A PROCEDURE FOR IMPROVING THE UPPER BOUND FOR THE NUMBER OF n-OMINOES

BY

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Abstract

An n-omino is a plane figure composed of n unit squares joined together along their edges. Every n-omino is generated by joining the edge of a unit square to the edge of a unit square in some (n-1)-omino so that the new square does not overlap any squares. Let t(n) denote the number of n-ominoes, then it is known that the sequence $((t(n))^{1/n} : n = 1,2,\ldots)$ increases to a limit Θ , and $3.72 < \Theta < 6.75$. A procedure exists for computing an increasing sequence of numbers bounded above by Θ . (Chandra recently showed that the limit of this sequence is Θ .) In the present work we give a procedure for computing a sequence of numbers bounded below by Θ . Whether or not the limit of this sequence is 0 remains an open question. By computing the first ten terms of our sequence, we have shown that $0 < 4.65$.

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1. Introduction

We begin with some definitions and a formulation of the problem treated in subsequent sections. Also included in this section is a brief indication of some of the known results dealing with the n-omino enumeration problem. Some of what follows together with more details may be found in [3] or [4].

Let C denote the set of all integer points in the Cartesian plane, that is, C = 1x1 where I denotes the set of all integers. Elements of C are called cells, and two cells are said to be connected if the distance between them in the Cartesian plane is 1 . The set of cells C may be regarded as the vertex set of an infinite planar graph R whose edges consist of all pairs of connected cells in C . For each natural number n , let R(n) denote the connected subgraphs of R having exactly n vertices. Clearly, R(n) has infinitely many elements for each number n , but we are only interested in certain equivalence classes defined on R(n) by means of the automorphism group $\mathcal{A}$ of R .

The automorphism group $\mathcal{A}$ of R consists of isometries of the plane which map C onto C ; more precisely, an element of $\mathcal{A}$ is the restriction of such an isometry to C . An important subgroup $\mathcal{T}$ of $\mathcal{A}$

corresponds to the set of translations of the plane which map C onto C . All of the elements of $\mathcal{S}$ may be formed by combining the elements $\mathcal{T}$ with combinations of some of the following isometries of the plane: reflection along the x-axis or y-axis, 90° , 180° , or 270° rotation about the origin.

Two elements of R(n) are said to belong to the same <u>translation class</u> if one of these elements can be transformed into the other by an element of $\mathcal{T}$. The set of all translation classes induced in R(n) by $\mathcal{T}$ is denoted T(n) . Representative elements of the translation classes induced in R(4) by 3 are shown in Figure 1. In the figure, boxes have been drawn around the cells of the animals, and the vertices and edges of the graphs have not been indicated in the conventional way.

Two elements of R(n) are said to be the <u>same</u> if one of them can be transformed into the other by an element of $\mathcal{S}$. The set of equivalence classes induced in R(n) by $\mathcal{S}$ is denoted S(n) . Representative elements of the equivalence classes induced in R(4) by $\mathcal{S}$ are shaded in Figure 1.

Let $t(n) = |T(n)|$ and s(n) = IS(n) | , then it follows from the definitions that

$$\frac{1}{8}\, t(n) \le s(n) \le t(n) \le 8s(n) \qquad (n = 1,2,\ldots) . \tag{1}$$

Furthermore, it was shown in [3] (i) that the limits

$$\Theta = \lim_{n\to\infty} (t(n))^{1/n} \ , \quad \Theta' = \lim_{n\to\infty} (s(n))^{1/n} \tag{2}$$

exist, (ii) $\Theta' = \Theta$, (iii) $\Theta \ge (t(n))^{1/n}$ for all n , (iv) and $\Theta > 3.72$. This last result was an improvement over $\Theta > 3.14$ and

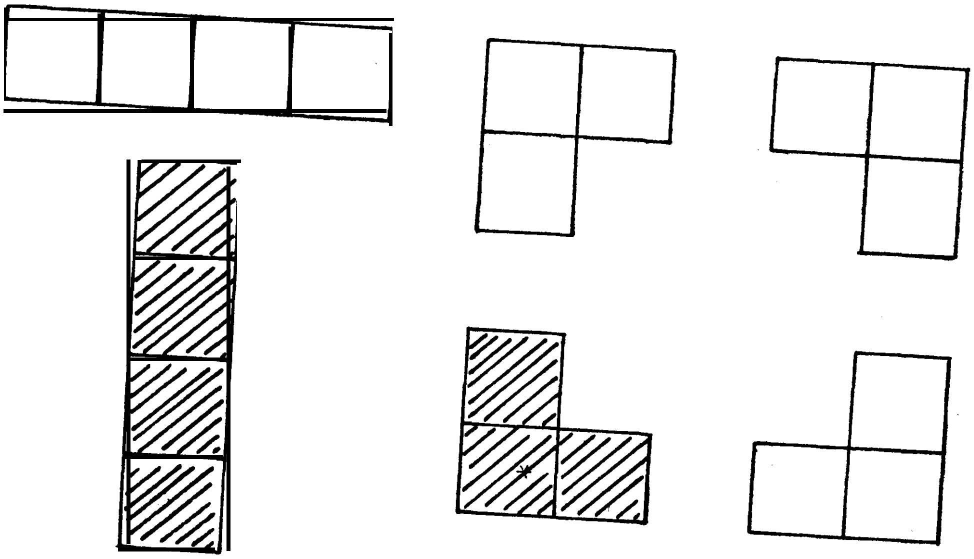


Figure 1. Representative elements of classes in T(4) .

$\Theta > 3.20$ given in Eden [1], and [5] respectively. Also/Eden showed that

$$(3) \qquad t(n) \leq \binom{3n}{n-1} ,$$

and since

$$(4) \qquad \lim_{n \to \infty} \left(\binom{3n}{n-1}\right)^{1/n} = \frac{27}{4} ,$$

it follows from his result that $\Theta \leq 6.75$. Thus, the best bounds on Θ after [3] were

$$3.72 < \Theta \leq 6.75 .$$

Read [6] (for more details see also [4]) gave a method for computing the generating function for the number of elements of T(n) involving n-ominoes whose cells occupy no more than r rows of cells in the plane. For example, when r = 2 this generating function is $(1+x^2)/(1-2x+x^4)$. In general, Read's method gives rise to a rational function $p_r(x)/q_r(x)$ with p_r and q_r relatively prime polynomials such that $q_r(0) = 1$. Thus, if the largest real root of $q_r(1/x)$ is α_r , then it follows that $\alpha_i \leq \alpha_{i+1} < \Theta$ for $i = 1,2,\ldots$. Therefore, this method leads to a procedure for improving lower bounds on Θ indefinitely. It might be remarked that the amount of work required by this method to improve the bound $3.72 < \Theta$ (proved by an entirely different method in [3]) appears to be prohibitive.

