\bar{\ell}(H)$, to $\ell(H)$ over some subinterval of $(-\infty,p)$ which would allow a relatively simple expression for $\bar{s}_m$. The approximation of [B-H-K] was not satisfactory for our purposes. It consisted of two tenth degree polynomial approximations, one for the region [-1,0] and the other for [0,.2]. In [B-H-K], H_o was fixed at zero and so their approximation lead to two expressions for $\bar{s}_m(H,H_o)$, one valid in [-1,0], and the other in [0,.2]. In our work H_o is arbitrary and will vary from run to run, so we must have a single representation for $\bar{s}_m$.

An adequate approximation to $\ell(H)$ over [-1,.22] was found by observing that (for k = 1.4) the singularity of R(H) at p is of order 12/7, and that $\ell(H)$ has the expansion

(3.3) $$1(H) = \sum_{j=o}^{\infty} b_j (p-H)^{\frac{2(j-6)}{7}}$$

Table 3.1

m	b_m
0	-1.77922350435@-01
1	-4.13643140387@-02
2	9.10662027300@-02
3	1.82057189808@-01
4	2.22090857643@-01
5	2.18571018174@-01
6	1.86381855591@-01
7	1.41067084695@-01
8	9.49481253260@-02
9	5.57240086360@-02
10	2.67431923433@-02
11	8.12726610830@-03
12	-1.89181661577@-03
13	-5.81565004750@-03
14	-6.07697894760@-03
15	-4.60812991016@-03
16	-2.70169778730@-03
17	-1.06914384192@-03
18	2.25378894809@-05
19	5.69218829880@-04
20	7.11356001550@-04
21	6.15142078990@-04
22	3.93011997122@-04
23	2.41854948571@-04
24	-2.35198016890@-06
25	5.99536217220@-05
26	-3.24517556322@-04
27	4.07221423060@-04
28	-8.80318119550@-04
29	1.48834642679@-03
30	-3.59333068761@-03
31	7.71610253620@-03
32	-1.38419726542@-02
33	2.48706741343@-02
34	-5.61248842780@-02
35	1.44175838135@-01
36	-3.51230674014@-01
37	7.85080811930@-01
38	-1.68397714918@+00
39	3.61290715632@+00
40	-7.50537518870@+00
41	1.31044606223@+01
42	-1.27367360421@+01

(a)

Single Precision (lo-digits)

m	b_m
0	-1.77922350433@-01
1	-4.13643140391@-02
2	9.10662027560@-02
3	1.82057189748@-01
4	2.22090857715@-01
5	2.18571018080@-01
6	1.86381855798@-01
7	1.41067084133@-01
8	9.49481263810@-02
9	5.57240074310@-02
10	2.67431922580@-02
11	8.12727107470@-03
12	-1.89183242604@-03
13	-5.81561429310@-03
14	-6.07705244360@-03
15	-4.60797048296@-03
16	-2.70204072313@-03
17	-1.06852395106@-03
18	2.16181237245@-05
19	5.70321287280@-04
20	7.11163148530@-04
21	6.10306531940@-04
22	4.10215296682@-04
23	2.07302986624@-04
24	5.16173904508@-05
25	-4.23578005217@-05
26	-8.18121992230@-05
27	-8.33970618110@-05
28	-6.47265781910@-05
29	-3.97693804231@-05
30	-1.73184490543@-05
31	-1.40752984364@-06
32	7.36139121330@-06
33	1.03263459349@-05
34	9.53460763530@-06
35	6.93341584640@-06
36	3.96830234562@-06
37	1.48765355960@-06
38	-1.70159746431@-07
39	-1.01371116270@-06
40	-1.23314879039@-06
41	-1.07119981352@-06
42	-7.43258576910@-07

(b)

Double Precision (20-digits)

Table 3.2

H	V	$\ell(H)$	$\bar{\ell}(H)$	RESIDUAL
-1.00	0.28167511771	0.99592948542	0.99592842099	1.06E-06
-0.99	0.28462335096	0.99574960836	0.99574867754	9.31E-07
-0.98	0.28760501266	0.99556146560	0.99556065433	8.11E-07
-0.97	0.29062056684	0.99536465588	0.99536395142	7.04E-07
-0.96	0.29367048664	0.99515875701	0.99515814775	6.09E-07
-0.95	0.29675525492	0.99494332463	0.99494280004	5.25E-07
-0.94	0.29987536417	0.99471789097	0.99471744152	4.49E-07
-0.93	0.30303131719	0.99448196347	0.99448158058	3.83E-07
-0.92	0.30622362709	0.99423502334	0.99423469920	3.24E-07
-0.91	0.30945281770	0.99397652406	0.99397625166	2.72E-07
-0.90	0.31271942401	0.99370588967	0.99370566268	2.27E-07
-0.89	0.31602399239	0.99342251315	0.99342232592	1.87E-07
-0.88	0.31936708099	0.99312575440	0.99312560183	1.53E-07
-0.87	0.32274926014	0.99281493842	0.99281481595	1.22E-07
-0.86	0.32617111272	0.99248935304	0.99248925660	9.64E-08
-0.85	0.32963323458	0.99214824679	0.99214817279	7.40E-08
-0.84	0.33313623498	0.99179082647	0.99179077166	5.48E-08
-0.83	0.33668073699	0.99141625444	0.99141621597	3.85E-08
-0.82	0.34026737802	0.99102364603	0.99102362139	2.46E-08
-0.81	0.34389681023	0.99061206653	0.99061205349	1.30E-08
-0.80	0.34756970110	0.99018052799	0.99018052460	3.40E-09
-0.79	0.35128673394	0.98972798590	0.98972799042	-4.52E-09
-0.78	0.35504860845	0.98925333550	0.98925334646	-1.10E-08
-0.77	0.35885604110	0.98875540802	0.98875542408	-1.61E-08
-0.76	0.36270976657	0.98823296630	0.98823298638	-2.01E-08
-0.75	0.36661053696	0.98768470053	0.98768472364	-2.31E-08
-0.74	0.37055912358	0.98710922337	0.98710924870	-2.53E-08
-0.73	0.37455631734	0.98650506473	0.98650509157	-2.68E-08
-0.72	0.37860292942	0.98587066627	0.98587069401	-2.77E-08
-0.71	0.38269979202	0.98520437540	0.98520440357	-2.82E-08
-0.70	0.38684775927	0.98450443888	0.98450446705	-2.82E-08
-0.69	0.39104770801	0.98376899584	0.98376902365	-2.78E-08
-0.68	0.39530053860	0.98299607032	0.98299609751	-2.72E-08
-0.67	0.39960717605	0.98218356320	0.98218358956	-2.64E-08
-0.66	0.40396857077	0.98132924348	0.98132926883	-2.53E-08
-0.65	0.40838569977	0.98043073886	0.98043076308	-2.42E-08
-0.64	0.41285956766	0.97948552555	0.97948554856	-2.30E-08
-0.63	0.41739120787	0.97849091728	0.97849093901	-2.17E-08
-0.62	0.42198168374	0.97744405330	0.97744407372	-2.04E-08
-0.61	0.42663208987	0.97634188549	0.97634190460	-1.91E-08
-0.60	0.43134355341	0.97518116428	0.97518118208	-1.78E-08
-0.59	0.43611723553	0.97395842344	0.97395843994	-1.65E-08
-0.58	0.44095433288	0.97266996343	0.97266997868	-1.52E-08
-0.57	0.44585607911	0.97131183347	0.97131184748	-1.40E-08
-0.56	0.45082374659	0.96987981187	0.96987982466	-1.28E-08
-0.55	0.45585864892	0.96836938440	0.96836939618	-1.18E-08
-0.54	0.46096214040	0.96677572165	0.96677573239	-1.07E-08
-0.53	0.46613562107	0.96509365270	0.96509366245	-9.75E-09
-0.52	0.47138053691	0.96331763782	0.96331764663	-8.82E-09
-0.51	0.47669838267	0.96144173773	0.96144174570	-7.97E-09

$\bar{\ell}(H)$, using the first 43 coefficients of the expansion of $\ell[H]$.

Table 3.2 (con't

H	V	$\ell(H)$	$\hat{\ell}(H)$	RESIDUAL
-0,50	0.48209070409	0,95945958045	0.95945958762	-7,18e-09
-0.49	0.48755910044	0.95736432450	0.95736433093	-6,43e-09
-0,48	0,49310522728	0.95514861883	0.95514862459	-5,75e-09
-0.47	0,49873079916	0,95280455865	0.95280456377	-5,12e-09
-0,46	0.50443759283	0,95032363681	0,95032364137	-4,56e-09
-0.45	0.51022745033	0.94769669038	0,94769669444	-4,06e-09
-0.44	0.51610228252	0.94491384151	0,94491384507	-3,56e-09
-0,43	0.52206407277	0.94196443222	0,94196443536	-3,15e-09
-0.42	0,52811488091	0.93883695240	0.93883695516	-2,76e-09
-0.41	0.53425684745	0.93551895985	0.93551896230	-2,45e-09
-0.40	0,54049219802	0.93199699184	0.93199699395	-2,11e-09
-0.39	0.54682324844	0.92825646652	0.92825646835	-1,84e-09
-0,38	0.55325240973	0.92428157376	0.92428157535	-1,59e-09
-0.37	0.55978219385	0.92005515317	0,92005515455	-1,38e-09
-0.36	0.56641521962	0.91555855810	0.91555855927	-1,17e-09
-0.35	0.57315421933	0,91077150368	0,91077150471	-1,03e-09
-0.34	0.58000204573	0.90567189672	0,90567189758	-8,51e-10
-0.33	0.58696167951	0.90023564480	0.90023564554	-7,40e-10
-0.32	0.59403623759	0.89443644203	0,89443644264	-6 ,00e-10
-0.31	0,60122898198	0.88824552763	0.88824552811	-4,86e-10
-0.30	0,60854332935	0.88163141380	0.88163141416	-3,62e-10
-0,29	0.61598286152	0.87455957821	0,87455957846	-2,53e-10
-0.28	0.62355133673	0.86699211577	0.86699211588	-1,15e-10
—-0,27	0.63125270214	0.85888734321	0.85868734317	4,18e-11
-0,26	0,63909110717	0.05019934976	0.85019934948	2,80e-10
-0.25	0,64707091832	0.84087748432	0,84087748369	6,24e-10
-0.24	0.65519673633	0.83086576865	0.83086576879	-1,46e-10
-0,23	0,66347341024	0,82010223053	0.82010223066	-1,33e-10
-0.22	0.67190606164	0.80851812710	0,80851812721	-1,11e-10
-0.21	0,68050010318	0.79603706029	0,79603706036	-6,55e-11
-0.20	0.68926126276	0.78257395031	0.78257395040	-9,28e-11
-0.19	0,69819560954	0,76803384757	0.76803384761	-4,18e-11
-0,18	0.70730958293	0,75231055135	0,75231055137	-2,55e-11
-0.17	0.71661002450	0.73528500001	0,73528500003	-2,36e-11
-0.16	0,72610421358	0,71682338751	0,71682338751	-1,82e-12
-0.15	0.73579990712	0.69677495230	0.69677495234	-3,82e-11
-0.14	0,74570538399	0,67496937213	0.67496937217	-3,64e-11
'0.13	0.75582949491	0,65121368205	0.65121368208	-3,46e-11
-0,12	0.76618171840	0.62528861428	0.62528861430	-2,00e-11
-0.11	0.77677222404	0.59694423179	0.59694423183	-4,73e-11
-0,10	0,78761194391	0,56589469700	0,56589469704	-4,00e-11
-0.09	0,79871265383	0,53181197270	0.53181197275	-5,28e-11
-0,08	0.81008706579	0.49431820057	0.49431820061	-3,46e-11
-0.07	0,82174893378	0.45297642975	0.45297642976	-1,82e-11
-0,06	0,83371317518	0,40727927411	0,40727927413	-2,55e-11
-0.05	0,84599601072	0,35663495387	0.35663495386	1,09e-11
-0.04	0.85861512670	0.30035000622	0.30035000626	-4,73e-11
-0,03	0,87158986355	0.23760772770	0.23760772766	4,18e-11
-0.02	0,88494143649	0.16744109659	0,16744109661	-2,36e-11
-0.01	0,89869319491	0,08869850144	0,08869850143	1,27e-11
0.00	0,91287092921	-0,00000000026	-0,00000000004	-2,22e-10
0,01	0.92750323574	-0,10031900480	-0,10031900490	1,06e-10
0.02	0.94262195453	-0.21428099099	-0,21428099107	8,00e-11
0.03	0.95826269685	-0.34436251289	-0.34436251279	-9,64e-11
0.04	0.97446548596	-0.49362198301	-0.49362199647	1,35e-08

Table 3.2 (cod-t)

H	V	$\ell(H)$	$\tilde{\ell}(H)$	RESIDUAL
0.05	0.99127555152	-0.66587160763	-0.66587161183	4.20e-09
0.06	1.00874429275	-0.86591152770	-0.86591152920	1.50e-09
0.07	1.02693050930	-1.09985392433	-1.09985392442	8.73e-11
0.08	1.04590194092	-1.37557841672	-1.37557841692	2.04e-10
0.09	1.06573724600	-1.70338381342	-1.70338381332	-1 .02e-10
0.10	1.08652857057	-2.09693954376	-2.09693954328	-4.80e-10
0.11	1 ,I0838493834	-2.57470596257	-2.57470596209	-4.80e-10
0.12	1.13143680239	-3.16210870461	-3.16210870343	-1.18e-09
0.13	1.15584227913	-3.89496426965	-3.89496427141	1.76e-09
0.14	1.18179588542	-4.82505765359	-a,82505765285	-7.42e-10
0.15	1.20954110887	-6.02957797020	-6.02957797010	-1 .31e-10
0.16	1.23938906804	-7.62781509130	-7.62781509290	1.59e-09
0.17	1.27174725139	-9.81233482410	-9.81233482030	-3.84e-09
0.18	1.30716580707	-12.91111808070	-12.91111808130	5.82e-10
0.19	1.34641635550	-17.52191991450	-17.52191991630	1.75e-09
0.20	I.39063599246	-24.83478987800	-24.83478990340	2.54e-08
0.21	1.44161618569	-37.52253776940	-37.52253910010	1.33e-06
0.22	1.50246314780	-62.74095549100	-62.74095589500	3.73e-09
0.23	1.57943776486	-125.95313124800	‘125.95313135300	1 .04e-07
0.24	1.68924770653	-387.33409698600	-387.33409694500	-4.10e-08
0.25	1.94538648089	-17160.52854390000	-17160.53556070000	7.02e-03

The first 43 coefficients, $b_o,\ldots,b_{42}$, were calculated in both single and double precision (10 and 20 decimal digit accuracy) on a B5500 computer; they are listed in Table 3.1. The residuals listed in Table 3.2 indicate that these coefficients yield a very good approximation to $\ell(H)$. The program which calculated these coefficients is included in the appendix. Equation (3.3) follows from (2.1) and (2.2) by substituting $5(1-\tau)$ for v^2 so that

$$(3.4) \qquad H = \int_{\sqrt{\frac{5}{6}}}^{v} (1-.2v^2)^{2.5} \frac{dv}{v} = -\frac{1}{2}\int_{\frac{5}{6}}^{\tau} \frac{\tau^{2.5}d\tau}{1-\tau} = -\frac{1}{2}\int_{\frac{5}{6}}^{\tau} (\tau^{5/2}+\tau^{7/2}+\ldots)d\tau$$

$$(3.5) \qquad p-H = \frac{\tau^{7/2}}{7} + \frac{\tau^{9/2}}{9} + \ldots$$

$$(3.6) \qquad R(H) = \frac{6\tau-5}{\tau - 6} .$$

Our approximation,

$$(3.7) \qquad \bar{\ell}(H) = \sum_{j=o}^{7} a_j (p-H)^{\frac{2j-12}{7}}$$

was found by using the Remez algorithm, as adapted for the B5500 computer by Golub and Smith [G-S], to calculate the best values, in the Chebyshev sense, for $a_o,a_1,\ldots,a_7$.

We now give a representation theorem for $\bar{s}_m$, our approximation to s_m based on $\bar{\ell}(H)$:

<u>Theorem 3.1</u>. Let the $\bar{s}_m$ be connected by the recurrence relation

$$(3.8)\qquad \overline{s}_m(H,H_o) = \int_{H_o}^{H}\int_{H_o}^{H_1} \overline{\ell}(H_2)\overline{s}_{m-2}(H_2,H_o)dH_2dH_1 \quad \text{for} \quad m \geq 2$$

where

$$(3.9)\qquad \overline{s}_o(H,H_o) = 1 \qquad \overline{s}_1(H,H_o) = H-H_o$$

$$(3.10)\qquad \overline{\ell}(H) = \sum_{j=o}^{N-1} a_j(p-H)^{\frac{2j-12}{7}} \quad \text{and} \quad 7 \leq N < \infty \ .$$

Then $\overline{s}_m$ can be expressed as

$$(3.11)\qquad \overline{s}_m(H,H_o) = \sum_{j=o}^{mN} c_{m,j}(p-H)^{j/7} \qquad m = 0,1,2,\ldots$$

where $c_{m,1} = c_{m,3} = c_{m,5} = 0$ for all m . The $c_{m,j}$ and $c_{m-2,j}$ are connected by the following recurrence relations:

$$(3.12)\qquad c_{m,j} = -\frac{7}{j}\beta_{m,j-2} \quad \text{for } j = 2,3,\ldots,mN \quad \text{with} \quad j \neq 7$$

$$(3.13)\qquad c_{m,7} = \sum_{j=o}^{mN-2} \beta_{m,j}(p-H_o)^{\frac{j-5}{7}}$$

$$(3.14)\qquad c_{m,o} = -\sum_{j=2}^{mN} c_{m,j}(p-H_o)^{\frac{j}{7}}$$

where

$$(3.15)\qquad \beta_{m,1} = \beta_{m,3} = \beta_{m,5} = 0$$

and for $j = 0,2,4,6,7,8,\ldots,mN-2$, with $[\cdot]$ denoting the greatest integer function,

$$\beta_{m,j} = \frac{7}{5-j} \sum_{k=[\frac{1}{2} + \frac{1}{2} \max\{0,j-(m-2)N\}]}^{[\frac{1}{2} \min\{j,2N-2\}]} a_k c_{m-2,j-2k} \tag{3.16}$$

Proof: Equation (3.11) holds for $m = 0,1$. We proceed by induction, assuming that (3.11)-(3.16) hold for, $m-2$ and proving them for m. We have

$$\overline{s}_{m-2}(H,H_o)\overline{\ell}(H) = \sum_{j=o}^{N-1} a_j (p-H)^{\frac{2j-12}{7}} \sum_{k=0}^{(m-2)N} c_{m-2,k} (p-H)^{\frac{k}{7}} \tag{3.17}$$

$$= \sum_{j=o}^{mN-2} \alpha_{m,j} (p-H)^{\frac{j-12}{7}}$$

where, for $j = 0,1,\ldots,mN-2$

$$\alpha_{m,j} = \sum_{k=[\frac{1}{2} + \frac{1}{2} \max\{0,j-(m-2)N\}]}^{[\frac{1}{2} \min\{j,2N-2\}]} a_k c_{m-2,j-2k} \tag{3.18}$$

Since (3.17) is to be integrated, we must show that $\alpha_{m,5} = 0$, so that the term $\alpha_{m,5}(p-H)^{-1}$ drops out of (3.17) and no $\log(p-H)$ terms enter. Part of our induction hypothesis is $c_{m-2,5-2k} = o$ for $k = 0,1,2$, and so

$$\alpha_{m,5} = \sum_{k=[\frac{1}{2} + \frac{1}{2} \max\{0,5-(m-2)N\}]}^{[\frac{1}{2} \min\{5,2N-2\}]} a_k c_{m-2,5-2k} = o \quad . \tag{3.19}$$

The rest follows as a formal calculation. Q.E.D.

This procedure of approximating a singular function which is to be integrated many times, is more general than it may at first appear. If a logarithmic term had appeared in the above, we would simply have started our series for $\tilde{\ell}(H)$ at $a_o(p-H)^{-\frac{12}{7}+\epsilon}$, for some suitably chosen small constant ϵ. (As a matter of fact, we have had to do just this in the implementation of the solution to the boundary value problem.)