An alternative procedure for improving lower bounds on Θ indefinitely was proposed by the late Leo Moser. Consider the set W(n) of translation classes of n-ominoes X such that X has exactly one cell in its bottom row and more than one cell in all other rows; also, one cell in the top row of X is to be distinguished from the other cells. For example, W(1) has one element, W(2) is empty, and W(3) has four elements. Figure 2 illustrates the elements of W(4) ; the distinguished cells in top rows are marked with a cross. Now we use elements of W(n) to construct elements of a set T*(n) consisting of translation classes of n-ominoes X such that X has exactly one cell in its bottom row and a distinguished cell in its top row. Let $t^*(n) = |T^*(n)|$, then it is easy to see that

$$(5) \qquad t(n-1) \leq t^*(n) \leq nt(n) \ ,$$

and this implies

$$(6) \qquad \lim_{n\to\infty} (t^*(n))^{1/n} = \theta \ .$$

Now we estimate t*(n) from below. Every element $X \in T^*(n)$ corresponds to a unique sequence $(X_1,\ldots,X_k)$ with $X_1 \in W(n_1),\ldots,X_k \in W(n_k)$ where k , $n_1,\ldots,n_k$ are certain numbers uniquely determined by X with $n = n_1 + \ldots + n_k$. This sequence is found by cutting X into pieces with lines running along the bottom of each row of X containing exactly one cell. The element X_i lies between the i-th and (i+1)-st of these lines, and the distinguished cell in the top row of X_i is either the distinguished cell of X (in case i = k), or it is the cell joined to the unique cell in the bottom row of X_{i+1} . Letting $w(n) = |W(n)|$, it follows that

$$(7) \qquad t^*(n) = \sum_{k=1}^{n} \sum w(n_1) \ldots w(n_k)$$

where the inner sum extends over all compositions $(n_1,\ldots,n_k)$ of n into k positive parts. If $(w^*(1),w^*(2),\ldots)$ is any sequence of

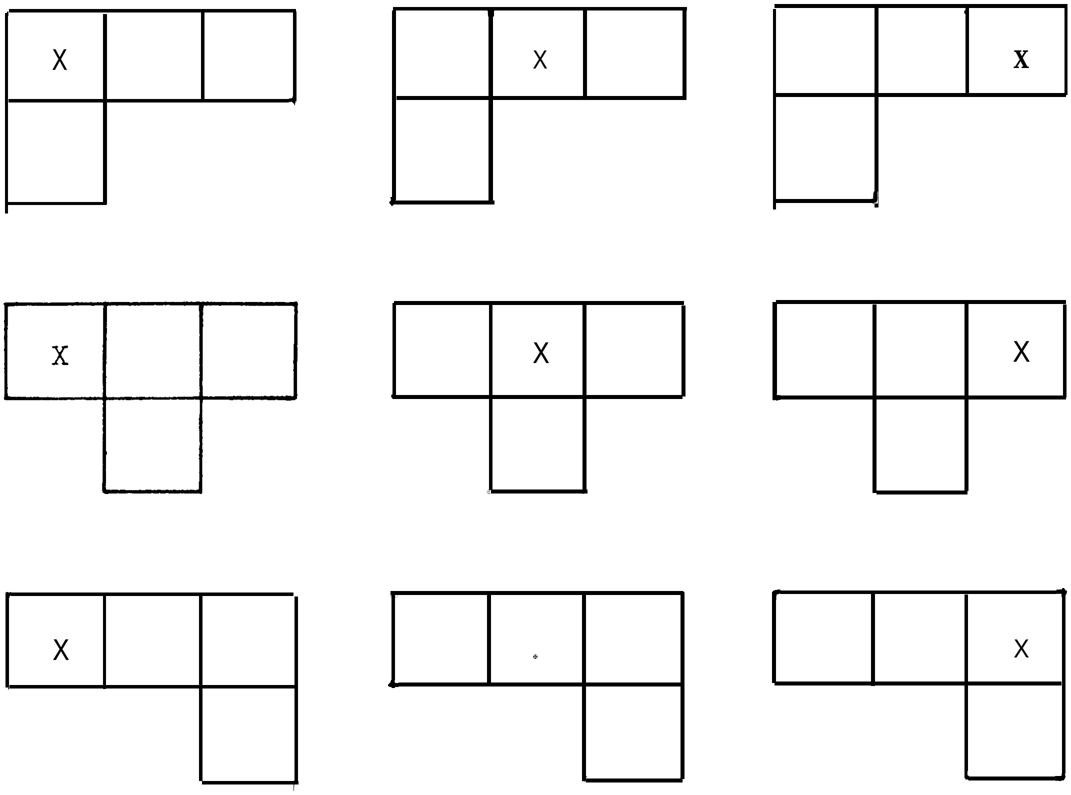


Figure 2. Elements of W(4) .

non-negative numbers such that $w^*(n) \leq w(n)$, then of course

$$(8) \qquad t^*(n) \geq \sum_{k=1}^{n} \sum w^*(n_1) \ldots w^*(n_k) .$$

Setting

$$(9) \qquad f(x) = \sum_{n=1}^{\infty} w^*(n)x^n ,$$

we have

$$(10) \qquad \frac{f(x)}{1 - f(x)} = \sum_{n=1}^{\infty} \left\{ \sum w^*(n_1) \ldots w^*(n_k) \right\} x^n$$

where the inner sum extends over all compositions $(n_1, \ldots, n_k)$ of n into exactly k positive parts for $k = 1,2,\ldots$. The coefficient of X^n in the power series in (10) is a lower bound for $t^*(n)$ so long as $1 \leq w^*(n) \leq w(n)$. Thus, if we define

$$(11) \qquad f_r(x) = \sum_{n=1}^{r} w(n)x^n ,$$

and define a sequence $(t^*_r(1), t^*_r(2), \ldots)$ by

$$(12) \qquad \frac{f_r(x)}{1 - f_r(x)} = \sum_{n=1}^{\infty} t^*_r(n)x^n ,$$

then it follows that

$$(13) \qquad t^*_{r\ 1}(n) \leq t^*_r(n) \leq t^*(n)$$

for $r = 1,2,\ldots$ and $n = 1,2,\ldots$. Furthermore, if we put

$$(14) \qquad \varphi_r = \lim_{n \to \infty} (t^*_r(n))^{1/n} ,$$

then $\varphi_1 \leq \varphi_2 \leq \ldots \leq \theta$. Finally, we come to the computation of φ_r .

Since $f_r(x)/(1-f_r(x))$ is a rational function which generates a sequence of increasing positive integers, it follows that φ_r is equal to the largest real root of the equation $f_r(1/x)-1 = 0$. Thus, Moser's procedure comes down to enumerating the sets $W(1),\ldots,W(r)$ to find φ_r . One has more and more work to find improvements by this method, and indeed, so far no one has had the ambition required to calculate φ_r for a large enough number r to improve the bound $3.72 < \Theta$.

So far we have seen two procedures for improving lower bounds on Θ indefinitely. No such procedure is known for improving the upper bound on Θ , and it is our goal in this paper to show that such a procedure exists. Furthermore, we shall achieve a considerable improvement over Eden's bound $\Theta \leq 6.75$. The next section deals with combinatorial aspects of this problem which lead to a technical problem involving generating functions. This problem is dealt with in the third section, and in the final section we discuss the calculations which lead to our new upper bound for Θ .

2. n-Ominoes Viewed as Sequences of Twigs

In this section we develop an idea which originates with Eden [1]. We begin with a description of this idea, reformulating it so that our development appears straightforward. The idea is that a unique planted plane tree E_X embedded in R may be associated with each n-amino X . The tree E_X is then interpreted as a sequence of "twigs", that is, certain small subtrees also embedded in R . Eden's set of twigs E (shown in Figure 3) is finite, and each $Y\epsilon E$ is assigned a weight $w(Y) = x^a y^b$, where a denotes the number of cells in Y less 1 , and b denotes the number of "dead" cells in Y . (Dead cells are colored black in Figure 3.) Let E^k denote the set of all sequences of elements of E having length k for $k = 0,1,\ldots,$ and define the <u>weight</u> of $\bar{Y}\epsilon E^k$ to be $W(\bar{Y}) = xw(Y_1)\ldots w(Y_k)$ where $\bar{Y} = (Y_1,\ldots,Y_k)$ for $k = 1,2,\ldots$, and define the weight of the empty sequence to be x .

It turns out that sequences of twigs corresponding to elements of T(n) have weight $x^n y^n$, and the sum of the weights of all finite sequences of elements of E is

$$(1)\qquad x\left\{1-\sum_{Y\epsilon E} w(Y)\right\}^{-1} = \sum_{k=0}^{\infty} x\left\{\sum_{Y\epsilon E} w(Y)\right\}^{k}$$

$$= \sum_{k=0}^{\infty}\ \sum_{\bar{Y}\epsilon E^k} W(\bar{Y}) \quad .$$

Since $(Y\epsilon E)w(Y) = y(1+x)^3$, the generating function given by (1) is

$$(2)\qquad \frac{X}{1-y(1+x)^3} = \sum_{n=0}^{\infty} xy^n(1+x)^{3n} = \sum_{n=0}^{\infty}\ \sum_{m=1}^{3n+1}\binom{3n}{m-1}x^m y^n \quad .$$

Thus, if it is shown that there exists an injection of T(n) into the set of finite sequences of E having weight $x^n y^n$, then we are

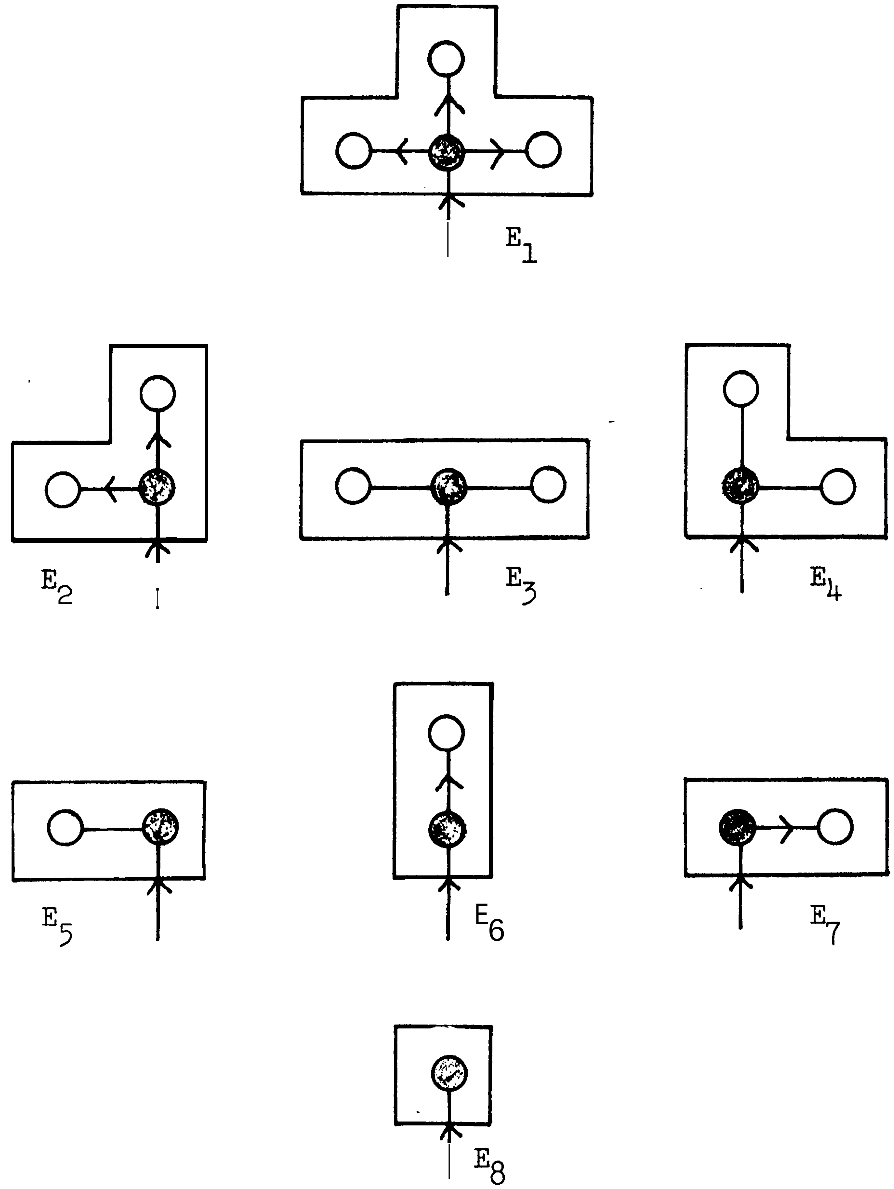


Figure 3. **Eden's** set of twigs E .

justified in concluding that the coefficient $e(n,n)$ of $x^n y^n$ in this power series is an upper bound for $t(n)$. Hence, if $1/\varepsilon$ is the radius of convergence of the "diagonal? power series of (2), that is, the power series

$$(3) \qquad \sum_{n=0}^{\infty} e(n,n) z^n \quad ,$$

then $\Theta \leq \varepsilon$ where 0 is defined in ($1.2). Rut,

$$(4) \qquad e(n,n) = \binom{3n}{n-1} \quad \text{and} \quad \varepsilon = \frac{27}{4} \quad ,$$

and this is Eden% result mentioned in §1.

It remains to describe an injection of $T(n)$ into the set of finite sequences of elements of E having weight $x^n y^n$. Suppose $X \epsilon R(n)$, then a spanning tree E_X of X which is at the same time a planted plane tree embedded in R may be defined as follows: Assign labels 1 and 0 to the left-most cell in the bottom row of X and the cell below this one respectively, then draw an edge from cell 0 into cell 1 . Now we define a process which generates a spanning tree of X assigning labels $1,\ldots,n$ to the vertices of X . The process consists of a sequence of n steps $P(1),\ldots,P(n)$ which may be described in general.