The values of a_j for j = 0,1,...,7 are listed in Table 3.3. It follows from the Remez algorithm that

$$(3.20) \qquad \max_{-1 \leq H \leq 22} |\ell(H) - \tilde{\ell}(H)| = 4.10533 \times 10^{-5}$$

and the values of $\ell(H)-\tilde{\ell}(H)$ in Table 3.4 confirm this result.

j	a_j
0	-0.1505866818
1	-0.4018655347
2	2.0945191543
3	-5.8821787341
4	10.9583158033
5	-10.7524447788
6	5.9416272229
7	-0.8198101027

Table 3.3

For computational purposes, it is useful to decompose $\tilde{s}_m$ of (3.11) into seven subsums, $X_{m,k}(H,H_o)$:

$$(3.21) \qquad X_{m,k} \equiv X_{m,k}(H,H_o) = \sum_{j=0}^{[\frac{mN-k}{7}]} c_{m,7j+k}(p-H)^j \qquad \text{for } k = 0,\dots,6$$

Table 3.4

H	$\ell(H)$	$\hat{\ell}(H)$	RESIDUAL
-1.00	0.9959294854	0.9959705387	-0.0000410532
-0.95	0.9949433246	0.9949469612	-0.0000036365
-0.90	0.9937058897	0.9936842046	0.0000216851
-0.85	0.9921482468	0.9921121400	0.0000361068
-0.80	0.9901805280	0.9901395036	0.0000410244
-0.75	0.9876847005	0.9876466686	0.0000380319
-0.70	0.9845044389	0.9844755284	0.0000289105
-0.65	0.9804307389	0.9804151337	0.0000156052
-0.60	0.9751811643	0.9751809776	0.0000001867
'0.55	0.9683693844	0.9683845936	-0.0000152092
-0.50	0.9594595805	0.9594880509	-0.0000284704
-0.45	0.9476966904	0.9477343230	-0.0000376326
-0.40	0.9319969918	0.9320380440	-0.0000410522
-0.35	0.9107715037	0.9108091401	-0.0000376363
-0.30	0.8816314138	0.6816585552	-0.0000271414
-0.25	0.8408774844	0.8408880189	-0.0000105345
-0.20	0.7825739504	0.7825643555	0.0000095949
-0.15	0.6967749524	0.6967462574	0.0000286950
-0.10	0.5658946971	0.5658544750	0.0000402221
-0.05	0.3566349540	0.3565983358	0.0000366182
.00	0.0000000004	-0.0000128543	0.0000128546
0.05	-0.6658716068	-0.6658472800	-0.0000243268
0.10	-2.0969395414	-2.0969003767	-0.0000391647
0.15	-6.0295779653	-6.0296000934	0.0000221281
0.20	-24.8347898435	-24.8347511783	-0.0000386653
0.22	-62.7409553870	-62.7409143510	-0.0000410362

(3.22) $$\overline{s}_m(H,H_o) = X_{m,o} + \sum_{k=1}^{6} X_{m,k}(p-H)^{k/7}$$

In this way the evaluation of $\overline{s}_m$ involves the calculation of up to the sixth power of $(p-H)^{1/7}$ and up to the $[\frac{mN}{7}]$-th power of (p-H), instead of the mN-th power of $(p-H)^{1/7}$. This calculation of $\overline{s}_m$ is roughly equivalent to the evaluation of seven $[\frac{mN}{7}]$-th degree polynomials in (p-H). For m = 10 and N = 8, 11-th degree polynomials are evaluated instead of 80-th degree polynomials. Thus, approximation with negative, fractional powers of the variable (p-H) has several beneficial side effects:

(1) More coefficients are used per unit degree of the approximation; e.g. a 2nd degree polynomial approximation has 3 arbitrary coefficients, whereas a 2nd degree approximation in powers of 2/7 has 8 arbitrary coefficients. Freedom to choose more coefficients aids in minimizing error.

(2) Beginning the fractional power expansion at a negative power again allows more coefficients.

These advantages more than compensate for the problems caused by the presence of a singularity of $\ell(H)$ near the domain of integration.

4. EXAMPLES

In this section, the (approximated) solutions to four initial value problems are presented in the form of tables and graphs in the hodograph and physical planes. This was done in the following way:

(1) the line $H = H_o$ was specified ($H_0 = -.2$ was used in all four examples), and the procedure FANDG was supplied for evaluating $\overline{f}(\theta)$, $\overline{g^{(1)}}(\theta)$ and their derivatives (these two functions are the initial values for the differential equation);

(2) the coefficients for $\overline{s}_m(H,H_o)$, for $m = 0,1,\ldots,41$ were computed, using the recurrence relations in Theorem 3.1;

(3) the coefficients for $\frac{d}{dH}\overline{s}_m(H,H_o)$ were computed from those of $\overline{s}_m(H,H_o)$;

(4) three streamlines were traced in the hodograph: $\psi(H,\theta) = \overline{\psi}(0,1.5)$, $\psi(H,\theta) = \overline{\psi}(.05,1.5)$ and $\psi(H,\theta) = \overline{\psi}(.1,1.5)$;

(5) these streamlines were numerically transformed into the physical plane, using the relations

$$(4.1)\qquad \begin{aligned} x &= \int \frac{\cos\theta}{\rho}\left[\frac{M^2-1}{v^2}\,\psi_\theta\, dv + \psi_v\, d\theta\right] \\ Y &= \int \frac{\sin\theta}{\rho}\left[\frac{M^2-1}{v^2}\,\psi_\theta\, dv + \psi_v\, d\theta\right] \end{aligned}$$

(See [B-H-K, p. 21] for further details and references.)

The values of H and θ making up a streamline $\psi(H,\theta) =$ constant, were chosen so that $|\overline{\psi}(H,\theta)-\text{constant}| \leq 10^{-5}$. During each calculation of $\overline{\psi}(H,\theta)$, terms in (2.8) were added in until the last term added was

$< 10^{-6} \times |(\text{the current value of the sum})|$. An average of six terms (involving $\overline{s}_0, \overline{s}_1, \ldots, \overline{s}_{11}$) of (2.8) was used in computing $\overline{\psi}(H,\theta)$ for these examples. Each example took about 13 minutes on the B5500, and used STRFNC about 1300 times.

In the first example, FANDG computed the initial values for the Ringleb solution:

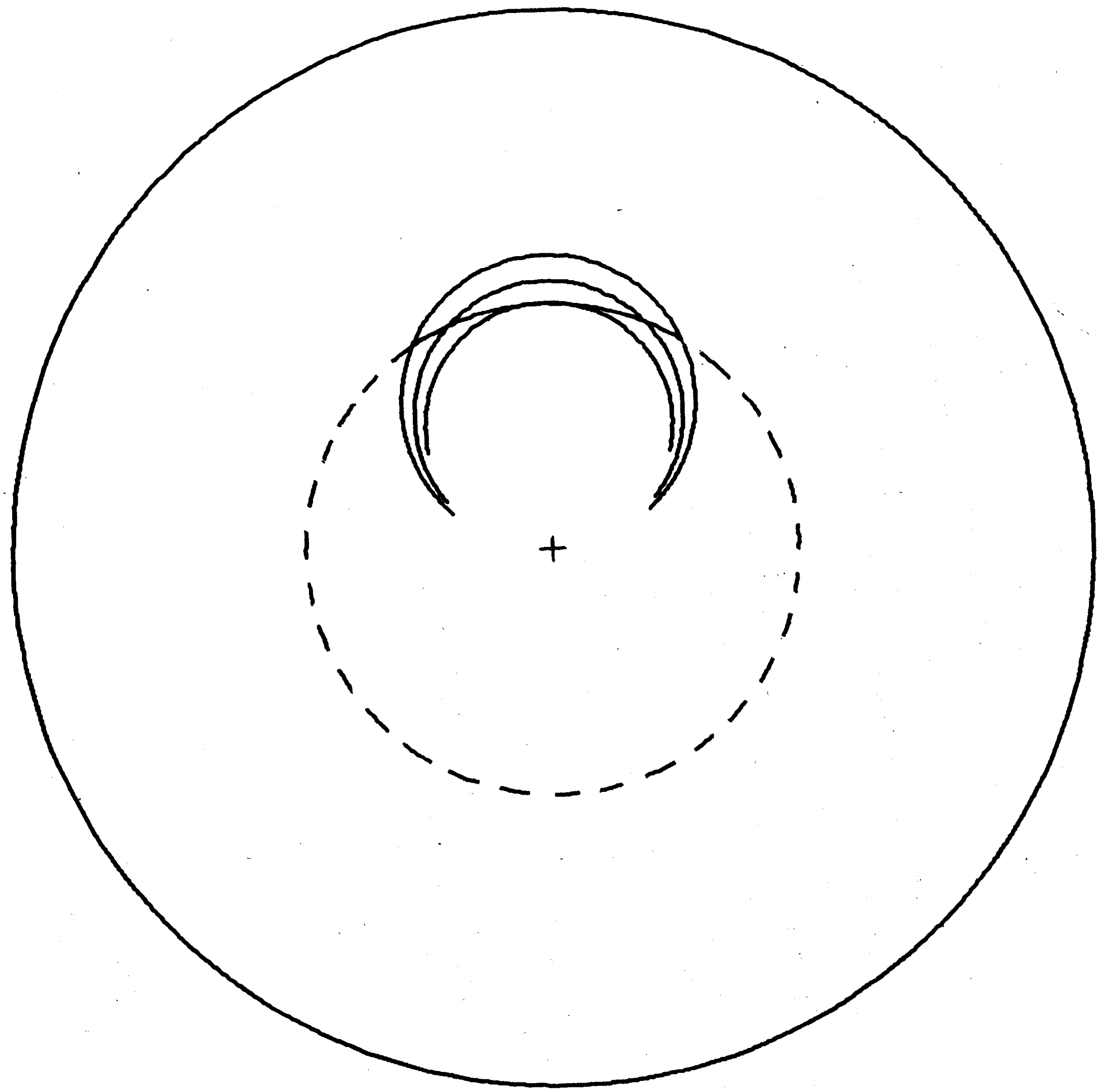
$$(4.2) \qquad f(e) = \frac{2.538 \sin r\theta}{v(H_o)} \;, \quad g^{(1)}(\theta) = -\frac{2.538 \sin r\theta}{v(H_o)(1-.2v^2(H_o))^{2.5}}$$

with $r = 1$. Examples 2, 3 and 4 used (4.2) with $r = .8$, 1.2 and 1.5 , respectively. A closed form solution for $\psi(H,\theta)$ in these last three examples is not known.

EXAMPLE1

RINGLEB SOLUTION

(r = 1)

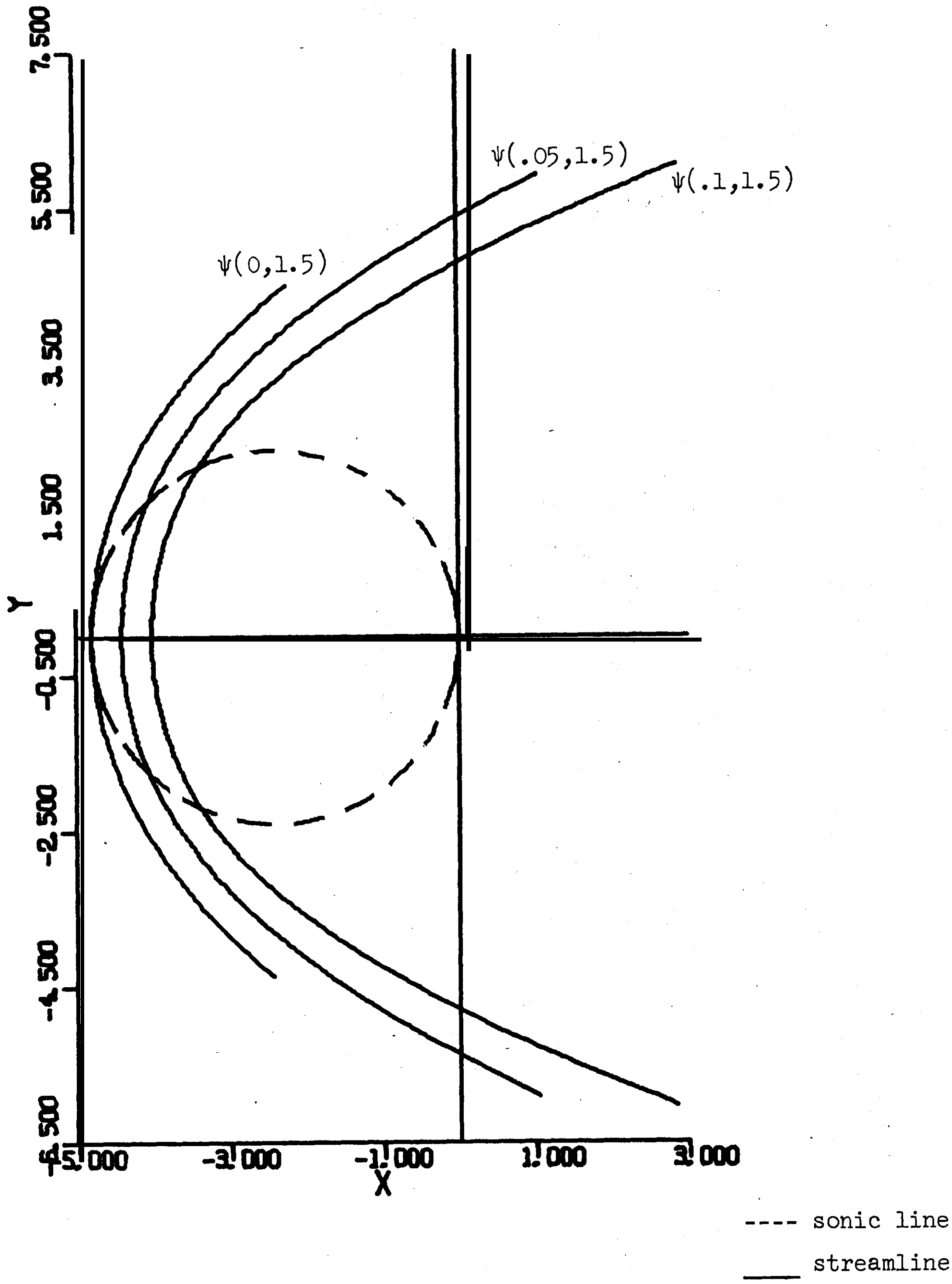


---- sonic line

____ streamline

HODOGRAPH PLANE

EXAMPLE I (con't.



PHYSICAL PLANE

EXAMPLE I (con't.)

$\psi(H,\theta)$ = 2.77327

H	θ	V	X	Y
-0.357181	0.670000	0.569304	-2.242512	4.519349
-0.246264	0.790000	0.650090	-3.116098	3.743652
-0.163745	0.910000	0.722525	-3.682120	3.102722
-0.102783	1.030000	0.784569	-4.078386	2.526454
-0.058670	1.150000	0.835328	-4.367205	1.973886
-0.028118	1.270000	0.874074	-4.575625	1.422097
-0.008889	1.390000	0.900247	-4.713453	0.861072
0.000000	1.500000	0.912871	-4.779586	0.338178
0.001588	1.570000	0.915163	-4.791589	0.003021
0.000478	1.630000	0.913560	-4.783195	-0.284338
-0.002931	1.690000	0.908669	-4.757582	-0.570692
-0.008703	1.750000	0.900508	-4.714820	-0.855155
-0.016940	1.810000	0.889106	-4.654953	-1.137109
-0.027792	1.870000	0.874504	-4.577906	-1.416323
-0.041457	1.930000	0.856755	-4.483338	-1.693063
-0.058181	1.990000	0.835923	-4.370461	-1.968194
-0.078271	2.050000	0.812082	-4.237791	-2.243268
-0.102096	2.110000	0.785318	-4.082835	-2.520634
-0.130099	2.170000	0.755729	-3.901659	-2.803553
-0.162807	2.230000	0.723419	-3.688295	-3.096367
-0.200855	2.290000	0.688506	-3.433854	-3.404747
-0.245001	2.350000	0.651115	-3.125191	-3.736064
-0.296167	2.410000	0.611380	-2.742776	-4.099968

EXAMPLE I (con?,)

$\psi(H,\theta)$ = 2.55392

H	θ	V	X	Y
-0.608494	0.444513	0.427338	1.069939	5.921173
-0.488494	0.513757	0.488389	-0.432149	5.145598
-0.368494	0.599576	0.560775	-1.602965	4.422127
-0.249132	0.710000	0.647770	-2.523155	3.721549
-0.152526	0.830000	0.733331	-3.144892	3.122730
-0.081516	0.950000	0.808345	-3.564248	2.607194
-0.029892	1.070000	0.871732	-3.869102	2.122954
0.006672	1.190000	0.922581	-4.097494	1.639303
0.031190	1.310000	0.960161	-4.263252	1.140478
0.045677	1.430000	0.983931	-4.368588	0.622701
0.050000	1.500000	0.991276	-4.401377	0.313843
0.051449	1.570000	0.993765	-4.412524	0.002462
0.050436	1.630000	0.992024	-4.404726	-0.264560
0.047323	1.690000	0.986713	-4.380988	-0.530016
0.042044	1.750000	0.977851	-4.341542	-0.792479
0.034492	1.810000	0.965470	-4.286697	-1.050833
0.024513	1.870000	0.949614	-4.216739	-1.304410
0.011902	1.930000	0.930340	-4.131775	-1.553100
-0.003598	1.990000	0.907719	-4.031549	-1.797441
-0.022304	2.050000	0.881830	-3.915211	-2.038697
-0.044599	2.110000	0.852768	-3.781024	-2.278924
-0.070943	2.170000	0.820637	-3.625991	-2.521052
-0.101882	2.230000	0.785552	-3.445338	-2.769003
-0.138072	2.290000	0.747640	-3.231767	-3.027870
-0.180295	2.350000	0.707038	-2.974333	-3.304213
-0.229501	2.410000	0.663891	-2.656649	-3.606528
-0.286848	2.470000	0.618354	-2.253924	-3.946017
-0.346848	2.524202	0.575301	-1.785603	-4.297146
-0.406848	2.571660	0.536212	-1.259398	-4.651499
-0.466848	2.613762	0.500521	-0.665216	-5.014157
-0.526848	2.651492	0.467781	0.007538	-5.388860
-0.586848	2.685571	0.437635	0.770293	-5.778577

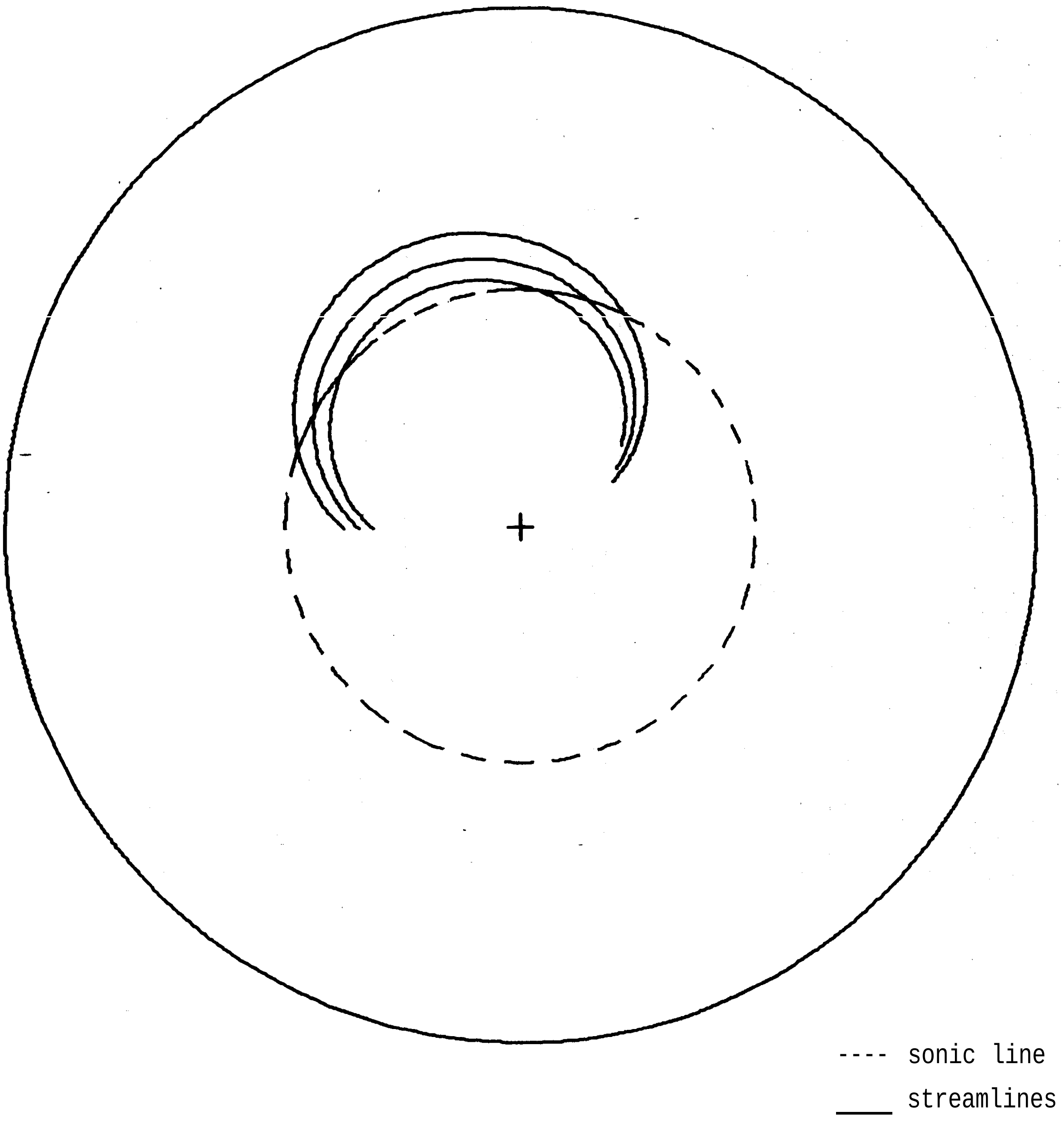
EXAMPLE I (con?.)