$P(i)$: An edge has been drawn from cell j_i into cell i . Three cells together with cell j_i surround cell i which for the moment we call a_i , b_i , c_i going clockwise around cell i from cell j_i . If a_i is a cell of X and has not been labelled earlier in this process, then an edge is drawn from cell i into a_i , and a_1 is assigned the successor of the last label used in this process. Repeat this for b_i and c_i , and go on to $P(i+1)$ or stop if $i = n$. It can be shown easily by induction that carrying out

$P(1),\ldots,P(n)$ creates a spanning tree of X which is also a planted plane tree embedded in R . At vertex i in this tree we find exactly one of the twigs shown in Figure 3, denote this twig by Y_i , and define $E_X = (Y_1,\ldots,Y_n)$. See Figure 4 for an example of a spanning tree created by this process; the sequence of twigs in this example is $(E_4, E_4, E_8\ E_3\ E_6\ E_8\ E_8\ E_6\ E_7\ E_8)$, and

$$xw(E_4)\ \ldots w(E_8) = x\cdot x^2y\cdot x^2y\cdot y\cdot x^2y\cdot xy\cdot y\cdot y\cdot xy\cdot xy\cdot y = x^{10}y^{10}$$

Now we show that the weight of $E_X = (Y_1,\ldots,Y_k)$ is x^ny^n for all $X\in R(n)$. To see this, we need the concept of the partial planted plane trees embedded in R which are formed by the sequences $(Y_1,\ldots,Y_k)$ for $k = 1,\ldots,n$. Modify step P(i) above by adding the operation of coloring cell i black. (Assume that all cells of X are white initially.) Carrying out modified steps $P(1),\ldots,P(k)$ gives rise to the partial planted plane tree having twigs $Y_1,\ldots,Y_k$. Suppose $xw(Y_1)\ \ldots\ w(Y_k) = x^ay^b$, then it is easy to show by induction that the number of black cells (which we call dead cells) in the partial tree is b, and the total number of cells in the partial tree is a . Since every cell of X is colored black after carrying out modified steps $P(1),\ldots,P(n)$, and since X has n cells altogether, it follows that $W(E_X) = x^ny^n$.

Finally, if $X,X'\in R(n)$, and X is a translation of X' , then E_X is a translation of $E_{X'}$. Thus, the spannipg tree of a representative element of a translation class of n-ominoes is representative of the spanning trees of all the n-ominoes in the translation class. This completes the description of an injection of T(n) into the set of finite sequences of elements of E having weight x^ny^n .

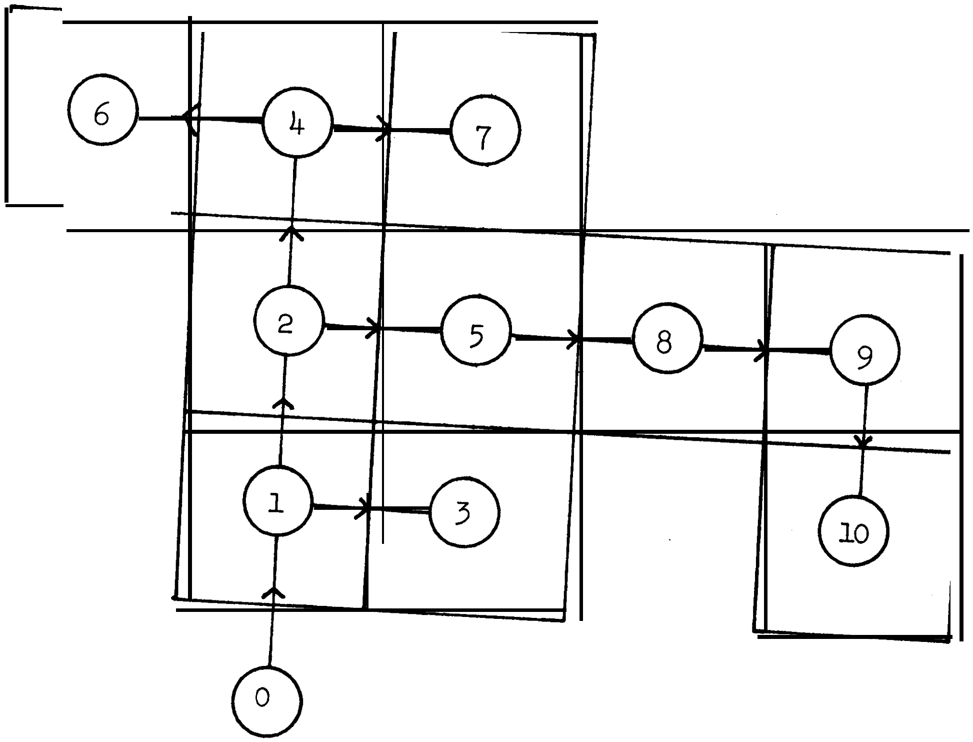


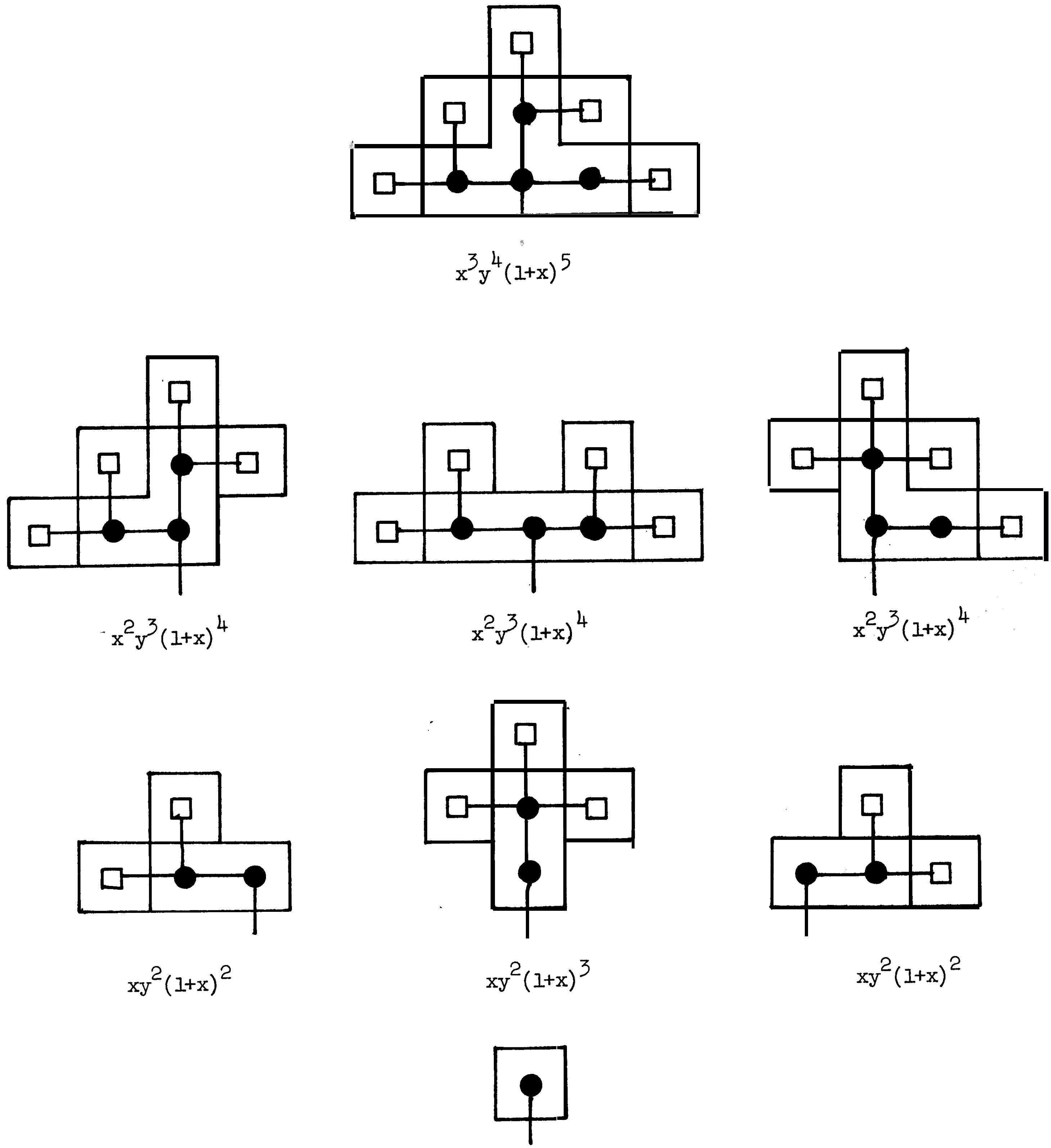
Figure 4. A spanning tree generated by Eden's method.

Our development of Eden's idea now follows naturally. The spanning tree E_X of $X \epsilon R(n)$ may be viewed as a sequence of elements selected from a set of "larger" twigs. For example, such a set of twigs may be defined for k = 1,2,... as follows: Let E(k) denote the set of all partial planted plane trees Z embedded in R such that (1) the dead cells of Z are connected to the root of Z with a path of length less than k , and (2) Z must be a sub-tree of the partial spanning tree of some polyomino. The <u>weight</u> of an element $Y \epsilon E(k)$ is defined to be $w_k(Y) = x^a y^b$ where a denotes the total number of cells in Y less 1 , and b denotes the total-number of dead cells in Y . The <u>weight</u> of a sequence $\bar{Y} = (Y_1, \ldots, Y_r)$ of elements of E(k) is defined to be $W_k(\bar{Y}) = xw_k(Y_1) \ldots w_k(Y_r)$. Every n-omino X gives rise to a unique sequence of elements of E(k) , and it can be shown by induction that the weight of such a sequence is $x^n y^n$. It follows from these definitions that E(1) = E_1 and the elements of E(1) are shown in Figure 3. The elements of E(2) are compactly represented by the drawings in Figure 5 which are interpreted as follows; Each drawing represents the collection of twigs having in common the dead cells marked as black vertices. The elements of each collection are obtained by including all subsets of the cells marked with square vertices as white cells of a twig. The sum of-the weights of all the twigs in each collection is written below each drawing.

Following (1), the sum of the weights of all finite sequences of elements of E(k) is given by

$$(5) \qquad x\left\{1 - \sum_{Y \epsilon E(k)} w(Y)\right\}^{-1} = \sum_{m,n=0}^{\infty} e_k(m,n) x^m y^n \quad ,$$

and the coefficient $e_k(n, n)$ of $x^n y^n$ is an upper bound for t(n) .



$x^3y^4(1+x)^5$

$x^2y^3(1+x)^4$ $x^2y^3(1+x)^4$ $x^2y^3(1+x)^4$

$xy^2(1+x)^2$ $xy^2(1+x)^3$ $xy^2(1+x)^2$

Y

Figure 5. Elements of E(2) .

Furthermore, it can be shown that $e_{k+1}(n,n) \leq e_k(n,n)$ for $k = 1,2,\ldots$ and all n, in fact, for any fixed k, strict inequality must hold for all sufficiently large n. (Since our final result does not depend on these claims, we shall not bother to prove them.) Thus, if $1/\varepsilon_k$ denotes the radius of convergence of the diagonal power series of the power series given in (5), we have $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \Theta$, where Θ is defined in (1.2). In the next section we show how to compute an upper bound for ε_k; in fact, it follows from the results proved there that $\varepsilon_1 = 6.75, \varepsilon_2 \leq 5.50$, and $\varepsilon_3 \leq 5.25$. The amount of work required by this procedure for $k = 4$ or 5 say, may not be prohibitive, and the upper bound for Θ might be further improved by this method. However, there is a set of twigs more efficient than the extension of Eden's set and it is the procedure associated with this set that we plan to push to the limits of our computing ambition.

There are eight L-shaped 4-sets of cells near a given cell u which we call L-contexts of u; rather than take space to define these 4-sets precisely, we merely picture them in Figure 6. Using this concept, we describe the set of twigs L shown in Figure 7. Each element of L is composed of the following things: (i) a root cell along with a specified L-context of this cell, (ii) a set (possibly empty) of open cells which is linearly ordered, and (iii) each open cell is assigned one of its L-contexts. In Figure 7 we have marked the L-context of a twig's root cell with asterisks, the root cell itself is colored black, the open cells are colored white, and the L-context assigned to each open cell is indicated with an L. Where necessary (that is, in twigs L_3 and L_5), the linear order assigned to the open cells of a twig is indicated by numbering them.

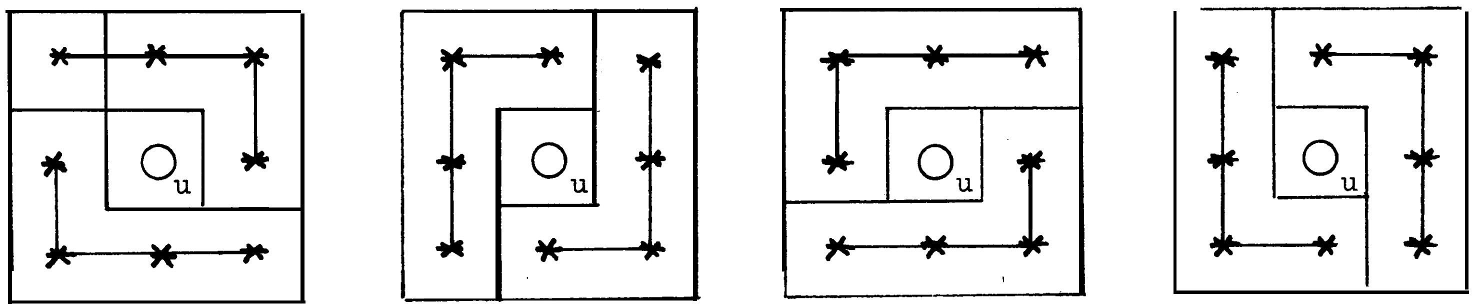


Figure 6. The eight L-contexts of cell u .