$\psi(H,\theta)$ = 2.33002

H	θ	V	X	Y
-0.695135	0.365072	0.388885	2.877149	6.042925
-0.575135	0.419169	0.443331	1.086153	5.306357
-0.455135	0.484400	0.507244	-0.299229	4.638221
-0.335135	0.565182	0.583374	-1.371249	4.022005
-0.214730	0.670000	0.676414	-2.210618	3.430994
-0.112822	0.790000	0.773758	-2.792562	2.914211
-0.03'8940	0.910000	0.859974	-3.175934	2.479842
0.014219	1.030000	0.933820	-3.455529	2.072781
0.051722	1.150000	0.994235	-3.671452	1.659033
0.077'119	1.270000	1.040351	-3.837091	1.219706
0.092822	1.390000	1.071503	-3.952436	0.749280
0.100000	1.500000	1.086529	-4.009656	0.296088
0.101277	1.570000	1.089257	-4.020179	0.002022
0.100385	1.630000	1.087349	-4.012814	-0.250218
0.097639	1.690000	1.081527	-3.990475	-0.500118
0.092973	1.750000	1.071814	-3.953605	-0.745516
0.086275	1.810000	1.058243	-3.902856	-0.984653
0.077386	1.870000	1.040863	-3.838958	-1.216335
0.066094	1.930000	1.019738	-3.762548	-1.440055
0.052132	1.990000	0.994943	-3.673967	-1.656067
0.035170	2.050000	0.966566	-3.573032	-1.865437
0.014809	2.110000	0.934712	-3.458758	-2.070062
-0.009428	2.170000	0.899493	-3.329036	"2.272702
-0.038112	2.230000	0.861037	-3.180191	-2.477026
-0.071924	2.290000	0.819483	-3.006377	-2.687728
-0.111678	2.350000	0.774978	-2.798648	-2.910730
~0.158355	2.410000	0.727685	-2.543488	-3.153551
-0.213152	2.470000	0.677773	-2.220378	-3.425923
-0.273469	2.526471	0.628566	-1.826024	-3.720823
-0.333469	2.575149	0.584534	-1.384301	-4.016389
-0.393469	2.618009	0.544616	-0.884102	-4.318997
-0.453469	2.656191	0.508209	-0.316058	-4.632029
-0.513469	2.690517	0.474845	0.329813	-4.958083
-0.573469	2.721602	0.444148	1.064386	-5.299320
-0.633469	2.749916	0.415812	1.899670	-5.657649
-0.693469	2.775832	0.389585	2.849030	-6.034841

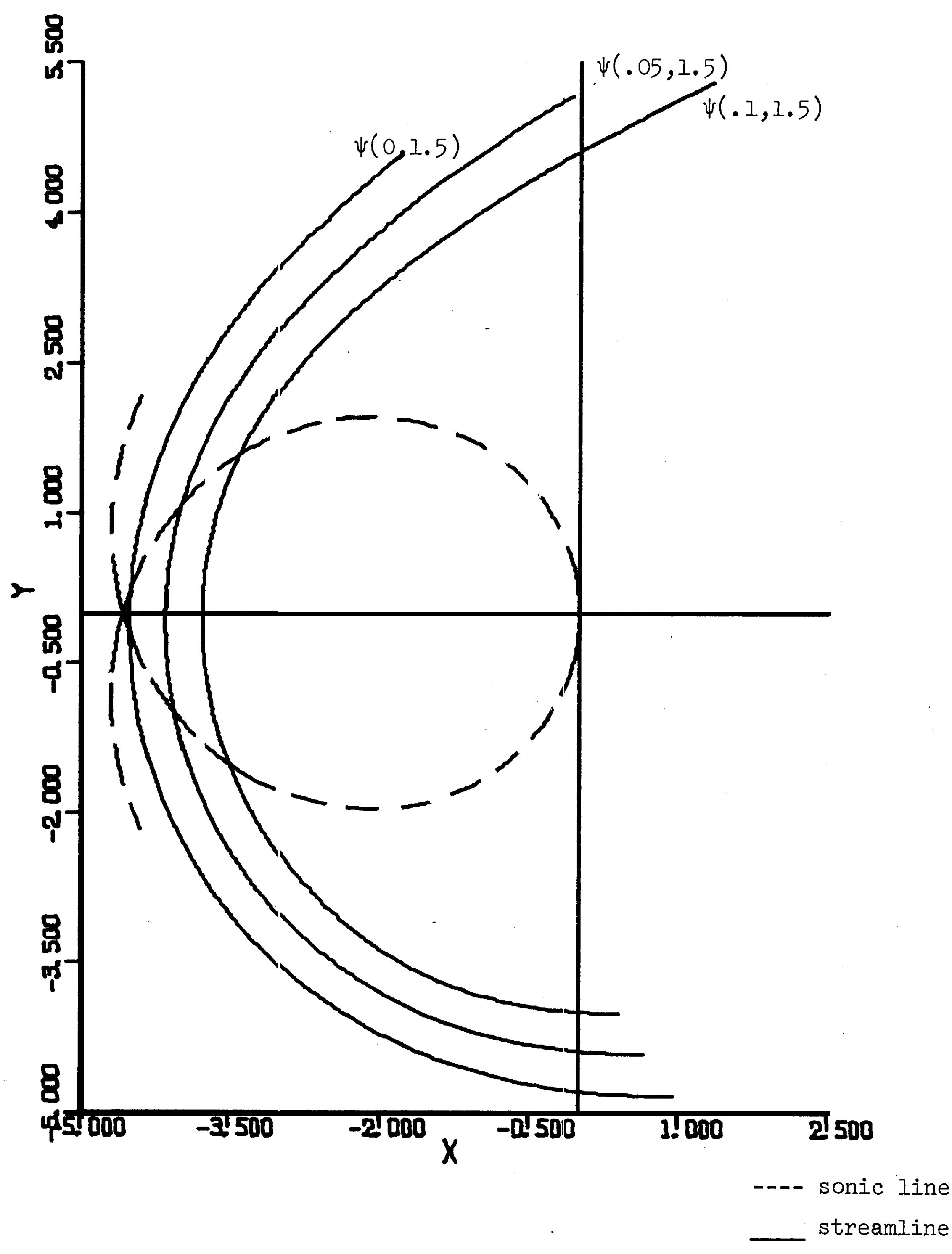
EXAMPLE II

$r = .8$



HODOGRAPH PLANE

EXAMPLE II (con't.)



PHYSICAL PLANE

EXAMPLE II (con't.)

$\psi(H,\theta)$ = 2.57676

H	θ	V	X	Y
-0.459174	0.679269	0.504913	-1.812304	4.573623
-0.339493	0.790000	0.580352	-2.727762	3.752560
-0.239967	0.910000	0.655224	-3.364548	3.031989
-0.163568	1.030000	0.722694	-3.795537	2.405731
-0.104582	1.150000	0.782613	-4.098271	1.827095
-0.059066	1.270000	0.834846	-4.309051	1.269647
-0.024239	1.390000	0.879234	-4.444491	0.718928
0.000000	1.500000	0.912871	-4.508365	0.214336
0.012136	1.570000	0.930691	-4.519884	-0.107177
0.020706	1.630000	0.943709	-4.511837	-0.382636
0.027702	1.690000	0.954620	-4.487262	-0.657376
0.033211	1.750000	0.963402	-4.444184	-0.930622
0.037303	1.810000	0.970038	-4.388683	-1.201443
0.040028	1.870000	0.974512	-4.314920	-1.468790
0.041421	1.930000	0.976815	-4.225152	-1.731545
0.041497	1.990000	0.976942	-4.119744	-1.988560
0.040259	2.050000	0.974892	-3.999158	-2.238703
0.037690	2.110000	0.970670	-3.863940	-2.480898
- 0.033760	2.170000	0.964286	-3.714694	-2.714156
0.028419	2.230000	0.955753	-3.552046	-2.937610
0.021600	2.290000	0.945089	-3.376607	-3.150526
0.013219	2.350000	0.932315	-3.188915	-3.353322
0.003167	2.410000	0.917455	-2.989392	-3.542562
-0.008685	2.470000	0.900533	-2.778270	-3.720954
-0.022495	2.530000	0.881574	-2.555531	-3.887329
-0.038448	2.590000	0.860605	-2.320820	-4.041614
-0.056767	2.650000	0.837649	-2.073343	-4.183795
-0.077711	2.710000	0.812730	-1.811749	-4.313869
-0.101592	2.770000	0.785869	-1.533953	-4.431768
-0.128776	2.830000	0.757084	-1.236917	-4.537276
-0.159699	2.890000	0.726393	-0.916324	-4.629886
-0.194881	2.950000	0.693813	-0.566118	-4.708608
-0.234948	3.010000	0.659359	-0.177819	-4.771668
-0.280655	3.070000	0.623051	0.260519	-4.016028
-0.332926	3.130000	0.584913	0.765920	-4.836609

EXAMPLE II (con?.)

ψ(H,θ) = 2.3662')

H	θ	V	X	Y
-0.585520	0.536424	0.438276	-0.075907	5.162146
-0.465520	0.613881	0.501277	-1.217096	4.426613
-0.345520	0.711196	0.576209	-2.129787	3.719365
-0.233100	9.830000	0.660891	-2.831001	3.043472
-0.147131	0.950000	0.738620	-3.295318	2.473098
-0.081466	1.070000	0.808401	-3.619632	1.958474
-0.931098	1.190000	0.870148	-3.851861	1.467296
0.007443	1.310000	0.923717	-4.013627	0.981091
0.036604	1.430000	0.968897	-4.113610	0.490122
0.050000	1.500000	0.991276	-4.144420	0.199956
0.061069	1.570000	1.010653	-4.154897	-0.092914
0.068867	1.630000	1.024832	-4.147507	-0.345590
0.075219	1.690000	1.036730	-4.124829	-0.599039
0.080214	1.750000	I.046317	-4.086746	-0.852327
0.083919	1.810000	1.053568	-4.033233	-1.104342
0.086384	1.870000	1.058460	-3.964392	-1.353834
0.087643	1.930000	1.060979	-3.880469	-1.599474
0.087712	1.990000	1.06111~	-3.781864	-1.839901
0.086592	2.050000	1.058876	-3.669119	-2.073783
0.084269	2.110000	1.054259	-3.542901	-2.299864
0.080711	2.170000	1.047283	-3.403968	-2.517012
0.075870	2.230000	1.037966	-3.253130	-2.724253
0.069679	2.290000	1.026336	-3.091193	-2.920793
0.062055	2.350000	1.012421	-2.91~910	-3.106034
0.052892	2.410000	0.996257	-2.736922	-3.279567
0.042060	2.470000	0.977878	-2.545694	-3.441163
0.029407	2.530000	0.957320	-2.345451	-3.590740
0.014748	2.590000	0.934619	-2.136110	-3.738369
-0.002135	2.650000	0.909607	-1.917195	-3.854155
-0.021498	2.710000	0.882916	-1.687735	-3.968263
-0.043648	2.770000	0.853971	-1.446136	-4.070814
-0.068947	2.830000	0.822994	-1.189993	-4.161810
-0.097826	2.890000	0.790003	-0.915842	-4.241017
-0.130801	2.950000	0.755011	-0.618780	-4.307807
-0.168494	3.010000	0.718028	-0.291913	-4.360904
-0.211659	3.070000	0.679063	0.074488	-4.397997
-0.261219	3.130000	0.638128	0.494314	-4.415107

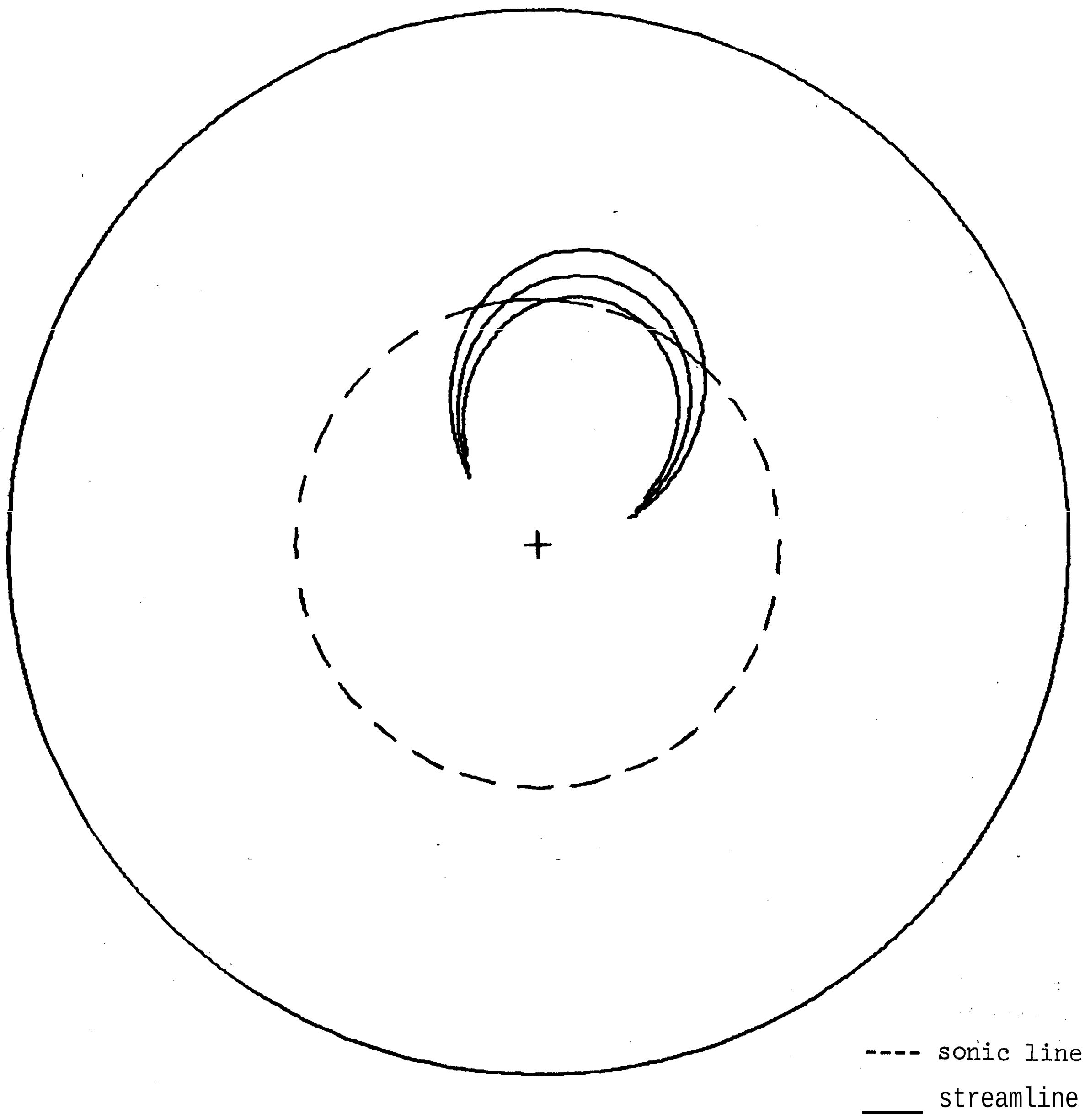
EXAMPLE II (con?.)

$\psi(H,\theta)$ = 2.15312

H	θ	V	X	Y
-0.666839	0.445646	0.400980	1.350616	5.393565
-0.546839	0.506108	0.457464	0.024044	4.612360
-0.426839	0.579952	0.523967	-1.023706	3.983691
-0.306839	0.673057	0.603528	-1.857063	3.384705
-0.192046	0.790000	0.696353	-2.506193	2.806470
-0.103968	0.910000	0.783281	-2.936194	2.319639
-0.037519	1.030000	0.861800	-3.235730	1.884086
0.0129'44	1.150000	0.931902	-3.453490	1.467489
0.051272	1.270000	0.993459	-3.610903	1.050679
0.080161	1.390000	1.046215	-3.716011	0.622565
0.100000	1.500000	1.086529	-3.767370	0.215792
0.109838	1.570000	1.108021	-3.776871	-0.050353
0.116742	1.630000	1.123786	-3.770074	-0.383363
0.122350	1.690000	1.137044	-3.749061	-0.517087
0.126748	1.750000	1.147746	-3.713520	-0.753406
0.130004	1.810000	1.155851	-3.663283	-0.989956
0.132166	1.870000	1.161326	-3.598372	-1.225189
0.133270	1.930000	1.164147	-3.519023	'I.457435
-0.133330	1.990000	1.164302	-3.425700	-1.684983
0.132349	2.050000	I.161792	-3.3190~1	-1.906160
0.130311	2.110000	1.156625	-3.200031	-2.119411
0.127185	2.170000	1.148825	-3.069550	-2.323363
0.122923	2.230000	1.138424	-2.928714	-2.516875
0.117460	2.290000	1.125461	-2.778609	-2.699071
0.110712	2.350000	1.109985	-2.620259	-2.869349
0.102574	2.410000	1.092048	-2.454558	-3.027369
0.092920	3.470000	1.071704	-2.282206	-3.173031
0.081597	2.530000	1.049009	-2.103650	-3.306'431
0.068426	2.590000	1.024017	-1.919025	-3.427820
0.053192	2.650000	0.936777	-1.728091	-3.537544
0.035643	2.710000	0.967334	-1.530160	-3.635988
0.015479	2.770000	0.935724	-1.324004	-3.723508
-0.007656	2.830000	0.901978	-1.107717	-3.800358
-0.034185	?.890000	0.866116	-0.878523	-3.866590
-0.064618	2.950090	0.829150	-0.632487	-3.921919
-0.099569	3.010000	0.788085	-0.364082	-3.965531
-0.139786	3.070000	0.745920	-0.065517	-3.995769
-0.186188	3.130000	0.701649	0.274318	-4.009630

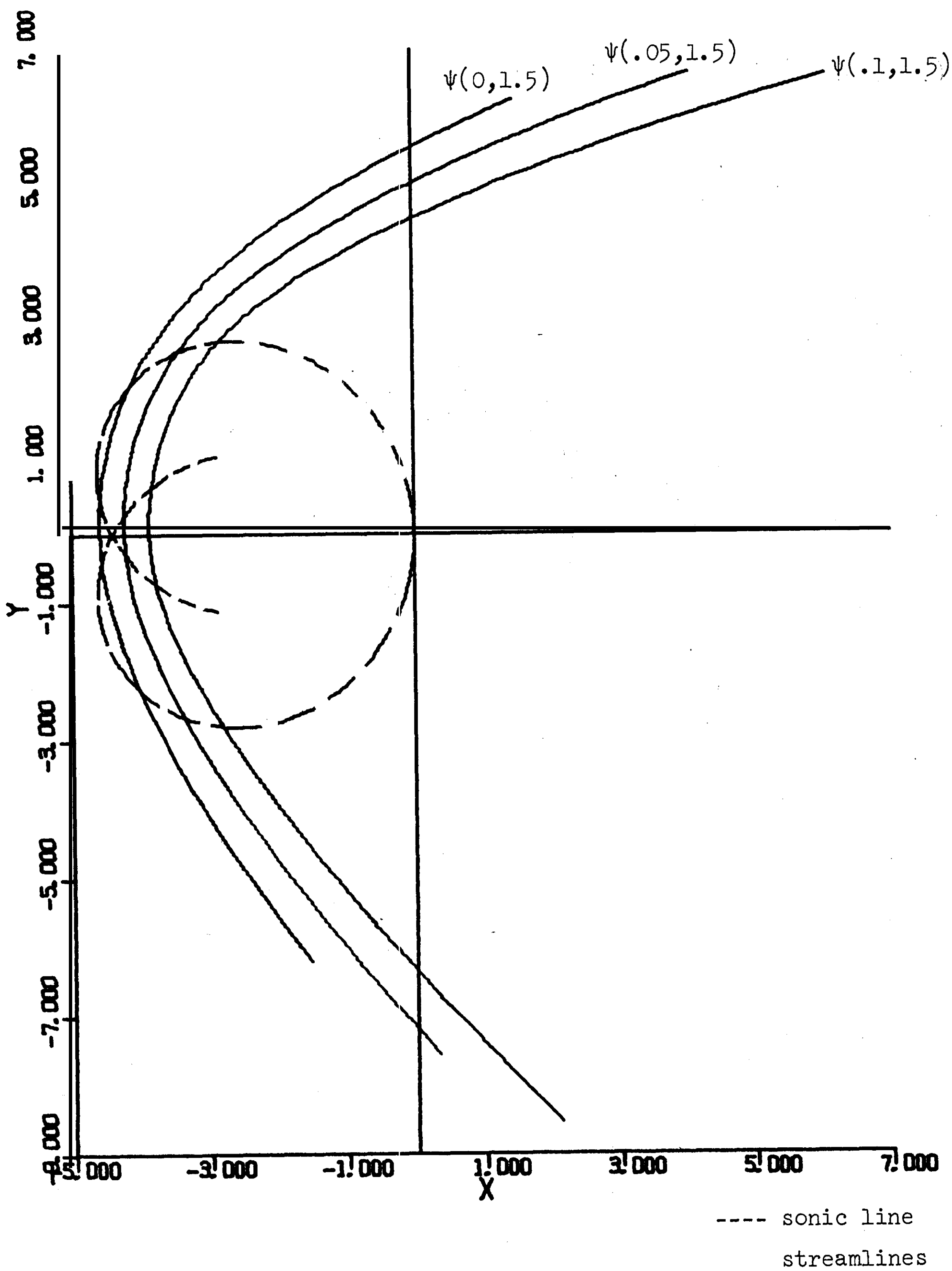
EXAMPLE III

r = 1.2



HODOGRAPH PLANE

EXAMPLE III (con't.)