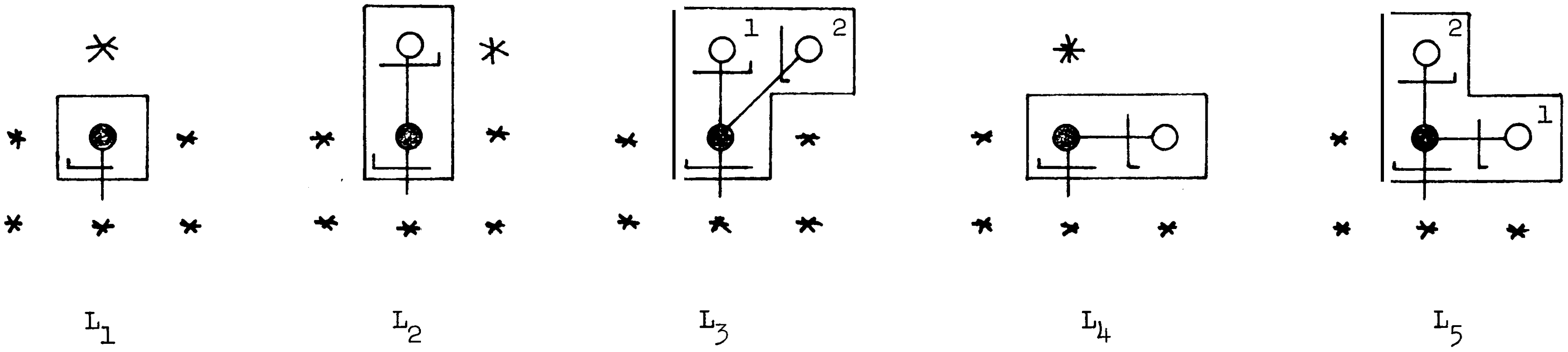


Figure 7. The set of twigs L

Every element $X \in R(n)$ corresponds to a unique n-term sequence of elements of L. Just as in Eden's method, this sequence is constructed algorithmically by assigning a linear order to the cells of X at the same time assigning an element of L to each cell of X . The leftmost cell u in the bottom row of X is cell 1 of X . The L-context U of u which consists of the cell to the left of u and the three cells below u form the L-context of a twig (which we will specify in a moment) whose root is u . Let w and y denote the cells connected to the right and above u , and let x denote the cell ($\neq$ u) connected to w and y . The twig assigned to u (that is, cell 1 of X) is (i) L_1 , if $w,x,y \notin X$, (ii) L_2 , if $w,x \notin X$, $y \in X$, (iii) L_3 , if $w \notin X$, $x,y \in X$, (iv) L_4 , if $y \notin X$, $w \in X$, (v) L_5 , if $w,y \in X$. It is easy to check that (i)-(v) cover all possible situations. The L-context of a twig is interpreted as a set of cells whose status of belonging or not belonging to X is known. This is the case with the root of the twig assigned to u . Note however that this is true for cell 2 of twigs L_3 and L_5 only after the twig assigned to cell 1 has been specified. Now the linear order assigned to the cells of X and the assignment of twigs to the cells of X is carried out by doing $Q(2),\ldots,Q(u)$ where Q(i) is defined as follows: Suppose labels $1,\ldots,j_{i\ 1}$ have been given to cells of X with $j_{i\ 1}$ the last label given any cell. Go to cell i of X which is the open cell of a twig assigned to yet another cell of X , and let the L-context specified be the L-context of the twig to be assigned to cell i . All previously labelled cell of X are deleted, and cell i is viewed as the root cell of some connected component of X . Now the twig assigned to cell i is determined in the same way as for cell 1 , and the open cell (or cells) belonging to this twig is (are) labelled $1+j_i$ (or $1+j_i$ and $2+j_i$ according to the linear order specified by the linear order of the open cells of the twig). Note that generally the L-context of cell i may require one to reflect and/or rotate the appropriate twig to be assigned to it. The sequence of elements of L assigned to X by this method is defined to be $L_X = (X_1,\ldots,X_n)$ where X_i is the twig assigned to cell i by Q(i) . The spanning tree and sequence of twigs generated by this method corresponding to the decomino shown in Figure 4 is shown in Figure 8.

Clearly, a common sequence is assigned to the elements of a translation class of n-ominoes, and n-ominoes belonging to different translation classes are assigned different sequences. Hence, there is an injection of T(n) into the set of n-sequences of L . Furthermore, if the elements of L are given the weights $w(L_1) = y$, $w(L_2) = xy$, $w(L_3) = x^2y$, $w(L_4) = xy$, $w(L_5) = x^2y$, and the weight of a sequence $\bar{X} = (X_1,X_2,\ldots)$ of elements of L is defined to be $W(\bar{X}) = xw(X_1)w(X_2)\ldots$, then $W(L_X) = x^ny^n$ to all $X \in R(n)$. Thus, there is an injection of T(n) into the set of sequences of L with weight x^ny^n .

Letting $\ell(m,n)$ denote the number of sequences of L having weight x^my^n , we can use (1) with L in place of E to find

$$(6) \qquad \sum_{m,n=0}^{\infty} \ell(m,n)x^my^n = \frac{x}{1-y(1+2x+2x^2)}$$

$$= \sum_{n=0}^{\infty} xy^n(1+2x+2x^2)^n \quad .$$

Thus, $\ell(n,n)$ is equal to the coefficient of x^{n-1} in $(1+2x+2x^2)^n$, that is,