PHYSICAL PLANE

EXAMPLE III (cod-t.)

$\psi(H,\theta) =$ 2.72611

H	6		X	Y
-0.580826	0.401327	0.440552	1.476017	6.254477
-0.460826	0.471021	0.503963	-0.240465	5.460012
-0.340826	0.557665	0.579432	-1.548473	4.726842
-0.221657	0.670000	0.670498	-2.550364	4.027697
-0.128987	0.790000	0.756867	-3.196180	3.453982
-0.063925	0.910000	0.828981	-3.629674	2.962596
-0.020542	1.030000	0.884206	-3.952407	2.492531
0.005389	1.150000	0.920698	-4.204219	2.010002
0.016573	1.270000	0.937384	-4.395289	1.503578
0.014228	1.390000	0.933835	-4.524504	0.977393
0.000000	1.500000	0.912871	-4.586746	0.485194
-0.015624	1.570000	0.890908	-4.598089	0.167774
-0.033559	1.630000	0.866930	-4.589958	-0.109205
-0.056159	1.690000	0.838393	-4.564332	-0.394863
-0.083970	1.750000	0.805538	-4.518983	-0.695669
-0.117638	1.810000	0.768661	-4.449672	-1.021129
-0.157922	1.870000	0.728103	-4.349059	-1.384601
-0.205709	1.930000	0.664239	-4.205072	-1.804590
-0.262066	1.990000	0.637460	-3.998271	-2.306958
-0.322974	2.045510	0.591920	-3.724098	-2.877056
-0.382974	2.093138	0.551330	-3.395088	-3.479841
-0.443974	3.135315	0.514346	-2.998829	-4.134372
-0.502974	2.173075	0.480479	-2.525377	-4.850369
-0.562974	2.207153	0.449339	-1.963393	-5.637446

EXAMPLE III (con?.)

$\psi(H,\theta)$ = 2.51901

H	θ	V	X	Y
-0.698208	0.317281	0.387596	4.071576	6.636595
-0.578208	0.369817	0.441828	1,844998	5.845047
-0.458208	0.432963	0.505469	0.155303	5.132771
-0.338208	0.5~0590	0.581241	-1.122548	4.485856
-0.218208	0,610000	0.673434	-2.090477	3,884287
-0.111629	0.730000	0.775031	-2.769276	3.351524
-0.036975	0,850000	0.862501	-3.198451	2,920697
0,013330	0.970000	0.932483	-3.512720	2.517254
0.044887	1.090000	0,982602	-3.765403	2,096468
0.061554	1.210000	1.011521	-3.969367	1,640385
0.06'5497	1,330000	1.018648	-4.120222	1,153924
0.057243	1.450000	1,003859	-4.211469	0.655248
0.050000	1.500000	0.991276	-4.231372	0,448243
0.035696	1.570000	0.967420	-4.241685	0.160864
0.019143	1.630000	0.941307	-4,234524	-0.084132
-0.001897	1.690000	0.910147	-4.212408	-0,331015
-0,028037	1.750000	0.874180	-4.174091	-0.585367
-0.059997	1.810000	0,833717	-4.116524	-0,855782
-0.098611	1,870000	0.789138	-4.433845	-1.154507
-0,144843	1.930000	0,740881	-3.915922	-1,498466
-0,199821	1,990000	0.689420	-3.746061	-I,911054
-0.260336	2.046083	0,638841	-3,516536	-2,387872
-0.320316	2.094216	0.593811	-3,237906	-2,897276
-0.380316	2.136639	0.553047	-2.900613	-3.453847
-0.440316	2.174500	0,515915	-2.495850	-4,063101
-0.500316	2.208595	0.481919	-2.013670	-4.736341
-0.560316	2.239507	0.450666	-1.442758	-5.481226
-0.620316	?,267677	0.421836	-0.770200	-6,307138
-0,680316	2.293449	0.395165	0,018777	-7.224454

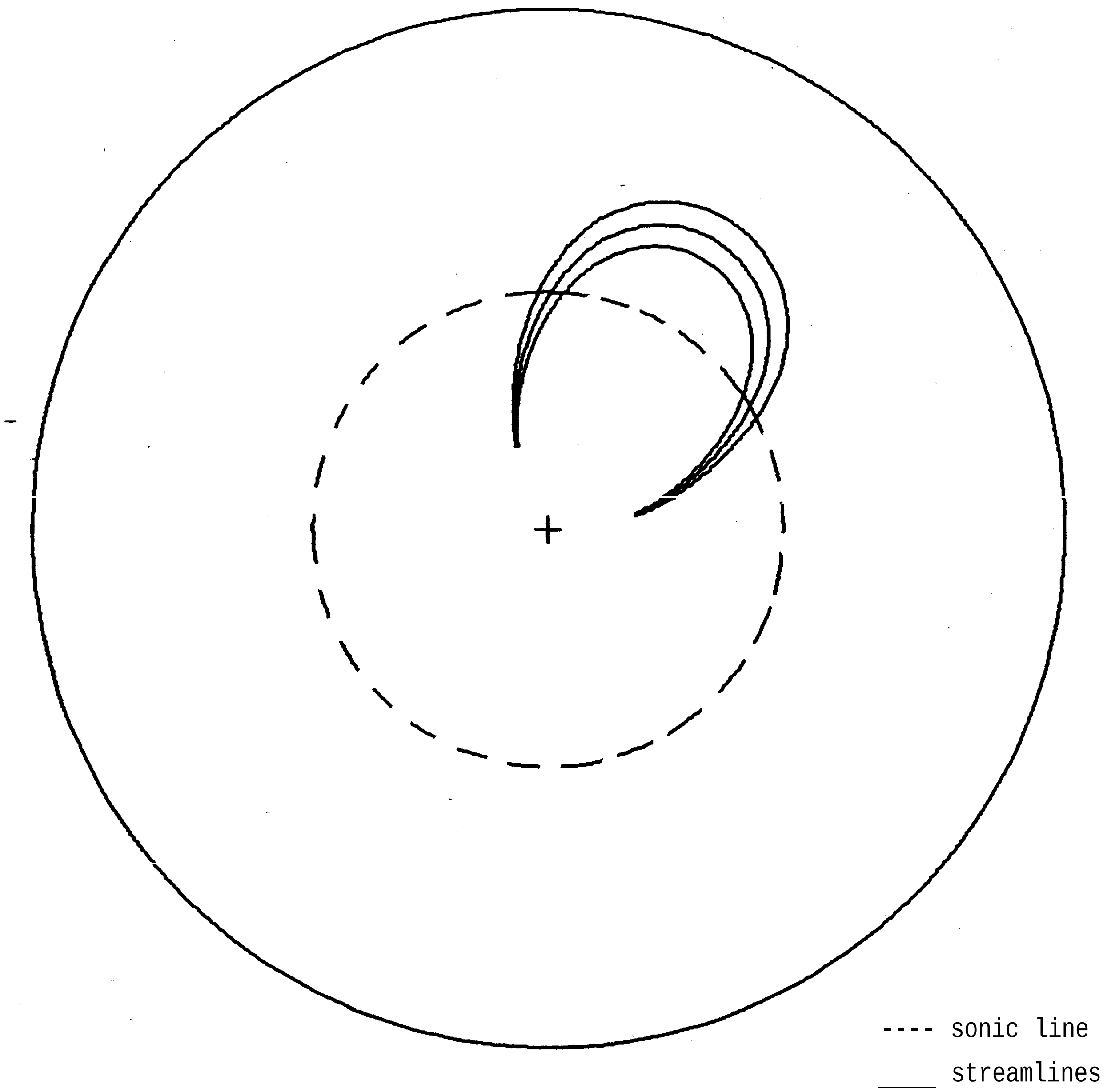
EXAMPLE III (con't.)

$\psi(H,\theta) = 2.30546$

H	θ	V	X	Y
-0.770631	0.264214	0.358614	6.105391	6.602550
-0.650631	0.307001	0.408105	3.491110	5.839439
-0.530631	0.357719	0.465807	1.507501	5.159072
-0.410631	0.418670	0.533866	0.010987	4.551301
-0.290631	0.493659	0.615509	-1.109627	4.005809
-0.170631	0.590000	0.716017	-1.944262	3.508361
-0.062023	0.710000	0.831268	-2.525989	3.070782
0.012382	0.830000	0.931059	-2.883756	2.725779
0.061465	0.950000	1.011362	-3.152513	2.394328
0.091925	1.070000	1.069863	-3.382926	2.026937
0.108397	1.190000	1.104804	-3.581824	1.604501
0.113544	1.310000	1.116410	-3.736953	1.137166
0.108219	1.430000	1.104410	-3.835264	0.654916
0.100000	1.500000	1.086529	-3.864520	0.380099
0.087496	1.570000	1.060685	-3.874064	0.115790
0.072863	1.630000	1.032278	-3.867752	-0.101634
0.054028	1.690000	0.998230	-3.848894	-0.312644
0.030302	1.750000	0.958743	-3.817395	-0.522012
0.000864	1.810000	0.914117	-3.771525	-0.737634
-0.035234	1.870000	0.864753	-3.706958	-0.970980
-0.079076	1.930000	0.811152	-3.615475	-1.237811
-0.131897	1.990000	0.753891	-3.482995	-1.559542
-0.192800	2.047961	0.695676	-3.294385	al.950235
-0.252800	2.096704	0.644822	-3.064468	-2.368264
-0.312800	2.139504	0.599203	-2.783622	-2.827966
-0.372800	2.177444	0.557943	-2.444118	-3.336567
-0.432800	2.211498	0.530386	-2.037334	-3.901006
-0.492800	2.242304	0.486020	-1.553520	-4.528516
-0.552800	2.270333	0.454442	-0.981566	-5.226971
-0.612800	2.295949	0.425324	-0.308770	-6.005129
-0.672800	2.319441	0.398396	0.479410	-6.872833
-0.732800	2.341042	0.373432	1.399636	-7.841207

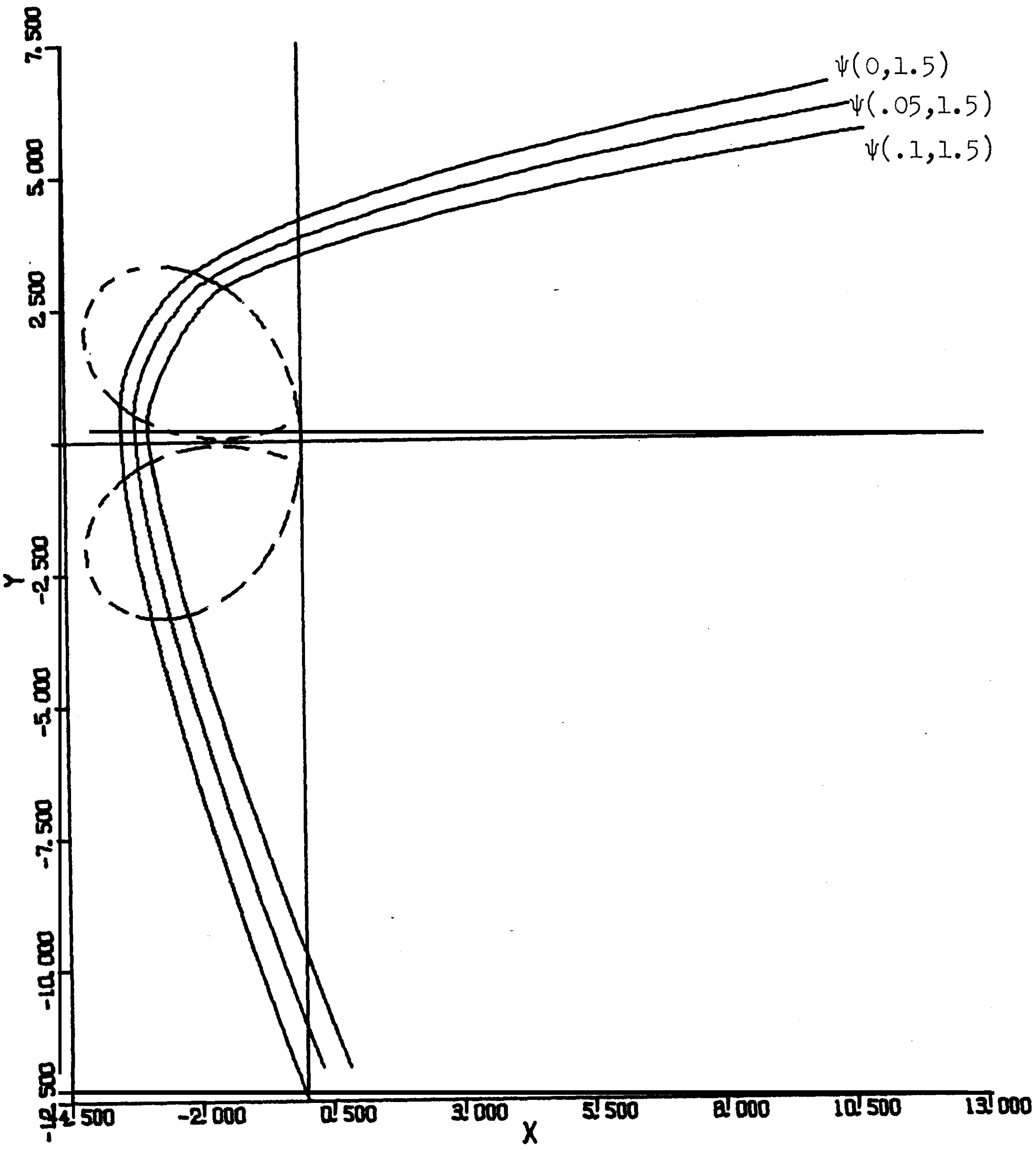
EXAMPLE IV

r = 1.5



HODOGRAPH PLANE

EXAMPLE IV (con't.)



PHYSICAL PLANE

EXAMPLE IV (con't.)

$\psi(H,\theta)$ = 2.20548

H	θ	V	X	Y
-0.803855	0.177423	0.346149	10.133879	6.638649
-0.683855	0.211565	0.393655	6.348335	5.898426
-0.563855	0.252018	0.448901	3.556194	5.244259
-0.443855	0.299932	0.513827	1.511290	4.669536
-0.323855	0.356902	0.591295	0.035407	4.170702
-0.203855	0.425672	0.685864	-0.999284	3.747860
-0.083855	0.512689	0.805669	-1.685890	3.404276
0.028592	0.630000	0.956028	-2.094014	3.146683
0.095673	0.750000	1.077408	-2.322963	2.958135
0.130230	0.870000	1.156421	-2.567860	2.699149
0.144563	0.990000	1.194216	-2.848825	2.320764
0.144240	1.110000	1.193324	-3.107798	1.869345
0.129135	1.230000	1.153673	-3.290854	1.439947
0.093386	1.350000	1.072666	-3.389280	1.102906
0.024388	1.470000	0.949419	-3.436601	0.808369
0.000000	1.500000	0.912871	-3.444460	0.715589
-0.068157	1.568212	0.823931	-3.454288	0.420090
-0.128157	1.616717	0.757720	-3.446610	0.093906
-0.188157	1.658598	0.699862	-3.419213	-0.306801
-0.248157	1.695758	0.648557	-3.367022	-0.790445
-0.308157	1.729251	0.602568	-3.284415	-1.365666
-0.368157	1.759714	0.560997	-3.165111	-2.042477
-0.428157	1.787568	0.523172	-3.001983	-2.832818
-0.488157	1.813108	0.488575	-2.786844	-3.750968
-0.548157	1.836562	0.456794	-2.510211	-4.813924
-0.608157	1.858113	0.427496	-2.161041	-6.041816
-0.668157	1.877919	0.400407	-1.726435	-7.458379
-0.728157	1.896116	0.375298	-1.191310	-9.091488
-0.788157	1.912830	0.351977	-0.538009	-10.973786

EXAMPLE IV (con?.)

$\psi(H,\theta)$ = 2.05042

H	θ	V	X	Y
-0.809536	0.163312	0.344066	10.531442	6.222300
-0.689536	0.194611	0.391244	6.700484	5.534332
-0.569536	0.231596	0.446085	3.876399	4.927689
-0.449536	0.275225	0.510498	1.809345	4.396350
-0.329536	0.326763	0.587287	0.318922	3.937328
-0.209536	0.388297	0.680902	-0.722624	3.551744
-0.089536	0.464555	0.799234	-1.402841	3.246541
0.029549	0.570000	0.957546	-1.783280	3.034989
0.109142	0.690000	1.106464	-1.944453	2.918646
0.149531	0.810000	1.208196	-2.130495	2.743130
0.168079	0.930000	1.265312	-2.409498	2.409642
0.173311	1.050000	1.283104	-2.717262	1.939420
0.167547	1.170000	1.263552	-2.966366	1.439379
0.148230	1.290000	1.204486	-3.110682	1.037878
0.106468	1.410000	1.100536	-3.166639	0.795663
0.050000	1.500000	0.991276	-3.182474	0.656730
-0.015592	1.570000	0.890953	-3.188156	0.472750
-0.075592	1.619553	0.815192	-3.181889	0.236330
-0.135592	1.661398	0.750141	-3.159602	-0.075402
-0.195592	1.698135	0.693178	-3.116196	-0.467620
-0.255592	1.731099	0.642591	-3.046418	-0.946099
-0.315592	1.761049	0.597192	-2.944595	-1.518306
-0.375592	1.788451	0.556118	-2.804390	-2.193758
-0.435592	1.813616	0.518720	-2.618576	-2.984265
-0.495592	1.836771	0.494492	-2.378814	-3.904186
-0.555592	1.858091	0.453035	-2.075416	-4.970742
-0.615592	1.877723	0.424024	-1.697079	-6.204394
-0.675592	1.895795	0.397192	-1.230587	-7.629297
-0.735592	1.912422	0.372315	-0.660481	-9.273836
-0.795592	1.927710	0.349203	0.031330	-11.171256

EXAMPLE IV (con't.)

$\psi(H,\theta)$ = 1.88713

H	θ	V	X	Y
-0.812365	0.149453	0.343035	10.808842	5.756544
LO.692365	0.177982	0.390050	6.959186	5.125183
-0.572365	0.211610	0.444691	4.122977	4.569640
-0.452365	0.251132	0.508851	2.048498	4.084485
-0.332365	0.297554	0.585306	0.554402	3.667215
-0.212365	0.352478	0.678453	-0.486357	3.319583
-0.092365	0.419485	0.796063	-1.156381	3.050582
0.027635	0.510000	0.954515	-1.497772	2.883754
0.121332	0.630000	1.134605	-1.563094	2.844354
0.166569	0.750000	1.260331	-1.643847	2.775171
0.187537	0.870000	1.336341	-1.887637	2.515059
0.195853	0.990000	1.371601	-2.237949	2.042332
0.195668	1.110000	1.370776	-2.574536	1.455648
0.186894	1.230000	1.333755	-2.802583	0.922246
0.165129	1.350000	1.255641	-2.900548	0.592628
0.118273	1.470000	1.127363	-2.917997	0.496613
0.100000	1.500000	1.086529	-2.918447	0.490643
0.042469	1.570000	0.978558	-2.919469	0.444509
-0.018443	1.623516	0.887055	-2.915455	0.315545
-0.078443	1.666003	0.811883	-2.899199	0.103279
-0.138443	1.702482	0.747267	-2.8615059	-0.192276
-0.198443	1.734869	0.690641	-2.808039	-0.572112
-0.258443	1.764167	0.640324	-2.723068	-1.040102
-0.318443	1.790947	0.595148	-2.604607	-1.602782
-0.378443	1.815564	0.554262	-2.446387	-2.269213
-0.438443	1.838254	0.517025	-2.241193	-3.050998
-0.498443	1.859193	0.482937	-1.980646	-3.962441
-0.558443	1.878519	0.451603	-1.654973	-5.020794
-0.618443	1.896349	0.422702	-1.252745	-6.246616
-0.678443	1.912787	0.395967	-0.760591	-7.664203
-0.738443	1.927931	0.371178	-0.162859	-9.303118
-0.798443	1.941870	0.348145	0.558768	-11.193819

5. ERROR ANALYSIS

Before proceding with a formal analysis, we present some empirical results. This will allow a more realistic evaluation of the error bounds to be proved. To do this we have used the well known Ringleb solution,

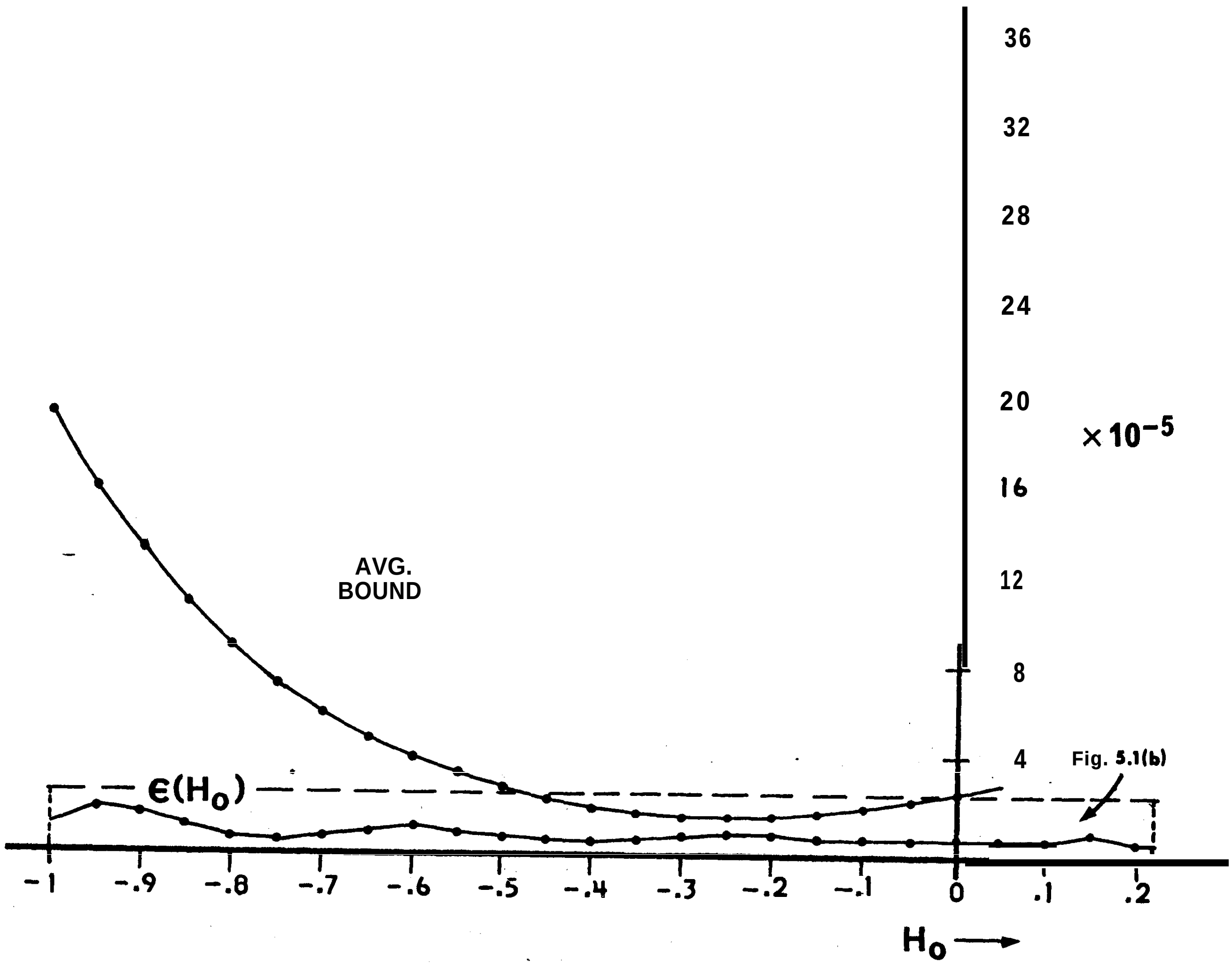
$$(5.1) \qquad \psi^R(H,\theta) = \frac{2.538}{v(H)} \sin \theta$$

of equation (2.5) to set up initial value problems for $H_o, H \epsilon [-1,.22]$. We have then used the program included in the Appendix to compute $\overline{\psi_7^R}(H,H_o,\theta)$ for $H,H_o = -1,-.95,\ldots,.2,.22$. Figure 5.1 is a graph of the average error, ϵ , versus H_0, where

$$(5.2) \qquad \epsilon(H_o) \equiv \frac{1}{26} \sum_{j=1}^{26} |\psi^R(H_j,1) - \overline{\psi_7^R}(H_j,H_o,1)|$$

and $\qquad H_1 = -1 , H_2 = -.95,\ldots,H_{25} = .20$ and $H_{26} = .22$.

Figure 5.2 contains graphs of $|\psi^R - \overline{\psi_7^R}|$ versus H , for several values of H_o . The maximum absolute error tabulated was 3.91×10^{-5} , occuring at H = .2 , $H_o = -.95$. The error bound on $|\psi^R - \overline{\psi_7^R}|$, given by the sum of formulae (5.35) and (5.46), was tabulated for $H_o = -1,-.95,\ldots,.05$ and $H = -1,-.95,\ldots,.2,.22$ (the omission of $H_o = .1,.15,.2$ and .22 will be explained shortly). The maximum value tabulated for this bound was 1.2×10^{-3} , occuring at H = .22 , $H_o = -1.0$. It is difficult to maximize this bound, as a function of H and H_o . However, a somewhat weaker bound, given by (5.54)+(5.55), can be maximized easily, yielding an upper bound (for all $H_o \epsilon [-1,.06593\ldots]$ and $H \epsilon [-1,.22]$) on the error in our approximate Ringleb solution of 3.3×10^{-3} .



36
32
28
24
20
×10⁻⁵
16
12
8
4
AVG.
BOUND
ε(Ho)
Fig. 5.1(b)
-1
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
.1
.2
Ho

FIG. 5.1(a)

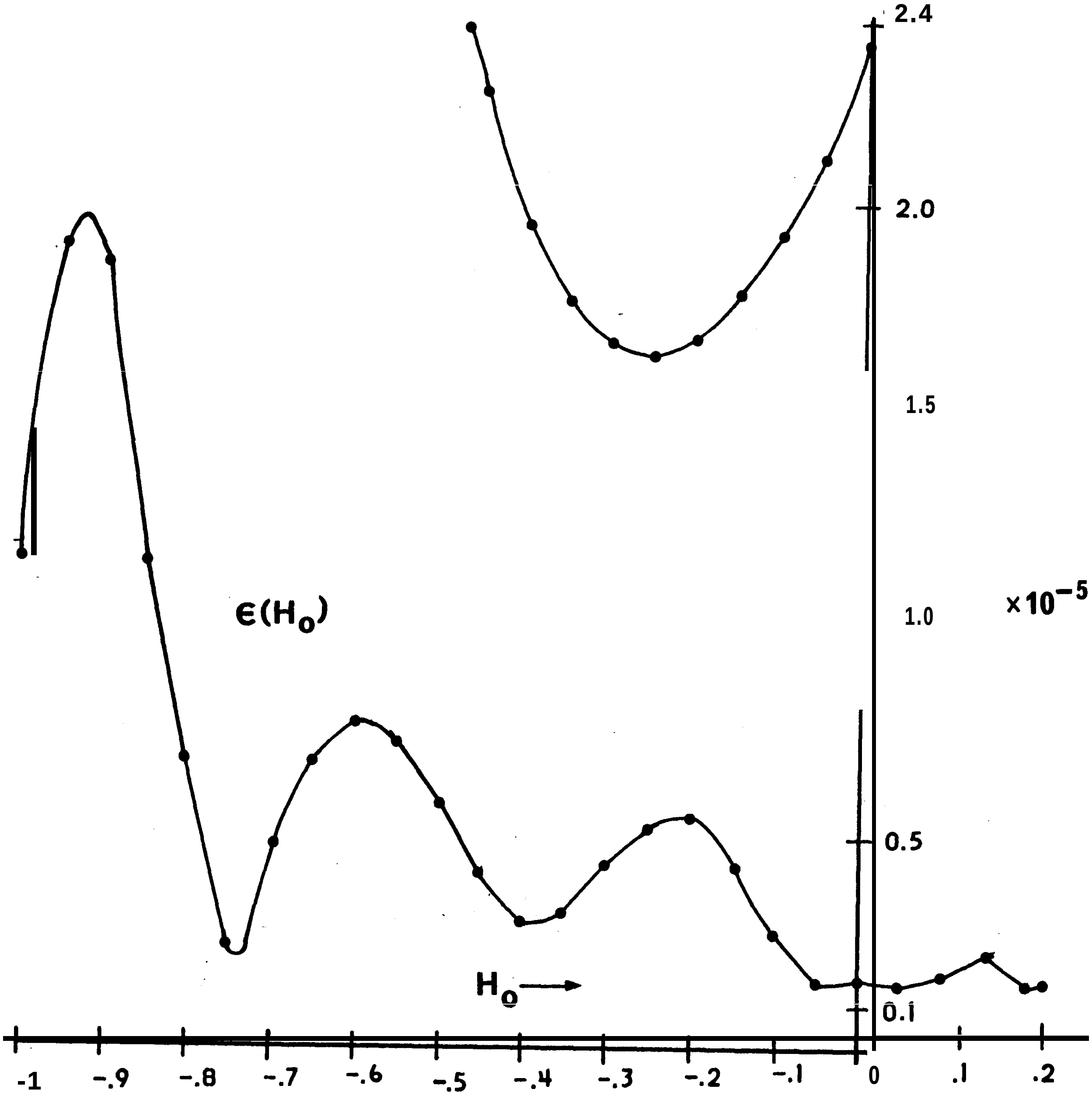
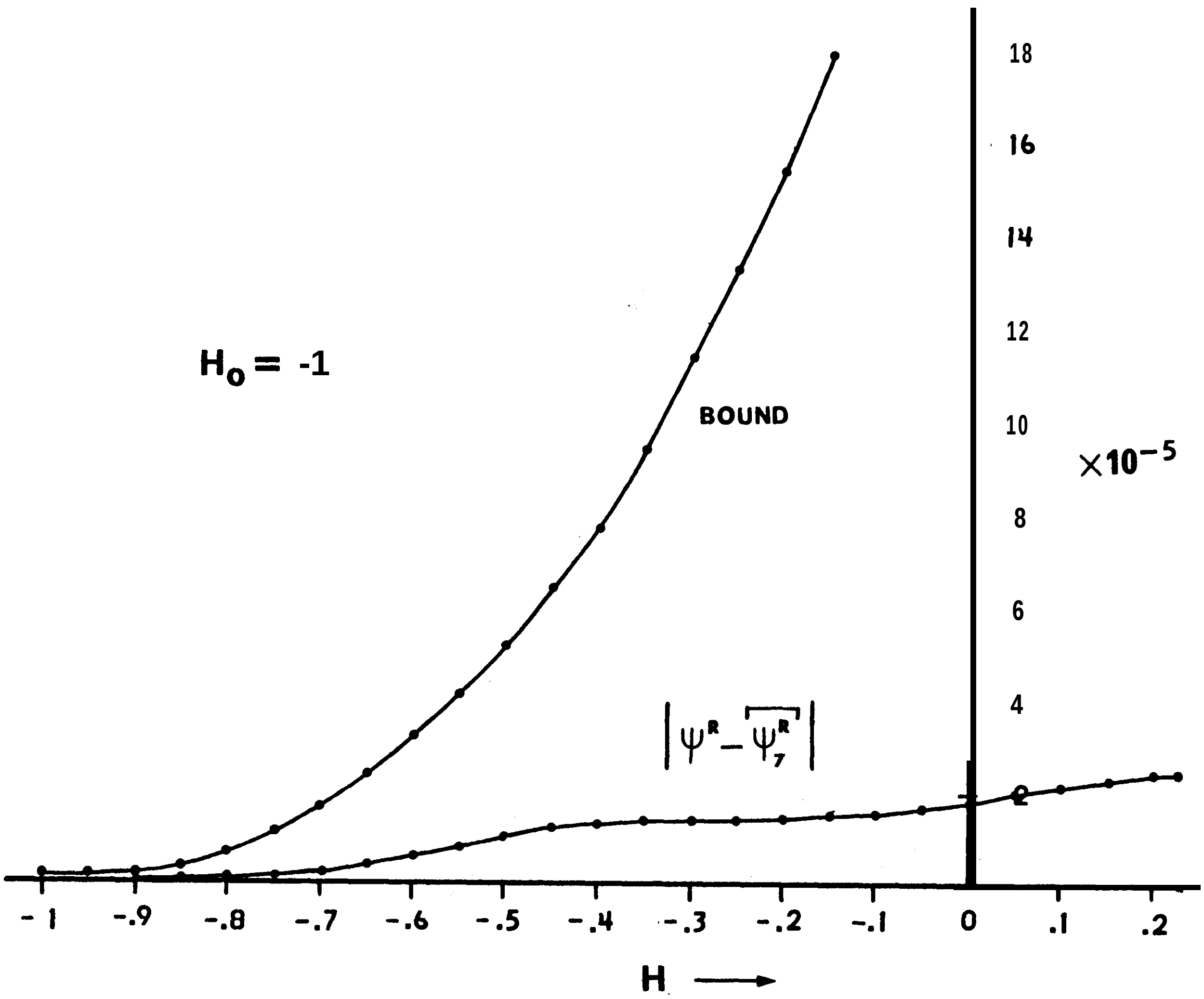
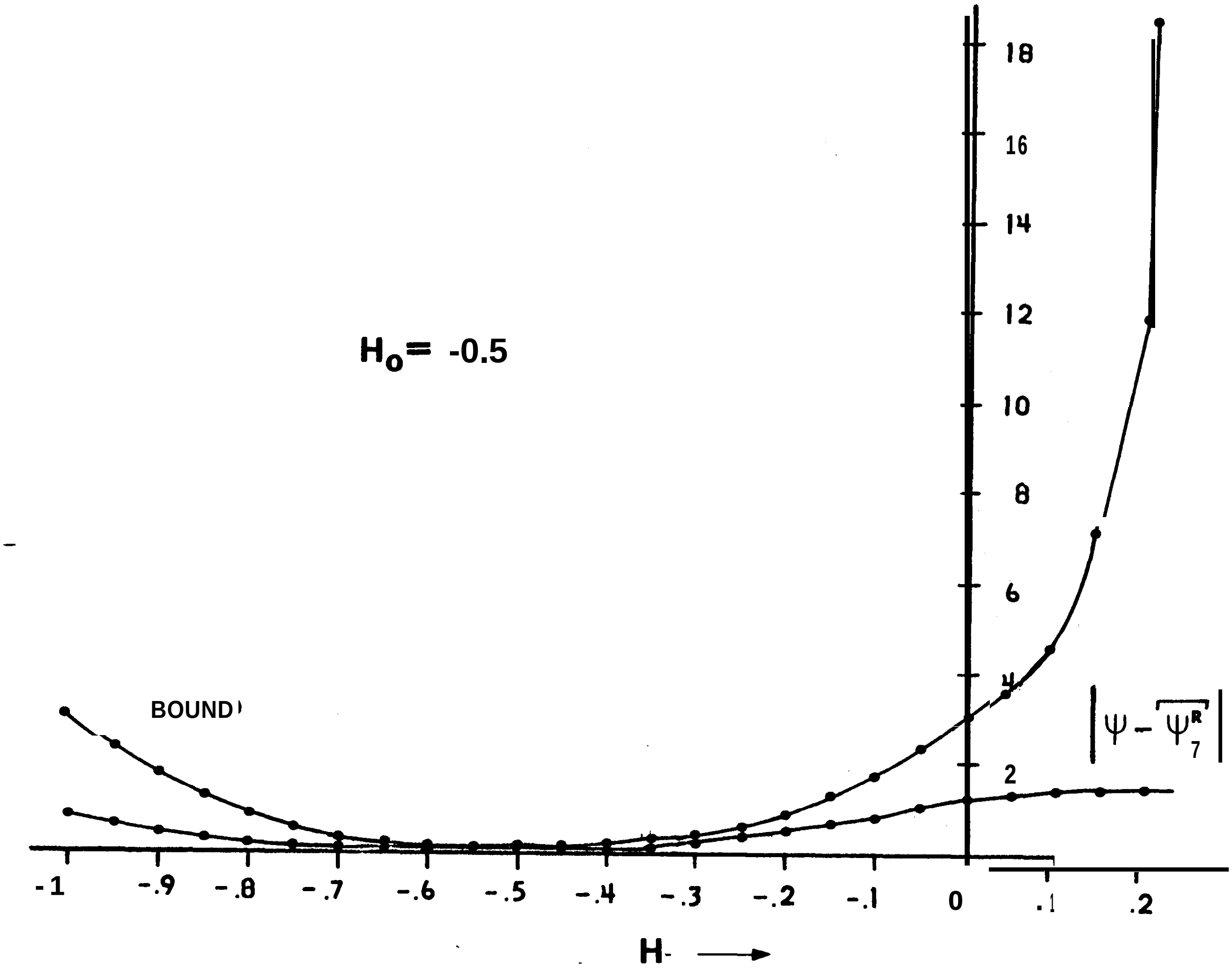


FIG 5.1(b)



18
16
14
12
10
8
6
4
2
$\times 10^{-5}$
$H_0 = -1$
BOUND
$\left| \psi^R - \overline{\psi_7^R} \right|$
-1
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
.1
.2
H →

FIG. 5.2 (a)



18
16
14
12
10
8
6
4
2
$H_o = -0.5$
BOUND
$\left|\psi - \overline{\psi^R_7}\right|$
-1
-.9
-.8
-.7
-.6
-.5
-.4
-.3
-.2
-.1
0
.1
.2
H →

FIG. 5.2 (b)

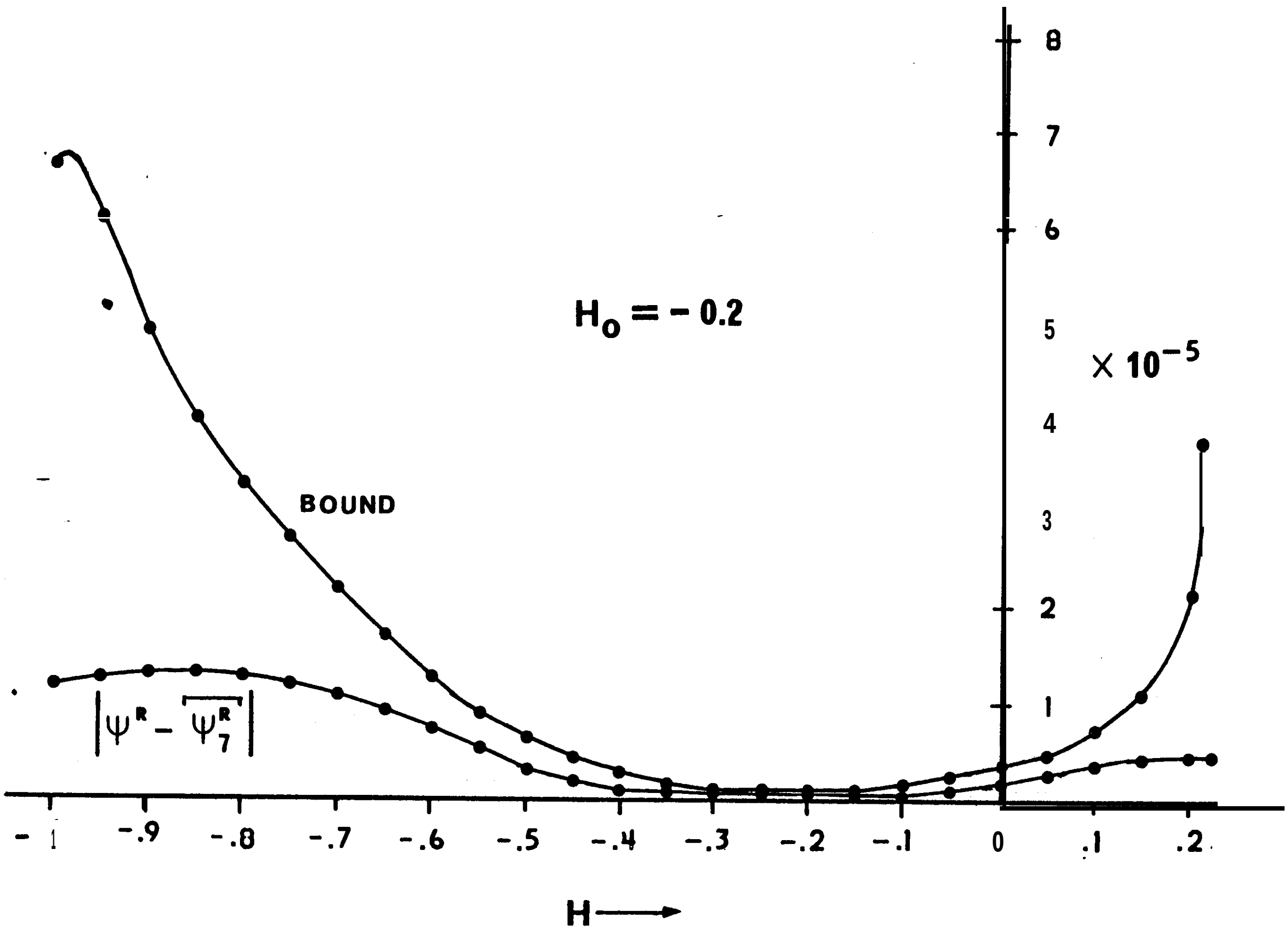


FIG. 5.2 (c)

The program which carried out the calculation of the error bound is included in the Appendix. These calculations were done only for $\theta = 1$ radian since the simple form of ψ^R and the fact that the error in $\overline{f^{(2j)}}$ and $\overline{g^{(2j+1)}}$ is very small in this case, make the <u>relative</u> error given by the formulae of this section, essentially independent of θ .