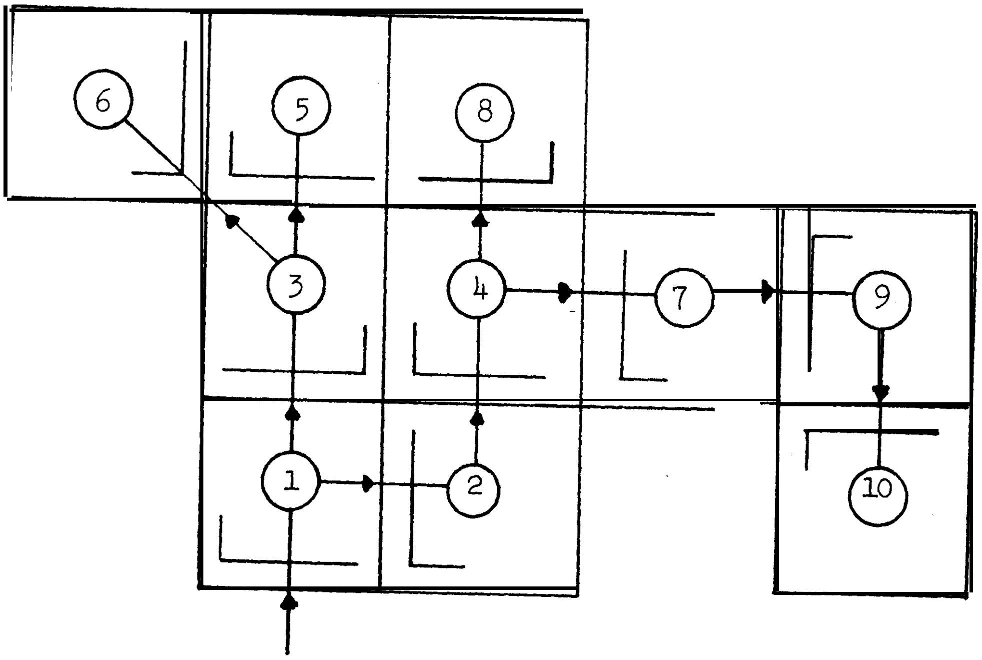


Figure 8. $L_X = (L_5, L_4, L_3, L_5, L_1, L_1, L_2, L_1, L_4, L_1)$.

(7) $$\ell(n,n) = \sum_{k=0}^{n} \binom{n}{k,k+1,n-2k-1} 2^{n-k-1} ,$$

but

(8) $$\sum_{k=0}^{n} \binom{n}{k,k+1,n-2k-1} 2^{n-k-1} = \frac{2^n}{\sqrt{n}} \sum_{k=0}^{n} \binom{n}{k,k+1,n-2k-1} \left(\frac{1}{\sqrt{2}}\right)^{k} \left(\frac{1}{\sqrt{2}}\right)^{k+1}$$

$$\leq \frac{2^n}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1\right)^{n} < (2+2\sqrt{2})^{n} .$$

It follows that

(9) $$\theta = \lim_{n\to\infty} (t(n))^{1/n} \leq \lim_{n\to\infty} (\ell(n,n))^{1/n} \leq 2+2\sqrt{2} < 4.83 ,$$

which is already a substantial improvement on Eden's bound $\theta \leq 6.75$. We can improve further on $\theta < 4.83$ using L in analogy to our improvement on $\theta < 6.75$ using E .

Consider the infinite set H of twigs generated from the set of twigs L in the following way. Every partial spanning tree of a polyomino is a member of H , where the spanning tree is generated with the procedure Q corresponding to the set of twigs L . Each twig $X \in H$ is like a polyomino except that (i) it has a unique root cell indicated, (ii) a particular L-context is associated with the root cell, (iii) a spanning tree of X is indicated, (iv) all nonterminal nodes of the spanning tree are dead cells, and (v) some of the terminal nodes of the spanning tree may be open cells, each with an associated L-context. Thus, the subset of H consisting of n-cell twigs with no open cells is isomorphic to T(n) .

A partial order $\leq$ may be defined on H as follows: For any $\bar{X},\bar{Y} \in H$ put $\bar{X} \leq \bar{Y}$ whenever

(i) $\bar{X}$ has fewer cells than $\bar{Y}$,

(ii) the root of $\bar{X}$ has the same L-context as the root of $\bar{Y}$,

(iii) the spanning tree of $\bar{X}$ is isomorphic to a subtree of $\bar{Y}$ rooted at the root of $\bar{Y}$.

In essence, $\bar{X} \leq \bar{Y}$ whenever $\bar{Y}$ can be "grown" from $\bar{X}$ by repeatedly applying the process Q to the open cells of X . The element $\emptyset$ (a twig of no cells) is considered to be the smallest element of H . The covering relation of H ordered by $\leq$ is a tree with root $\emptyset$ . . A finite subset C ($\neq \{\emptyset\}$) of H is called a <u>cut</u> if every element of H is comparable to some element of C (for example, the set L forms a cut of H). A cut is said to be <u>minimal</u> if no other cut is properly contained in it.

Given a minimal cut C , it is easy to show that the spanning tree of any n-omino $\bar{X}$ can be uniquely decomposed into a spanning tree corresponding to elements of C . The set of twigs corresponding to X are ordered by the label assigned to their root by the process Q . The set C of twigs thus forms a "complete" set of building blocks, that is, a set of twigs capable of constructing any n-omino. Furthermore, using the weight function w defined on L , we define the weight of an element $\bar{X} = (X_1,\ldots,X_k)$ of C to be $w(\bar{X}) = w(X_1)\ldots w(X_k)$, and the weight of a sequence $\bar{Y} = (Y_1,\ldots,Y_j)$ of elements of C is defined to be $W(\bar{Y}) = xw(Y_1)\ldots w(Y_j)$. Thus, if C_X denotes the sequence of elements of C corresponding to $X \in R(n)$, then $W(L_X) = W(C_X)$. Hence, there is an injective mapping of T(n) into the set of all sequences of C having weight $x^n y^n$.

Next, suppose C and C' are minimal cut sets, and every element of C is dominated by some element of C' , then we write $C < C'$. It is rather easy to prove that if $\ell_C(m,n)$ denotes the number of sequences of C with weight $x^m y^n$, and if $C < C'$, then $\ell_C(n,n) \geq \ell_{C'}(n,n)$. The point is, if sequences of elements of C and C' are converted into sequences of elements of L , then the sequences giving rise to the number $\ell_{C'}(n,n)$ constitute a subset of the sequences giving rise to the number $\ell_C(n,n)$. Thus, for each sequence $(C_1, C_2, \ldots)$ of minimal cuts with $C_1 < C_2 < \ldots$, we have

$$\ell_{C_1}(n,n) \geq \ell_{C_2}(n,n) \geq \ldots \geq t(n) \quad . \tag{10}$$

To calculate $\ell_{C_i}(n,n)$ we use (1) with C_i in place of E :

$$\frac{x}{1 - \sum_{T \in C_i} w(T)} = \sum_{m,n=0}^{\infty} \ell_{C_i}(m,n) x^m y^n \quad . \tag{11}$$

Thus, estimating $\ell_{C_i}(n,n)$ in (11) presents us with the problem of estimating from below the radius of convergence of the "diagonal function" of a rational double power series. We want to use the fact (implied by (10)) that if $1/h_i$ denotes the radius of convergence of $\ell_{C_i}(n,n)x^n$, then

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \theta \quad . \tag{12}$$

This is the problem treated in the next section.