Let us proceed with a formal error analysis. The error involved in our computation draws from three sources:

(1) <u>truncation</u> -- we have truncated the infinite series of (2.7), for $\psi(H,\theta)$, to yield $\psi_n(H,H_o,\theta)$;

(2) <u>function approximation</u> -- we have permitted the use of $\overline{f^{(2j)}}$, $\overline{g^{(2j+1)}}$, for $j = 0,1,\ldots,n$, and $\overline{\ell}$, to yield $\overline{\psi}_n(H,H_o,\theta)$; and

(3) <u>roundoff</u> -- computations are done in fixed length, finite precision arithmetic.

Errors of types (2) and (3) can be confused easily: type (2) errors are due to the fact that the formulae used to calculate certain functions would not give exact values, even if exact arithmetic were used; type (3) errors are due to the inexactness of computer arithmetic. Confusion may arise when inexact formulae are computed with inexact arithmetic.

Roundoff error has been no problem in our work, partly because we are using 10 digits for our essentially 5 digit calculations. We shall not consider roundoff error here. The following analysis provides bounds, as functions of H, H_o and θ , for the truncation and approximation errors. A series of five lemmas are required. The first three lemmas present rough bounds based on (2.9), itself a rather rough bound on $|s_m|$. The derivation of these bounds utilizes only one property of $\ell(H)$, that for $H \epsilon [\alpha,\beta]$, $|\ell(H)| \leq c^2$. In this paper, we deal with $[\alpha,\beta] \subseteq [-1,.22]$, for which $c^2 \leq 62.47$. When evaluating our bounds for particular H and H_o, we of course choose $[\alpha,\beta] = [H_o,H]$.

Let $a = .0659262218\ldots$. Then we have

(5.3) $$\ell(a) = -1 .$$

When $H_o < a < H$ or $H < a < H_o$, the first bounds are poor. Lemmas 5.4 and 5.5 give considerably improved bounds, valid for $H_o \leq a < H$. In the Ringleb computation considered, these new bounds were as must as 10^{10} better than the old bounds. The case $H < a < H_o$ could be treated similarly, but this will not be done here. (This is why the cases $H_0 = .1,.15,.2,.22$ were omitted from the bound calculations summarized in Graphs 5.1 and 5.2.) The improved bounds depend on one further property of $\ell(H)$, that $|\ell(H)| < 1$ for $H \epsilon [\alpha,a]$. Thus, all the bounds given are valid for any function, 1(H), whose graph lies within the darkened area of Figure 5.3; the first bounds are valid for any $\ell(H)$ whose graph lies within the dashed rectangle.

In order to present simple a priori bounds, we assume that, for fixed θ, $f^{(2j)}(\theta)$ and $g^{(2j+1)}(\theta)$ grow (with j) no faster than geometrically. However, the derivatives of even analytic functions can grow much faster than this. (If $h(\theta)$ is analytic then, by Cauchy's formula, $|h^{(j)}(\theta)| \leq \max|h(\theta)| j! r^{-j-1}$, where r is the minimum distance

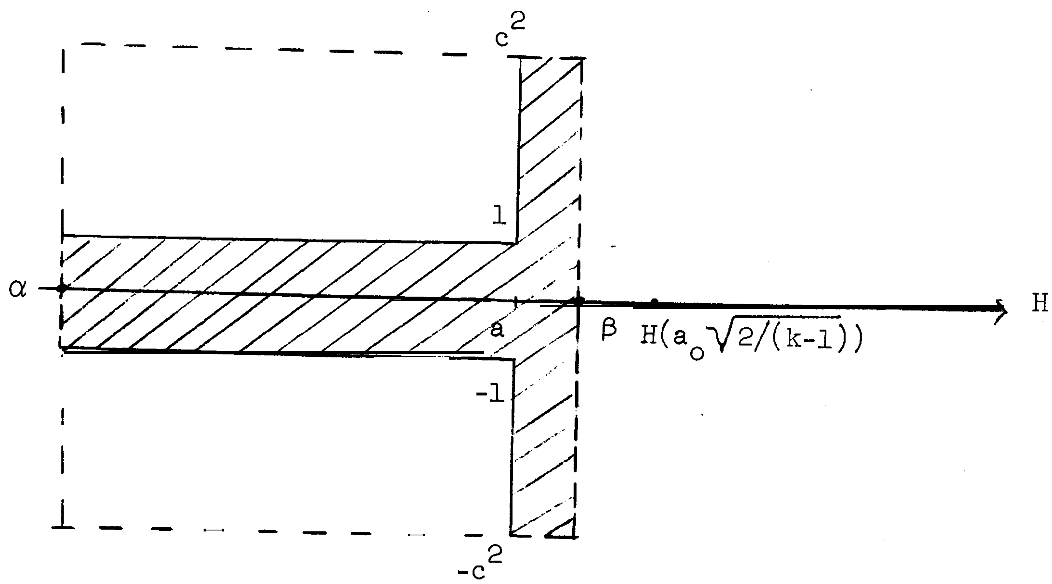


Figure 5.3

of θ from the boundary of some domain within which h is analytic; the maximum of $|h(\theta)|$ is to be taken over the same domain from which r is computed.) The bound on the approximation error also involves terms which must bound the error caused by $\overline{f^{(2j)}}$ and $\overline{g^{(2j+1)}}$ for $j \leq n$. If these errors can be assumed negligible (of if a bound can be found), then an a posteriori bound on the error due to function approximation can be computed, while the stream function, ψ, is being computed, without any assumptions about the growth of $f^{(2j)}$ and $g^{(2j+1)}$. This is not possible for the truncation error; we must have definite knowledge of the growth of $f^{(2j)}$ and $g^{(2j+1)}$, as $j \to \infty$, in order to bound it. And a bound on the approximation error is of no value without a bound on the truncation error. The usual heuristic solution to this problem is to let the program determine when to truncate the series for ψ dynamically, on the basis of the size of the last term computed; when the last term is small relative to the current value of the series, the truncation error would be assumed negligible. (Our program allows the user to decide whether a fixed number of terms or the heuristic stopping criterion is to be used.)

In the following, we assume that $c > 0$, and we let T_n and A_n denote the truncation and function approximation errors involved in (2.8), respectively, so that

(5.4) $$T_n(H,H_o,\theta) \equiv \psi(H,\theta) - \psi_n(H,H_o,\theta)$$

(5.5) $$A_n(H,H_o,\theta) \equiv \psi_n(H,H_o,\theta) - \overline{\psi_n}(H,H_o,\theta)$$

where ψ_n denotes the series for $\overline{\psi_n}$ with the approximation symbols, $\overline{}$, removed.

<u>Lemma 5.1.</u> Let θ be fixed. Suppose there exist constants r_f , r_g , B_f and B_g for which

$$(5.6)\qquad |f^{(2j)}(\theta)| \le r_f^{2j} B_f \ , \quad |g^{(2j+1)}(\theta)| \le r_g^{2j+1} B_g$$

for $j \ge n+1$.

Let an upper bound function, U_n , be defined by

$$(5.7)\qquad U_n(h,x) \equiv B_h \frac{(r_h x)^n}{n!} \cosh r_h x$$

where h can be f or g . Then we have

$$(5.8)\qquad |T_n(H,H_o,\theta)| \le U_{2n+2}(f,c|H-H_o|) + \frac{1}{c} U_{2n+3}(g,c|H-H_o|)$$

for all $H,H_o \epsilon [\alpha,\beta]$.

<u>Proof:</u> By definition,

$$(5.9)\qquad |T_n(H,H_o,\theta)| = |\sum_{j=n+1}^{\infty} (-1)^j \{s_{2j}(H,H_o) f^{(2j)}(\theta) + s_{2j+1}(H,H_o) g^{(2j+1)}(\theta)\}|$$

$$(5.10)\qquad \le \sum_{j=n+1}^{\infty} \{B_f r_f^{2j} |s_{2j}(H,H_o)| + B_g r_g^{2j+1} |s_{2j+1}(H,H_o)|\} \ .$$

If we apply the bound,

$$(5.11)\qquad |s_m(H,H_o)| \le \frac{c^{2[\frac{m}{2}]} |H-H_o|^m}{m!} = S_m(H,H_o)$$

to (5.10), we obtain

$$(5.12)\qquad |T_n(H,H_o,\theta)| \le B_f \sum_{j=n+1}^{\infty} \frac{x_f^{2j}}{(2j)!} + \frac{B_g}{c} \sum_{j=n+1}^{\infty} \frac{x_g^{2j+1}}{(2j+1)!}$$

where $x_h = r_h c|H-H_o|$ for $h = f,g$. The two series in (5.12) are just the remainders of Maclaurin expansions of $\cosh x_f$ and $\sinh x_g$, truncated after 2n+2 and after 2n+3 terms, respectively; (5.8) is derived by substituting (an upper bound on) the Taylor form of the remainder. Q.E.D.

Let us define

(5.13) $E_m(H,H_o) \equiv s_m(H,H_o)-\bar{s}_m(H,H_o)$

(5.14) $D(H) \equiv \ell(H)-\bar{\ell}(H)$

(5.15) $\delta \equiv \max_{H\in[\alpha,\beta]} |D(H)|$.

To facilitate the following proofs, let us define regions I , II , and III in the H_1,H_2- plane, as pictured in Figure 5.4.

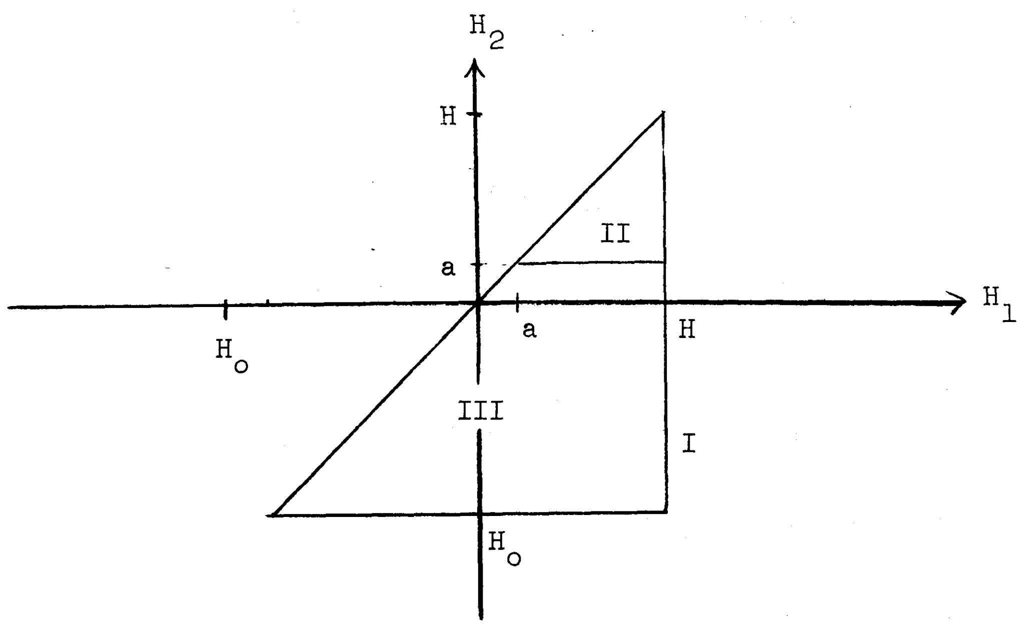


Figure 5.4

Region I is the union of II and III. We thus have

$$\iint_{II} F(H_1,H_2)dH_2dH_1 = \int_a^H \int_a^{H_1} F(H_1,H_2)dH_2dH_1$$

$$(5.16)\qquad \iint_{III} F(H_1,H_2)dH_2dH_1 = \int_{H_o}^{a} \int_{H_o}^{H_1} F(H_1,H_2)dH_2dH_1 + \int_a^H \int_{H_o}^{a} F(H_1,H_2)dH_2dH_1$$

$$\iint_{I} F(H_1,H_2)dH_2dH_1 = \iint_{II} F(H_1,H_2)dH_2dH_1 + \iint_{III} F(H_1,H_2)dH_2dH_1$$

<u>Lemma 5.2.</u> We have

$$(5.17)\qquad |E_m(H,H_o)| \le \frac{\delta}{c^2}\left[\frac{m}{2}\right]s_m(H,H_o)(1+\delta c^{-2})^{\frac{m}{2}} \quad \text{for } m \ge 0 .$$

<u>Proof:</u> The proof is by induction. $E_o = E_1 = 0$ and so (5.17) is true for m = 0,1. We assume it is true for m-2 and prove it for m .

$$(5.18)\qquad E_m(H,H_o) = \iint_I \{\ell(H_2)s_{m-2}(H_2,H_o)-\bar{\ell}(H_2)\bar{s}_{m-2}(H_2,H_o)\}dH_2dH_1 .$$

Adding and subtracting $\bar{\ell}s_{m-2}$ from the quantity in braces yields

$$(5.19)\ |E_m(H,H_o)| \le \left|\iint_I D(H_2)s_{m-2}(H_2,H_o)dH_2dH_1\right| + \left|\iint_I E_{m-2}(H_2,H_o)\bar{\ell}(H_2)dH_2dH_1\right|$$

$$(5.20)\qquad |E_m(H,H_o)| \le \frac{\delta c^{2[\frac{m}{2}]}|H-H_o|^m}{c^2\ m!} + (c^2+\delta)\left(\left[\frac{m}{2}\right]-1\right)\frac{\delta c^{2[\frac{m-2}{2}]}|H-H_o|^m}{c^2\ m!}(1+\delta c^{-2})^{\frac{m}{2}-1}$$

and (5.17) follows directly from this. Q.E.D.

Lemma 5.3. Let θ be fixed, and let constants C_f , D_f , C_g , D_g , c_f , c_g , d_f and d_g satisfy

$$(5.21) \qquad C_f c_f^{2j} \geq |f^{(2j)}| \quad , \quad C_g c_g^{2j+1} \geq |g^{(2j+1)}|$$

$$(5.22) \qquad D_f d_f^{2j} \geq |f^{(2j)} - \widetilde{f^{(2j)}}| \quad , \quad D_g d_b^{2j+1} \geq |g^{(2j+1)} - \widetilde{g^{(2j+1)}}|$$

for $j = 0,1,\ldots,n$.

Let us define bounding functions, F and G , by

$$(5.23) \qquad F(k,x,y) \equiv \frac{\delta}{2k}(C_f x \sinh x + D_f y \sinh y) + D_f \cosh y$$

$$(5.24) \qquad G(k,x,y) \equiv \frac{\delta}{2k}(C_g x(\cosh x - 1) + D_g y(\cosh y - 1)) + D_g \sinh y$$

Then we have, with $z = (1+\delta c^{-2})^{1/2}|H-H_o|c$,

$$(5.25) \qquad |A_n(H,H_o,\theta)| \leq F(c^2, c_f z, d_f z) + \frac{1}{c} G(c^2, c_g z, d_g z) \; .$$

This bound is independent of n .

Proof: By definition, we have

$$(5.26) \qquad |A_n(H,H_o,\theta)| = \left|\sum_{j=0}^{n}(-1)^j\{s_{2j}f^{(2j)} - \tilde{s}_{2j}\widetilde{f^{(2j)}} + s_{2j+1}g^{(2j+1)} - \tilde{s}_{2j+1}\widetilde{g^{(2j+1)}}\}\right|$$

Adding and subtracting $\tilde{s}_{2j}f^{(2j)}$ and $\tilde{s}_{2j+1}g^{(2j+1)}$, applying the triangle inequality and using the fact that $|\tilde{s}_m| \leq |E_m| + |s_m|$ yields

$$(5.27)\quad |A_n(H,H_o,\theta)| \le \sum_{j=0}^{\infty} \{(C_f c_f^{2j} + D_f d_f^{2j})|E_{2j}| + D_f d_f^{2j} |s_{2j}|$$

$$+ (C_g c_g^{2j+1} + D_g d_g^{2j+1})|E_{2j+1}| + D_g d_g^{2j+1} |s_{2j+1}|\}$$

Applying (5.11) and (5.17) to this yields

$$(5.28)\quad |A_n(H,H_o,\theta)| \le \frac{\delta}{2c^2}\left(C_f x_f \sum_{j=1}^{\infty} \frac{x_f^{2j-1}}{(2j-1)!} + D_f y_f \sum_{j=1}^{\infty} \frac{y_f^{2j-1}}{(2j-1)!}\right)$$

$$+ D_f \sum_{j=0}^{\infty} \frac{(y_f(1+\delta c^{-2})^{-\frac{1}{2}})^{2j}}{(2j)!}$$

$$+ \frac{\delta}{2c^2}\left(C_g \frac{x_g}{c} \sum_{j=1}^{\infty} \frac{x_g^{2j}}{(2j)!} + D_g \frac{y_g}{cc} \sum_{j=1}^{\infty} \frac{y_g^{2j}}{(2j)!}\right)$$

$$+ \frac{D_g}{c} \sum_{j=0}^{\infty} \frac{(y_g(1+\delta c^{-2})^{-\frac{1}{2}})^{2j+1}}{(2j+1)!}$$

where x_f , y_f , x_g , y_g are suitably defined; (5.25) follows directly from this.

Q.E.D.

The above bounds on T_n and A_n are reasonable as long as $[\alpha,\beta]$ is such that c remains small. But as $\beta \to .25125..$ we have $c \to \infty$. The reason our bounds can be bad is that the constant c multiplies the whole of $|H-H_o|$ in our bound of (2.9):

$$(5.29)\qquad |s_m(H,H_o)| \le \frac{(c|H-H_o|)^m}{m!} \delta_m^{-1}$$

When $H_o \ll a \ll H$ ("$\ll$" means "much less than"), then c is large, and so is $|H-H_o|$. It does not seem fair that, in this case, c should multiply all of $|H-H_o|$ since c is only needed to bound ℓ in $[a,H]$; a bound of unity suffices in $[H_o,a]$. Thus we may expect to be able to replace $c|H-H_o|$ by $c(H-a)+a-H_o$ in this case. Indeed, this can be done if the factor of 6," is removed, as can be proved from the following, stronger result.

<u>Lemma 5.4.</u> Let $h_o = H_o-a$ and $h = H-a$. We have, for $H_o \le a \le H$,

$$(5.30) \quad |s_m(H,H_o)| \le .5\left(1 + \frac{1}{c}\right) \frac{(ch-h_o)^m}{m!} + .5\left(1 - \frac{1}{c}\right) \frac{(-ch-h_o)^m}{m!} \equiv S_m^*(H,H_o)$$

with equality holding for $m = 0,1$. Further, this bound holds if a is replaced by any number between H_0 and a ; if a is replaced by H_o or $c = 1$, then (5.30) reduces to (5.29). Also, we have

$$(5.31) \quad S_m(H,H_o) > S_m^*(H,H_o) \qquad \text{for } H_o < a < H \text{ and } m \ge 2 .$$

<u>Proof</u>: The proof is by induction. Equality is achieved when $m = 0$ and 1 . Assuming (5.30) for m-2 , we prove it for m as follows:

$$(5.32) \quad |s_m| = \left|\iint_{I} \ell(H_2)s_{m-2}(H_2,H_o)dH_2dH_1\right|$$

$$\le \iint_{II} S_{m-2}(H_2,H_o)dH_2dH_1 + \iint_{III} c^2S^*_{m-2}(H_2,H_o)dH_2dH_1 \equiv X_m(H,H_o)$$

The first double integral requires $\ell s_{m\ 2}$ to be evaluated only for $H \le a$, and so (5.29) may be used with $c = 1$; c^2 times (5.30) was used for ℓs_{m-2} in the second integral. It follows that

$$(5.33) \qquad X_2(H,H_o) = \frac{h_o^2}{2} - h_o h + \frac{(ch)^2}{2} = S_2^*(H,H_o)$$

$$(5.34) \qquad X_m(H,H_o) = \frac{(-h_o)^m}{m!} + \frac{h(-h_o)^{m-1}}{(m-1)!}$$

$$+ .5(1 + \frac{1}{c})\{\frac{(ch-h_o)^m}{m!} - \frac{(-h_o)^m}{m!} - \frac{ch(-h_o)^{m-1}}{(m-1)!}\}$$

$$+ .5(1 - \frac{1}{c})\{\frac{(-ch-h_o)^m}{m!} - \frac{(-h_o)^m}{m!} - \frac{ch(-h_o)^{m-1}}{(m-1)!}\} = S_m^*(H,H_o) .$$

The inequality $S_m > S_m^*$ can be proved by expanding $(ch-ch_o)^m$ and $(ch \pm h_o)^m$.

Q.E.D.

The case $H \leq a \leq H_o$ can be dealt with in a similar manner, but this will not be pursued here. The bound on T_n corresponding to this new bound is

$$(5.35) \quad |T_n(H,H_o,\theta)| \leq .5(1 + \frac{1}{c})\{U_{2n+2}(f,ch-h_o)+U_{2n+3}(g,ch-h_o)\}$$

$$+ .5(1 - \frac{1}{c})\{U_{2n+2}(f,-ch-h_o)+U_{2n+3}(g,-ch-h_o)\}$$

$$\text{for } H_o - a = h_o \leq 0 \leq h = H-a .$$

To get a new bound on E_m and A_n we prove the following generalization of (5.17).

Lemma 5.5. If $E_m(H,H_o)$ is defined as in (5.13) then

$$(5.36) \quad |E_m(H,H_o)| \leq \frac{\delta}{c^2} (1+\delta)^{\frac{m}{2}}\{[\frac{m}{2}]S_m^*(H,H_o) = \frac{h_o(c^2-1)}{2^{\sigma(m)}} (S_{m-1}^*(H,H_o) - \frac{(ch)^{m-1}}{(m-1)!} \sigma(m))\}$$

$$\text{for } m \geq 0 \text{ and } H_o \leq a < H$$

where $S^*_{-1} \equiv 0$ and $\sigma(m) = 0$ if m is even, and = 1 if m is odd. Further, this holds if a is replaced by any number in $[H_o,a]$; if a is replaced by H_o and $(1+\delta)^{m/2}$ by $(1+\delta c^{-2})^{m/2}$, or if $c = 1$, then this reduces to (5.17).

Proof: Again, the proof is by induction; (5.36) holds for m = 0,1 . We assume it for m-2 , and work on the two terms on the right side of (5.19):

$$(5.37)\quad \left|\iint_I D(H_2)s_{m-2}(H_2,H_o)dH_2dH_1\right|$$

$$\leq \delta\{ \iint_{II} S_{m-2}(H_2,H_o)dH_2dH_1 + \iint_{III} S^*_{m-2}(H_2,H_o)dH_2dH_1\}$$

$$(5.38)\quad \left|\iint_I \ell(H_2)E_{m-2}(H_2,H_o)dH_2dH_1\right|$$

$$\leq (1+\delta)\{ \iint_{II} |E_{m-2}(H_2,H_o)|dH_2dH_1 + \iint_{III} c^2|E_{m-2}(H_2,H_o)|dH_2dH_1\}$$

$$(5.39)\quad \leq \delta(1+\delta)^{\frac{m}{2}}\{([\tfrac{m}{2}]-1)\{ \iint_{II} S_{m-2}(H_2,H_o)dH_2dH_1 + \iint_{III} S^*_{m-2}(H_2,H_o)dH_2dH_1\}$$

$$- \frac{h_o(c^2-1)}{2^{\sigma(m)}} \iint_{III} (S^*_{m-3}(H_2,H_o) - \frac{(ch_2)^{m-3}}{(m-3)!}\sigma(m))dH_2dH_1\}$$

where $h_2 = H_2 - a$.

Multiplying the right side of (5.37) by $(1+\delta)^{m/2}$, adding the result to (5.39) and simplifying yields

$$(5.40)\quad |E_m(H,H_o)| \leq \frac{\delta}{c^2}(1+\delta)^{\frac{m}{2}}\{ [\tfrac{m}{2}] \{S_m^*(H,H_o)+(c^2-1)(\frac{h(-h_o)^{m-1}}{(m-1)!} + \frac{(-h_o)^m}{m!})\}$$

$$- \frac{h_o(c^2-1)}{2^{\sigma(m)}} \{S_{m-1}^*(H,H_o) - \frac{h(-h_o)^{m-2}}{(m-2)!} - \frac{(-h_o)^{m-1}}{(m-1)!} - \sigma(m)\,\frac{(ch)^{m-1}}{(m-1)!}\}\}$$

$$(5.41)\qquad |E_m(H,H_o)| \leq$$

$$\frac{\delta}{c^2}\,(1+\delta)^{\frac{m}{2}}\{[\tfrac{m}{2}]S_m^*(H,H_o) - \frac{h_o(c^2-1)}{2^{\sigma(m)}}\,(S_{m-1}^*(H,H_o) - \frac{(ch)^{m-1}}{(m-1)!}\,\sigma(m))$$

$$- \frac{(c^2-1)}{2^{\sigma(m)}}\,\{\frac{h(-h_o)^{m-1}}{(m-2)!} + \frac{(-h_o)^m}{(m-1)!} - \frac{2^{\sigma(m)}[\frac{m}{2}]}{m-1}\,\frac{h(-h_o)^{m-1}}{(m-2)!} - \frac{2^{\sigma(m)}[\frac{m}{2}]}{m}\,\frac{(-h_o)^m}{(m-1)!}\}\}\,.$$

Since $[\frac{m}{2}] < m-1 < m$ for $m > 2$, we see that the last quantity in braces is > 0, and so we may replace it by zero without disturbing our inequality. The result is just (5.36). Q.E.D.

Various weaker, but simpler, bounds can be proved, two of the simplest (and weakest) being $\delta(1+\delta)^{\frac{m}{2}}[\frac{m}{2}]S_m^*(H, H_o)$ and $\delta[\frac{m}{2}]\,\frac{((ch-h_o)\sqrt{1+\delta})^m}{m!}$. The new bound on E_m provides the following bound on A_n: let bounding functions F_i and G_i be defined by

$$(5.42)\quad F_1(k,x,y) \equiv (1 + \tfrac{1}{c})(x+b(x+y))k \sinh kx+(1 - \tfrac{1}{c})(y+b(x+y))k \sinh ky$$

$$(5.43)\quad F_2(x,y) \equiv \frac{\delta}{4c^2}\{C_f F_1(c_f,x,y)+D_f F_1(d_f,x,y)\}$$

$$+\frac{D_f}{2}\{(1+\frac{1}{c})\cosh d_f x + (1-\frac{1}{c})\cosh d_f y\}$$

$$(5.44)\quad G_1(k,x,y) \equiv (1+\frac{1}{c})(x+\frac{b}{2}(x+y))k(\cosh kx-1)$$

$$+(1-\frac{1}{c})(y+\frac{b}{2}(x+y))k(\cosh ky-1)-b(x+y)(\cosh(k(x-y)/2)-1)$$

$$(5.45)\quad G_2(x,y) \equiv \frac{\delta}{4c^2}\{C_g G_1(c_g,x,y)+D_g G_1(d_g,x,y)\}$$

$$+\frac{D_g}{2}\{(1+\frac{1}{c})\sinh d_g x +(1-\frac{1}{c})\sinh d_g y\}$$

-where $b = c^2-1$. Then it follows from (5.27) that

$$(5.46)\quad |A_n(H,H_o,\theta)| \leq F_2(x,y)+G_2(x,y)$$

where

$$(5.47)\quad x = (ch-h_o)\sqrt{1+\delta} \text{ and } y = (-ch-h_0)\sqrt{1+\delta} .$$

Our new bounds, (5.35) and (5.46), reduce to the old bounds when either $c = 1$ or a is replaced by H_o , $(1+\delta)^{m/2}$ by $(1+\delta c^{-2})^{m/2}$ and, if $H_o > H$, then H and H_o are interchanged. For this reason, our program for calculating these bounds is written only for (5.35) and (5.46); for the case $H \leq .05$, the old bounds are derived by the replacement just described. For the Ringleb computation, all growth constants are 1 , and

$$C_f = B_f = |2.538 \sin(1)/v(H_o)| \tag{5.48}$$

$$C_g = B_g = \left| \frac{2.538 \sin(1)}{v(H_o)(1-.2v^2(H_o))^{2.5}} \right| \tag{5.49}$$

$$D_h = 10^{-9} B_h \qquad \text{for } h = f,g \tag{5.50}$$

$$\delta = 4.10533 \times 10^{-5} . \tag{5.51}$$

The bounds

$$|s_m(H,H_o)| \le \frac{(ch-h_o)^m}{m!} \qquad \text{for } H_o \le a \le H \tag{5.52}$$

$$|E_m(H,H_o)| \le \delta[\tfrac{m}{2}] \frac{((ch-h_o)\sqrt{1+\delta})^m}{m!} \qquad \text{for } H_o \le a < H \tag{5.53}$$

can be used to derive simpler bounds on A_n and T_n :

$$|A_n(H,H_o)| \le F(1,c_f z,d_f z)+G(1,c_g z,d_g z) \tag{5.54}$$

$$|T_n(H,H_o,\theta)| \le U_{2n+2}(f,ch-h_o)+U_{2n+3}(g,ch-h_o) \tag{5.55}$$

where $z = (ch-h_o)\sqrt{1+\delta}$ and F and G are given by (5.23) and (5.24). As $ch-h_o$ increases and H_o decreases, these bounds increase. Thus they attain their maxima when $H = \beta$ and $H_o = \alpha$. For the Ringleb computation described above, this implies

[redacted] for [redacted] and [redacted] , the bound being calculated at $H = .22$ and $H_o = -1$. The disadvantage of these simpler bounds is that, when a is replaced by H_o , they do not reduce to our old bounds; a factor of c^2 is lost. Thus, as $H_o \to a$ from below, while $H > a$, these bounds will become several orders of magnitude worse than our more complex bounds. (If β were closer to .25125..., then c^2 would be even larger, and this loss would be more drastic.)

APPENDIX

Three programs, written in B5500 Extended Algol, are discussed and listed in this section. The first program calculates the coefficients of the expansion of $\ell(H)$ about its singularity. Double-precision (about 20 digits accuracy) was required to calculate the first 43 coefficients. (This is the only place in these programs in which double-precision was used.) The coefficients generated in this way could be used to obtain a more accurate approximation to $\psi(H,\theta)$, valid over a wider interval of H values, than that given by the 8 term Chebyshev approximation to $\ell(H)$ used in the third program discussed here. The second program includes procedures capable of computing the error bounds derived in Section 5. A driver program uses these procedures to calculate the error bounds for our approximation in the case of the Ringleb solution. The output of this program was used to prepare the graphs in Section 5. The third program calculates our approximation to $\psi(H,\theta)$. Given H_o , it uses a truncated expansion of 1(H) to generate coefficients for polynomial-like approximations to the $s_m(H,H_o)$. These are used by the procedure STRFNC to evaluate $\tilde{\psi}(H,\theta)$, $\tilde{\psi}_H(H,\theta)$ and $\tilde{\psi}_\theta(H,\theta)$, for given H and θ . STRFNC calls upon the user-supplied procedure FANDG to obtain values of the initial value functions $f(\theta)$ and $g^{(1)}(\theta)$, and their derivatives. The driver program given here is set up to form our approximation to the Ringleb solution, and to tabulate tables of the actual error in this approximation. These data were also used in the preparation of the graphs in Section 5.

We have an explicit representation for H as a function of v :

$$\text{(A.1)} \quad H(v) = .251251\ldots + \sqrt{\tau}(\tau^2/5 + \tau/3 + 1) - \log\left(\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\right)$$

where $\tau \equiv 1-.2v^2$.

In these programs, v(H) was found by Newton-Raphson iteration, using (A.1). The procedure SPEED does just this. However, if the values of of the $s_m(H,H_o)$ and of $v(H_o)$ are available, then v(H) can be computed more efficiently by using the relation

$$\text{(A.2)} \quad v(H) = \frac{v(H_o)}{\sum_{j=0}^{\infty} \{s_{2j}(H,H_o) - V s_{2j+1}(H,H_o)\}}$$

where $V \equiv (1-.2v^2(H_o))^{-2.5}$.