3. The Diagonal of a Rational Function

In this section we tell how to find a lower bound for the circle of convergence of the diagonal of a double power series which represents a rational function. More precisely, suppose $P(x,y)$ and $Q(x, y)$ are polynomials with integer coefficients such that $P(0,0) = 1$ and $Q(x,y)/P(x,y)$ is in reduced form, then consider the representation of $Q(x,y)/P(x,y)$ as a power series

$$(1) \qquad F(x,y) = \frac{Q(x,y)}{P(x,y)} = \sum_{m,n=0}^{\infty} f(m,n)x^m y^n \ .$$

The diagonal of $F(x,y)$ is defined to be

$$(2) \qquad F_D(z) = \sum_{n=0}^{\infty} f(n,n)z^n \ .$$

In Section 2 we encountered the problem of determining an upper bound for

$$(3) \qquad \varphi = \liminf_{n\to\infty} (f(n,n))^{1/n} \ ,$$

that is, φ^{-1} is a lower bound for the radius of convergence of $F_D(z)$. To solve this problem, we use the integral representation for $F_D(z)$ given in [2].

We can suppose there exist positive constants α and β such that the power series in (1) represents $F(x,y)$ for all x and y such that $|x| < \alpha$, $|y| < \beta$. Thus, the function $F(s,zs^{-1})s^{-1}$ is represented by the Laurent power series

$$(4) \qquad F(s,zs^{-1})s^{-1} = \sum_{m,n=0}^{\infty} f(m,n)z^n s^{m-n-1}$$

inside the circular annulus $A = \{s: |s| < \alpha, \ |z||s^{-1}| < \beta\} =$ $\{s: |z|\beta^{-1} < |s| < \alpha\}$ which is not empty provided $|z| < \alpha\beta$. Note that the residue of $F(s,zs^{-1})s^{-1}$ at $s = 0$ is just $F_D(z)$. Thus, if C is a circle inside A with its center at $s = 0$, then we can apply the residue theorem to conclude that

$$(5) \qquad F_D(z) = \frac{1}{2\pi i}\int_C F(s,zs^{-1})s^{-1}ds \;;$$

furthermore, the integral on the right is the sum of the residues of $F(s,zs^{-1})s^{-1}$ at the singularities enclosed by C. Now we take into account the special form of $F(x,y)$.

There exist numbers u,v with $w = u+v$, together with polynomials $P_0(z),\ldots,P_w(z)$ with integer coefficients such that

$$(6) \qquad P(x,y) = \sum_{j=0}^{v} y^j P_{v-j}(xy) + \sum_{j=1}^{u} x^j P_{v+j}(xy) \;.$$

Using this form of $P(x,y)$, we have

$$(7) \qquad F(s,zs^{-1})s^{-1} = \frac{s^{v-1}Q(s,zs^{-1})}{\sum_{j=0}^{w} P_j(z)s^j} \;.$$

There exist functions $\pi_1 = \pi_1(z),\ldots,\pi_w = \pi_w(z)$ such that

$$(8) \qquad \sum_{j=0}^{w} P_j(z)s^j = P_w(z)\prod_{j=1}^{w}(s-\pi_j) \;;$$

in fact, the functions $\pi_1,\ldots,\pi_w$ are distinct provided the polynomial on the left in (8) is irreducible. We shall assume the π's are distinct and treat this case only.

Suppose π_j is inside C for $j = 1,\ldots,t$, and π_j is outside C for $j = t+1,\ldots,w$. Then we can combine (5), (7) and (8), and sum the residues of $F(s,zs^{-1})s^{-1}$ at π_j for $j = 1,\ldots,t$ to find

(9) $$F_D(z) = \sum_{j=1}^{t} \pi_j^{v-1} Q(\pi_j, z\pi_j^{-1}) \quad P_w(z) \prod_{\substack{k=1\\k\neq j}}^{w} (\pi_j - \pi_k) \quad {}^{-1} \quad .$$

Hence, the singularities of $F_D(z)$ form a subset of the roots of the equation

(10) $$R(z) = P_w(z) \prod_{\substack{j,k=1\\j\neq k}}^{w} (\pi_j(z) - \pi_k(z)) = 0 ,$$

and a lower bound for the radius of convergence of $F_D(z)$ is the minimum modulus of all these roots. If it is known that $\{f(n,n): n = 0,1,\dots\}$ is an increasing sequence of integers, a lower bound for the radius of convergence of $F_D(z)$ is the smallest real root of (10). Note that if the π's are not all distinct, then the product in (10) is 0 , and this test fails.

Finally, $R(z)$, the function defined by the left member of (10), is symmetric in $\pi_1, \dots, \pi_w$, so $R(z)$ can be expressed as a polynomial in $P_0/P_w, \dots, P_w/P_{w-1}$. Consulting Uspensky [7, pages 277-291], we see that $R(z)$ is closely related to the descriminant of $s^v P(s, zs^{-1})$ regarded as a polynomial in s . Furthermore, the descriminant of a polynomial can be computed in terms of its coefficients by means of Sylvester's determinant.. Applying the formulas given in Uspensky, we find

(11) $$(P_w(z))^{2w-2} R(z) = \det M(z) ,$$

where M is a $(2w-1) \times (2w-1)$ matrix whose first $w-1$ rows consist of cyclic shifts of $(P_w, P_{w-1}, \dots, P_0, 0, \dots, 0)$, and the next w rows consist of cyclic shifts of $(wP_w, (w-1)P_{w-1}, \dots, P_1, 0,$ [illegible] .*,0) .

Thus we are led to the following conclusion: If $F(x,y)$ is a rational function with the form given in (1), and if the diagonal

$F_D(z)$ of $F(x,y)$ generates an increasing sequence of integers, then an upper bound for φ defined in (3) is the largest real root of the polynomial equation

$$\det M(1/z) = 0 \ . \tag{12}$$

4. Computational Results

A sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of upper bounds for Θ were computed corresponding to a sequence of increasingly larger minimal cut sets $C_1, C_2, \ldots$ of H . The minimal cut C_i is defined to contain all twigs having at most i dead cells. Thus, we clearly have $C_i < C_{i+1}$, $i = 1, 2, \ldots$.

The computation was performed on the PDP-10 at Stanford University% Artificial Intelligence Center with a program written in SAIL, an ALGOL dialect. The results are summarized below. The computation of λ_{10} required approximately one hour of computer time. In addition, the largest real root of equation 12 of Section 3 for C_{10} was found to be **7.005** . Since λ_{10} must be smaller than λ_9 , however, we disregard this root as a spurious one introduced by replacing $F_D(z)$ by $R(z)$, and select the smaller root 4.65 . Fran the table we conclude that

$$(1) \qquad \lim_{n \to \infty} (t(n))^{1/n} = \Theta < 4.65 \ .$$

i	$\lvert C_i \rvert$	λ_i
1	5	4.83
2	21	4.83
3	93	4.83
4	409	4.80
5	1803	4.77
6	7929	4.74
7	34928	4.72
8	151897	4.70
9	656363	4.67
10	2821227	4.65

Table 1

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