Equation (A.2) can be derived most easily by equating the Ringleb solution, $\psi^R(H,\theta) = \frac{\sin\theta}{v(H)}$, to the solution, as given by (2.7), of the initial value problem, $f(\theta) = \frac{\sin\theta}{v(H_o)}$ and $g^{(1)}(\theta) = -\frac{\sin\theta}{v(H_o)} V$. When given an interval, I , of H values in which (A.2) is to be used, we can use the bounds on $|s_j(H,H_o)|$ given in Section 5, along with the fact that the denominator in (A.2) has values ranging between $\min\limits_{H,H_o \in I} \frac{v(H_o)}{v(H)}$ and $\max\limits_{H,H_o \in I} \frac{v(H_o)}{v(H)}$, to decide how many terms are needed for the denominator sum in order to make the truncation error less than the approximation error caused by using $\tilde{s}_j(H,H_o)$.

```
COMMENT      THE FOLLOWING THREE PROCEDURES SHOULD BE CONSIDERED
             GLOBAL TO THE FOLLOWING THREE PROGRAMS (THEY MAY BE INSERT-
             ED AFTER THE FIRST BEGIN OF EACH PROGRAM);

REAL PROCEDURE SPEED(X);    VALUE X;    REAL X;
BEGIN     REAL C, V;
   REAL PROCEDURE H(V);   VALUE V;   REAL V;
    BEGIN     REAL TAU,SQTAU;     DEFINE CONST=0.2512511361#;
     COMMENT     CONST CAN BE EVALUATED BY THE FOLLOWING TWO STATEMENTS,
         APPEARING IN THE MAIN PROGRAM:  CONST←0;
         COMMENT                          CONST←-H(SQRT(5/6));
      TAU←1-.2×V*2;     SQTAU← SQRT(TAU);
      H← SQTAU×(TAU*2/5+TAU/3+1) - .5×LN((1+SQTAU)/(1-SQTAU)) + CONST
    END H;
   V← IF X<0 THEN .4 ELSE 1.2;
WHILE ABS(C←H(V)-X)>@-9 DO  V← V - C×V/(1-.2×V*2)*2.5;
   SPEED ← ABS(V)
END SPEED;

REAL PROCEDURE MAX(X,Y);   REAL X,Y;
 MAX←IF X<Y THEN Y ELSE X;

REAL PROCEDURE MIN(X,Y);   REAL X,Y;
 MIN←IF X<Y THEN X ELSE Y;



COMMENT      THE FOLLOWING PROGRAM CALCULATES THE COEFFICIENTS FOR AN
             EXPANSION OF L(H) ABOUT ITS SINGULARITY AT .2512511361;

BEGIN
DEFINE  N=60 #, CONST=.2512511361 #;
ARRAY  A,A7,A7L[0:N], B,BL[0:N,0:N], CK,CKL[0:13];
INTEGER  MG,J,K,M,Q,Q6,Q7;
REAL  SUM, SUML, HH, V, TAU, L, LL;
 FILE OUT CARDS 0 (2,10);

COMMENT     N+1  COEFFICIENTS ARE TO BE COMPUTED (N MUST BE > 13),
            A[] IS WHERE THESE COEFFICIENTS WILL BE STORED.
            A7[M] = A[M]×7*(2×(M-6)/7),
            MG+1 TERMS WILL BE USED TO EVALUATE THE APPROXIMATION;

REAL PROCEDURE LH(H);   VALUE H;    REAL H;
BEGIN   REAL  SUM;    INTEGER  M;
     COMMENT   THIS EVALUATES THE TRUNCATED EXPANSION FOR L(H);
      SUM←0;
      FOR M←0 STEP 1 UNTIL MG DO SUM←SUM + A[M]×(CONST-H)*(2×(M-6)/7);
      LH←SUM;
   END;
```

```
COMMENT   WE  CALCULATE  A713  FIRST.  THIS IS DONE BYSERIES  REVERSION  ,
          USING THE  RELATIONS
             L(H)=  (6×TAU - 5 )/TAU*6
                 = A7[0]×X*(-12/7) + A7[1]×X*(-10/7) + . , .
             X  =  7×(CONST-H) = TAU*(7/2) + 7/9×TAU*(9/2) +
                                              7/11×TAU*(11/2) +...          .
          HIGHPRECISION  IS  NEEDED  FOR  THE  COMPUTATION  OF  THESE
          COEFFICIENTS, BECAUSE  THE  A7[M]  BECOME  SMALL  QUICKLY,  AND  MUCH
          CANCELLATION OCCURS;

FUR  M←0 STEP 1 UNTIL  N DO   B[M,0]←1;

FUR  K←1 STEP 1 UNTIL  N  DO
   BEGIN
     SUM←SUML←0;
    COMMENT  THEFOLLOWING DOUBLE  LOOP IS  EQUIVALENT  TO
     FOR  J←0  STEP  1 UYTIL  K  DO  SUM←SUM +49/((2×J+7)×(2×(K-J)+7));

     FOR  J←0  STEP  1 UNTIL K  00
        DOUBLE(49,0,  J,0,  J,0,  +,  7,0,  +,  K,0, J,0, -, 2,0, ×, 7,0,
           +,×,/,  SUM,SUML,  +,←,  SUM,SUML);
     DOUBLE( SUM, SUML,  ←,  B[13,K],BL[13,K]);
   END;

FOR K←1 STEP 1  UNTIL  N  DO
   BEGIN
    COMMENT      CALCULATE THE C[Q,K ]/S;
     FOR Q←8,9,10,13 DO
        BEGIN
           SUM←SUML←0;   Q6←(Q+6) QIV 2;    Q7←(Q+7) DIV 2;
           FOR J←1  STEP  1 UNTIL K-1 DO
              DOUBLE(B[Q6,J],BL[Q6,J], B[Q7,K-J],BL[Q7,K-J], ×,
                 SUM,SUML, +, ←, SUM,SUML);
           DOUBLE(SUM,SUML, ←,CK[Q],CKL[Q]);
        END;
    COMMENT      THE FOLLOWING DOUBLE  INSTRUCTION  IS EQUIVALENT  TO
     B[7,K]← (B[13,K] -CK[13]- CK[10] - CK[9] - 3×CK[8])/7;

     DOUBLE(B[13,K],BL[13,K],  CK[13],CKL[13], -, CK[10],CKL[10], -,
        CK[9],CKL[9], -, 3,0, CK[8],CKL[8], ×, -, 7,0, /,
        ←, B[7,K],BL[7,K]);

     FUR Q←8,9,10,13 DO
        BEGIN
           Q6←(Q+6) DIV 2 1    Q7←(Q+7) DIV 2;
          COMMENT  THE FOLLOWING DOUBLE  INSTRUCTION  IS
           B[Q,K]←CK[Q] + B[Q6,K]+B[Q7,K];

           DOUBLE(CK[Q],CKL[Q], B[Q6,K],BL[Q6,K], +, B[Q7,K],BL[Q7,K],
              +, ←, B[Q,K],BL[Q,K] );
        END;
   END;

COMMENT      CALCULATE  B[11,12,14,15,...,N;1,2,3,...,N];
```

```
FUR  M←11,12,14 STEP 1 UNTIL  N DO
   FOR K←1 STEP 1 UNTIL  N  DO
      BEGIN
         SUM←SUML←0;
         FUR J←0  STEP  1  UNTIL  K  DO
            DOUBLE(B[7,J],BL[7,J], B[M-1,K-J],BL[M-1,K-J], ×, SUM,SUML,
               +, ←, SUM,SUML);
         DOUBLE(SUM,SUML, ←, B[M,K],BL[M,K]);
      END;

CUMMENT      CALCULATE     B[0,1,2,3,4,5: 1,2,3,...,N];

FOR J←1 STEP 1  UNTIL  6  DO FOR K←1  STEP  1  UNTIL  N  DO
   BEGIN
      SUM←0;    Q6←6-J;    Q7←6+J;    SUML←0;
      F O R M←0 STEP 1 UNTIL K-1 DO
         DOUBLE(B[Q6,M],BL[Q6,M], B[Q7,K-M],BL[Q7,K-M], ×, SUM,SUML, +,
            ←, SUM,SUML);
      DOUBLE(-SUM,-SUML, ←, B[Q6,K],BL[Q6,K]);
   END;

COMMENT      B[M,K]    CALCULATIONS  ARE  NOW  DONE;

A7[0]←-5;    A[0]←-5×7*(-12/7);
DOUBLE(6,0, A7[0],A7L[0], B[0,1],BL[0,1], ×, -, ←, A7[1],A7L[1]);
A[1]←A7[1]×7*(-10/7);

FUR  M←2 STEP 1  UNTIL  N DO
   BEGIN
      SUM←SUML←0;
      FOR  J←0  STEP  1 UNTIL  M-1 DO
         DUUBLE(A7[J],A7L[J], B[J,M-J],BL[J,M-J], ×, SUM,SUML, +,
            ←, SUM,SUML);
      DUUBLE(-SUM,-SUML, ←, A7[M],A7L[M]);
      A[M]←-SUM×7*(2×(M-6)/7);
   ENOJ


WRITE(CARUS,<3E20.11>,FOR M←0 STEP 1 UNTIL N    DO A[M]);
WRITE(<"M", X19, "A[M]", X8, "A[M]/7*(2×(M-6)/7)">);
FOR M←0 STEP 1 UNTIL  N  DO WRITE(<I2, 2E25.11>, M, A[M], A7[M]);
WRITE([PAGE]);          MG←42;
WRITE(<"MG= ", I2///>, MG);
WRITE(<X3, "H",X13,  "V",  X250  "L(H)",  X20,  "*L(H)*",  X9,  "RESIDUAL">);
FOR  HH←-1 STEP  .01 UNTIL .2501 00
   WRITE(<F5.2, F20.11, 2R25.11,E15.2>,    HH, (V←SPEED(HH)),
            (L←(6×(TAU←(1-.2×V×V))-5)/TAU*6), (LL←LH(HH)), L-LL);
END.
```

```
COMMENT     THE FOLLOWING PROGRAM IS SET UP TO EVALUATE BOUNDS ON THE
            TRUNCATION AND APPROXIMATION ERROR FOR THE RINGLEB SOLUTION.
            HOWEVER, THE PROCEDURES NEEDED ARE PROGRAMMED IN GENERAL;

BEGIN    REAL  C, B, C2, DELTA, DELTA1, A, AA;
ARRAY RH, BH, CH, Dh, LCH, LDH[0:1];

COMMENT  C2 = MAX(ABS(L(X))   FOR X RETWEEN H AND HO,,
          C = SQRT(C2),
          A = INVERSE[L(-1)] = .0659262218.
          RH, LCH, LDH  AREGROWTH FACTORS,
         A PROCEDURE TO EVALUATE L(H) MUST BE PROVIDED.
         THE FOLLOWING PRUCEEDURES ARE ALL THAT IS NEEDED TO EVALUATE
         THEROUGH OR THE IMPRCVED ROUNDS,  THE BOUND IS GIVEN BY
                      T(N, HH, HHO) + AN(HH, HHO)     .
         IF THE ROUGH BOUND IS DESIRED, WE MUST HAVE
            HH = MAX(H,HO)      HHO= MIN(H,HO)     AA= HHO
            DELTA1= 1+DELTA/C2.
         IF THE IMPROVED HOUND IS DESIREd, THEN WE MUST HAVE
            HHO=HO≤A≤H=HH      DELTA1=SQRT(1+DELTA)      B=(C2-1 )×DELTA1
            AA=A;

REAL PROCEDURE SINH(X);    REAL X;    SINH←.5×(EXP(X)-EXP(-X));
REAL PROCEDURE COSH(X);    REAL X;    COSH←.5×(EXP(X)+EXP(-X));

REAL  PROCEDURE U(N,H,X);    INTEGER N,H;    REAL X;
IF X=0 THEN U←0 ELSE
BEGIN   REAL SUM;    INTEGER Ii
   SUM←0;    FUR I←2 STEP 1 UNTIL  N  DOSUM←SUM+LN(I);
   SUM←N×LN(RH[H]×ABS(X)) - SUM;
   U←BH[H]×EXP(SUM)×COSH(RH[H]×X)×SIGN(X)*(N-2×(N DIV 2));
END U;

REAL  PROCEDURE T(N,H,HO);    INTEGER N;    REAL H,HO;
 BEGIN    REAL X,Y;
   X←( C×(H-AA) - (HO-AA))×DELTA1;
   Y←(-C×(H-AA) - (HO-AA))×DELTA1;
   T←.5×((1+1/C)×(U(2×N+2,0,X)+ U(2×N+3,1,X)) +(1-1/C)×(U(2×N+2,0,Y)
          +U(2×N+3,1,Y)));
 END T;

REAL PROCEDURE F 1(K,  X, Y);    REAL-  K, X, Y;
 BEGIN    REAL BB;
   BB←  B×(X+Y);
   F1←(1+1/C)×(X+BB)×K×SINH(K×X) + (1-1/C)×(Y+BB)×K×SINH(K×Y);
 END F1;

REAL  PROCEDURE F2(X,Y);    HEAL  X, Y;
 F2←DELTA/(4×C2)× (CH[0]×F1(LCH[0],X,Y) + DH[0]×F1(LDH[0],X,Y))
   + DH[0]/2 x ((1+1/C)×COSH(LDH[0]×X) + (1-1/C)×COSH(LDH[0]×Y));

·REAL  PROCEDURE G1(K, X, Y);    REAL  K, X, Y;
 BEGIN    REAL BB;
   BB←. B×(X+Y)/2;
   G1←(1+1/C)×(X+BB)×K×(COSH(K×X)-1) +  (1-l/C)x(ytBB)xKx(COSH(KxY)-1)
```

```
          -BB×(COSH(K×(X-Y)/2)-1);
 END G1;

REAL PROCEDURE G2(X,Y);    REAL  X,Y;
 G2←DELTA/(4×C2)× (CH[1]×G1(LCH[1],X,Y) + DH[1]×G1(LDH[1],X,Y))
   + DH[1]/2 × ((1+1/C)×SINH(LDH[1]×X) + (1-1/C)×SINH(LDH[1]×Y)); ,

REAL PROCEDURE AN(H, HO);   REAL H,HO;
 BEGIN    REAL X,Y;
   X←( C×(H-AA) - (HO-AA))×DELTA1;
   Y←(-C×(H-AA) - (HO-AA))×DELTA1;
   AN← F2(X,Y) t G2(X,Y);
 END AN;

COMMENT  IN WHAT FOLLOWS, THESE PROCEDURES ARE APPLIED TO OUR LCH) AND
         THE RINGLEB SOLUTION;

REAL  THT, H, HO, HH, HHO, ERRCR, AVGEPS, VO, TN, ANN, ERNEW, EROLD;
INTEGER Ii

REAL PROCEDURE L(X);   VALUE X;    REAL XI
BEGIN       REAL V, TAU;
   V←SPEED(X);      TAU←1-.2×V*2;     L← (6×TAU-5)/TAU*6
END L ;

A←0.0659262218;   RH[0]←RH[1]←LCH[0]←LCH[1]←LDH[0]←LDH[1]←1;
THT←1;   DELTA←4.10533@-5;
 FORHO←-1.0 STEP .05 UNTIL .05 00
 BEGIN
   VO← SPEED(HO);
   BH[0]←CH[0]←ABS(2.538/VO × SIN(THT));
   BH[1]←CH[1]←ABS(2.538×(1-.2×VO*2)*(-2.5) /VO × SIN(THT));
   FOR I←0,1 DO DH[I]←BH[I]×@-8;
   WRITE(<". H O       H           BOUND                TN'      ",
       "            AN",/>);
   AVGEPS←0;
   FOR H←-1 STEP .05 UNTIL .2, .22 DO
    BEGIN
      C2←MAX(@-8, MAX(ABS(L(H)), ABS(L(HO))));
      HH←MAX(H, HO);   HHO←MIN(H,HO);
      DELTA1← SQRT(1+DELTA/(IF H>.05 THEN 1 ELSE C2));
      B← (C2-1);                     C← SQRT(C2);
      AA←IF H>.05 THEN A ELSE HHO;
      TN←T(7, HH, HHO);   ANN←AN(HH, HHO);   ERNEW←ERROR←TN + ANN;
      If H>.05 THEN
       BEGIN
         AA+ HHO;   DELTA1← SQRT(1+DELTA/C2);
         EROLD←T(7, HH, HHO) + AN(HH,HHO);
         ERROR←MIN(ERNEW, EROLD);    TN←ERNEW;    ANN←EROLD;
       END;
      AVGEPS←AVGEPS + ERROR;
      WRITE(<2(F6.2,X4), X5, 3(E12.5, X8)>, HO, H, ERROR, TN, ANN);
    END;
   WRITE(<"AVG ERR = ", E12.5>, AVGEPS/26);      WRITE([PAGE]);
END   END.
```

```
COMMENT     THE FOLLOWING PROGRAM IS SET UP TO FORM AND EVALUATE OUR
            APPROXIMATIUN TO PSI FOR THE RINGLEB SOLUTION, ANO TO MAKE A
            TABLE OF THE OBSERVED ERRORS IN THIS APPROXIMATION1

BEGIN   REAL HO, C, SUM, KM1, CMIH057, OLDH, OLDTHT, CF, CG;
   INTEGER M, M2, M2N7, MN7MI2, MMAX, N, N7, NN12, J, K, UP, MN7, IP,
                  NPSITRUNCMAX;          LABEL EXIT;

  COMMENT     NPSITRUNCMAX+1 IS THE MAXIMUM NUMBER OF TERMS WHICH KILL
                             BE USED IN OUR TRUNCATED SERIES FOR PSI
                             (SEE COMMENTS IN THE PROCEDURE STRFNC).
              N7 = N + 7    IS THE NUMBER OF TERMS To BE USED TO APPROX-
                            IMATE L(H).
              A[]           CONTAINS THE N7 COEFFICIENTS FOR THIS APPROX;

   NPSITRUNCMAX←20;      MMAX←2×NPSITRUNCMAX+1;
   N← 1;      N7←N+7;      NN12←N+N+12;
   BEGIN
      ARRAY  SCOEF,SPRIME[0:MMAX, 0:MMAX×N7], A[0:N+6];

REAL  VO, AIVO;
PROCEDURE FANDG(FVAL, GVAL, THT, OLDM, M);     VALUE M, OLDM, THT;
 INTEGER M, OLDM;      REAL THT;       ARRAY FVAL, GVAL[0];
 BEGIN      HEAL  SN, CS, X, Y, Z;       INTEGER IP;
  COMMENT      THIS PROCEDURE IS TO BE SUPPLIED BY THE USER.   IT IS TO
            CALCULATE THE INITIAL VALUES, F(T)=PSI(HO,T)  AND
            G1(T)=D(PSI(HO,T))/DH,   AND THEIR DERIVATIVES  AT  T=THT.
            FROM THE OLDM/TH AND UP TO THE M/TH DERIVATIVE OF F AND G ARE
            TO BE CALCULATEO AND STORED IN FVAL,GVAL[OLDM,...,M], WHERE
            G1= D(G)/DT .    IF OLDM>0 THEN THE 0/TH,1/TH,...,OLDM-1/TH
            DERIVATIVES WILL BE IN FVAL,GVAL[0,1,...,OLDM-1].
            WHEN OLDM=0, M WILL BE ≥ 2  ( THIS FACT IS EXPLOITED IN THE
            SAMPLE PROCEDURE GIVEN HERE);

  If OLDM=0  THEN
    BEGIN
      SN← SIN(THT);      CS← COS(THT);     X← 2.538/VO;
      FVAL[0]← Y← X×SN;      FVAL[1]← Z← X×CS;
      GVAL[0]← Z×ATVO;      GVAL[1]← -Y×ATVO;
    END;
  FOR IP←MAX(OLDM,2) STEP 1 UNTIL M DO
    BEGIN
      FVAL[IP]←-FVAL[IP-2];
      GVAL[IP]←-GVAL[IP-2]
    END;
END  FANDGJ


REAL  PROCEDURE SMVAL(H, SM, M, FUJ);     VALUE  H,M, FUJ;
   HEAL  Hi       INTEGER  M, FUJI      ARRAY  SM[0];
 BEGIN      HEAL  HORNER, CMIH;      INTEGER R, T, Jr K;
  COMMENT      LET   T=M×N7-FUJ.     THEN THIS PROCEDURE EVALUATES
               SMVAL = SM[0] t SM[1]×(C-H)*(2/7) +...+ SM[T]×(C-H)*(2×T/7);

   T←M×N7-FUJ;      R← T MOD 71
```

```
      SUM←0;    CMIH←C-H;     K←T-6;       IF K<0 THEN K←0;
      FOR T←T STEP -1 UNTIL K DO
         BEGIN
            HORNER←SM[T];
            FOR J←T-7 STEP -7 UNTIL R DO    HORNER←HORNER×CMIH + SM[J];
            SUM←SUM+HORNER×CMIH*(R/7);
            R←R-1;     IF R<0 THEN R←6
         END EVALUEATION OF SM;
      SMVAL←SUM
   END OF SMVAL;


   PROCEDURE DIFFSM(SM, DEGSM, SMPRIME);     VALUE DEGSM;
     , INTEGER DEGSM;     ARRAY SM, SMPRIME[0];
      FOR IP←DEGSM STEP -1 UNTIL 2 DO SMPRIME[IP-2]←-(IP/7)×SM[IP];


   PROCEDURE STRFNC(PSI, H, THT, DPDT, DPDH, MUP, EPS, TOOBIG);
    VALUE H, THT, MUP, EPS;    REAL PSI, H, THT, DPDT, DPDH, EPS;
    INTEGER MUP;    LABEL TOOBIG;
   BEGIN      OWN REAL TEMP;    OWN INTEGER OLDM, OLDMH, OLDMT, MP1, MP2;
      OWN REAL ARRAY S, DS, FVAL, GVAL[0:MMAX+1];
      INTEGER MUP1, MUP2, M;    REAL LASTERM;

     COMMENT     VALUES ARE RETURNED IN PSI, DPDT, AND DPDH.
          IF MUP≥0   THEN MUP+1 TERMS ARE USED TO EVALUATE OUR APPROXIMATE
            PSI.IF THE LAST TERM IS >EPS×ABS(PSI) THEN AN ERROR RETURN TO
            TOOBIG IS EXECUTED. ALL INTERMEDIATE RESULTS ARE SAVED, AND
            ANOTHER CALL, WITH MUP INCREASED, WILL CONTINUE THE
            COMPUTATION.
          IF MUP=-1   THEN TERMS ARE ADDED IN TO PSI UNTIL THE LAST TERM
            IS ≤ EPS×ABS(PSI).   IF THIS HAS NOT HAPPENED AFTER
            NPSITRUNCMAX+1 TERMS HAVE BEEN ADDED IN, THEN AN ERROR RETURN
            TO TOOBIG IS EXECUTED.  NO RECOVERY IS POSSIBLE, SINCE THE
            REQUIRED COEFFICIENTS FOR SM ARE NOT AVAILABLE.   THE ENTIRE
            RUN MUST BE REDONE, WITH A LARGER NPSITRUNCMAX;

      LASTERM←@20;
      IF H≠OLDH  THEN
       BEGIN
         TEMP←(C-H)*(-.71428571428571);          COMMENT I.E.,  *(-5/7);
         OLDMH←OLDM←0
       END;
      IF THT≠OLDTHT THEN OLDMT←OLDM←0;
      IF OLDM=0  THEN  PSI← DPDT← DPDH← 0;
      IF MUP≥0 THEN
       BEGIN
         MUP1←2×MUP+1;     MUP2←MUP1+1;
         IF MP1≥OLDMH  THEN
          BEGIN
            FOR M←OLDMH  STEP 1 UNTIL MUP1 DO
             BEGIN
              S[M]← SMVAL(H, SCOEF[M,*], M, 0);
              DS[M]←SMVAL(H,SPRIME[M,*], M, 2)  x  TEMP
             END;
```

```
          ULDMH←MP2
        END;
       IF MP2≥ULDMT THEN
        BEGIN
          FANDG(FVAL, GVAL, THT, OLDMT, MP2);        OLDMT←MP2+1
        END;
       WHILE MUP1≥OLDM DO
        BEGIN
          wit OLDM+1;    MP2←OLDM+2;
          LASTERM← S[OLDM]×FVAL[OLDM] + S[MP1]×GVAL[MP1];
          PSI← LASTEHM - PSI;
          DPDH← DS[OLDM]×FVAL[OLDM] + DS[MP1]×GVAL[MP1]  - DPDH;
          DPDT←  S[OLDM]×FVAL[MP1 ] +  S[MP1]×GVAL[MP2]  - DPDT;
          OLDM← MP2
        END;
       I F ABS(LASTERM)>EPS×ABS(PSI) THEN GO TO TOOBIG;
     END ELSE  WHILE ABS(LASTERM)>EPS×PSI DO
        BEGIN
          MP1←ULDM+1;    MP2←OLDM+2;
          It MP1>MMAX THEN GO TO TOOBIG;
          It MP1 ≥OLDMH THEN
           BEGIN
             FOR M←OLDM, MP1 DO
              BEGIN
               S[M]← SMVAL(H, SCOEF[M,*], M, 0);
               DS[M]←SMVAL(H,SPRIME[M,*], M, 2) x TEMP
              END;
             OLDMH← MP2
           END;
          It MP2≥ULDMT THEN
           BEGIN
             FANDG(FVAL, GVAL, THT, OLDMT, MP2);    OLDMT← MP2+1
           END;
          LASTERM← S[OLDM]×FVAL[OLDM] + S[MP1]×GVAL[MP1];
          PSI← LASTERM - PSI;
          DPDH← DS[OLDM]×FVAL[OLDM] + DS[MP1]×GVAL[MP1]  - DPDH;
          DPDT←  S[OLDM]×FVAL[MP1 I + S[MP1]×GVAL[MP2]  - DPDT;
          OLDM← Mp2
        END;
    IF OLDM=4×(OLDM DIV 4) THEN
     BEGIN  PSI←-PSI;  DPDT←-DPDT;  DPDH←-DPDH   END;
END STRFNC;


COMMENT   THE FOLLOWING 5LINES ARE PART OF THE (USER) SAMPLE PROGRAM;
REAL    MAXEPS, MAXH, MAXHO, PSI, DPDT, DPDH, H, THT;
MAXEPS ← MAXH←MAXHO←0;   THT←1;
FOR   HO←-1 STEP .05 UNTIL .2,.22 DO
  BEGIN
   VO← SPEED(HO);     ATVO← (1-.2×VO*2)*(-2.5);
```

```
ULDH←OLDTHT←@30;      COMMENT    INITIALIZATION  FOR STRFNC;
                                                              COMMENT
************************************************************************
  C O E F F I C I E N T   C A L C U L A T I O N   F O R   SM(H)"S;

    C←0.2512511361;
    F I L L  A[*]WITH -.1505866818, -.4018655347, 2.0945191543,
       -5.8821787341, 10.95831580, -10.7524447788, 5.9416272229,
       -.8198101027;
    CMIH057←(C-HO)*(-5/7);
    SCOEF[0,0]←1;    SCOEF[1,0]←C-HO;    SCOEF[1,7]←-1;
    M2N7←-N7;    MN7←N7;
    FOR M←2  STEP  1 UNTIL  MMAX  DO
    BEGIN

      COMMENT    STEP  I: CALCULATE  BETA[M,J]   AND  STORE  IN SCOEF[M,J];

      MN7←MN7 + N7;    MN7MI2←MN7-2;    M2←M-2;    M2N7←M2N7+N7;
      FOR J←0,2,4,6  STEP  1 UNTIL  MN7MI2 DO
      BEGIN    SUM←0;
         Kt  (1+ MAX(0, J-M2N7)) DIV  2;
         UP←MIN(J,NN12) DIV  2;
         FOR K←K  STEP  1 UNTIL  UP  DO  SUM←SUM + A[K]×SCOEF[M2,J-K-K];
         SCOEF[M,J]←(7/(5-J))×SUM;
      END OF BETAMJ CALCULATIONS;
      SCOEF[M,1]←SCOEF[M,3]←SCOEF[M,5]←0;

      COMMENT    STEP  II:   CALCULATE  K(M-1);

      KM1← SMVAL(HO, SCOEF[M,*], M,  2)  × CMIH057;

      COMMENT    STEP  III:   CALCULATC  SCOEF[M,J],J:=1,...,M(N+7);

      FOR  JtMN7       STEP  -1 UNTIL 2 DO
         SCOEF[M,J]←(-7/J)×SCOEF[M,J-2];
      SCOEF[M,7]←KM1;    SCOEF[M,0]←SCOEF[M,1]←0;

      COMMENT    STEP  IV:   CALCULATE  SCOEF[M,0]=-KM;

      SCOEF[M,0]← -SMVAL(HO, SCOEF[M,*], M,  0)
    END ;
  FOR  M←0 STEP 1 UNTIL   MMAX   DO
    DIFFSM(SCOEF[M,*],M×N7, SPRIME[M,*]);
                                                              COMMENT
     E N D   O F   C O E F F I C I E N T   C A L C U L A T I O N
************************************************************************;
```

```
COMMENT      THE REMAINDER IS SAMPLE  PROGRAM;
BEGIN   REAL AVGEPS, X, Y;      INTEGER  MUP;
    MUP←7;
    WRI TE(<" HO          H              SPEED                    MACH  NO. ",
              "               PSI(H,1)           *PSI*(H,HO,1) ",
             "                PSI = *PSI*",/>);         AVGEPS←0;
   FOR H←-1 STEP .05 UNT IL  .20, .22 on
    BEGIN
      REAL PROCEDURE M(V);   VALUE Vi    REAL  V;
       M← V/SQRT(1-.2×V*2);
      LABEL TOOBIG, AROUND;
      Gil TO AROUND;
    TOOBIG:
      WRITE(<"INCREASING MUP",I4>, (MUP←MUP+1));
       IF  MUP>NPSITRUNCMAX  THEN  GO TO EXIT;
    AROUND:
      STRFNC(PSI, H, THT, DPDT, DPDH, MUP, @-4, TOOBIG);
      WRITE(<2(F6.2,X4), 4(R15.8,X5),  X5,  E12.5>,
            HO, H, (Y←SPEED(H)), M(Y), (X←2.538/Y×SIN(THT)), PSI,X-PSI);
      AVGEPS←AVGEPS + ABS(PSI-X);
      IF ABS(PSI-X)>MAXEPS THEN
        BEGIN   MAXEPS←ABS(PSI-X);   MAXH←H;   MAXHO←HO   END;
    END;
   WRITE(<"AVG ERR = ", E12.5>, AVGEPS/26);
   WRITE([PAGE]);
 END   END;
WRITE(<"MAX ERR = ", E12.5, "   AT H= ", F6.2, "  AND  HO = ", F6.2>,
   MAXEPS, MAXH, MAXHO);
    END;
EXIT:
END.
```

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