DEGREES AND MATCHINGS

BY

V. CHVATAL

STAN-CS-72-268

MARCH 1972

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

DEGREES AND MATCHINGS

by

V. Chvátal
Computer Science Department
Stanford University

Abstract

Let n , b , d be positive integers. D. Hanson proposed to evaluate $f(n,b,d)$, the largest possible number of edges in a graph with n vertices having no vertex of degree greater than d and no set of more than b independent edges. Using the alternating path method, he found partial results in this direction. We complete Hanson's work; our proof technique has a linear programming flavor and uses Berge's matching formula.

AMS(MOS) subject classification numbers (1970): Primary 05C35
Secondary 90C10

Keywords and phrases: Degrees, matchings, independent edges, Berge's matching formula.

This research was supported by the National Science Foundation under grant number GJ-992, and the Office of Naval Research under grant number N-00014-67-A-0112-0057 NR 044-402.

1. Introduction

Erdös and Rado [5] proved that given any positive integers n, k there is always an integer a with the following property: if F is any family of more than a sets, each of cardinality n, then some k members of F have pairwise the same intersection. Let us denote the smallest such a by $\varphi(n,k)$. Some results on $\varphi(n,k)$ can be found in [5], [1] and [3]. Obviously, $\varphi(2,k)$ is the maximum number of edges in a graph containing no vertex of degree greater than k-1 and no set of more than k-1 independent edges. The values of $\varphi(2,k)$ have been determined by N. Sauer (to appear):

$$\varphi(2,k) = \begin{cases} k(k-1) & \text{if } k \text{ is odd,} \\ (k-1)^2 + \frac{1}{2}k - 1 & \text{if } k \text{ is even.} \end{cases} \quad (1)$$

D. Hanson [6] considered a little more general problem. By an (n,b,d)-graph we shall mean a graph G such that

(i) G has n vertices,

(ii) G contains no set of more than b independent edges,

(iii) G contains no vertex of degree greater than d.

The largest possible number of edges of an (n,b,d)-graph will be denoted by f(n,b,d). In the Greek alphabet notation of [7], f(n,b,d) is the maximum of q(G) subject to the constraints

$$p(G) = n, \quad \beta_1(G) \le b, \quad \Delta(G) \le d.$$

Obviously, $f(n,b,d) = f(n,b,n-1)$ whenever $d > n-1$. Similarly, $f(n,b,d) = f(2b+1,b,d)$ whenever $n < 2b+1$. Hence we can restrict ourselves to the case $n > d+1$, $n \geq 2b+1$.

Apart from the most difficult case (d odd and < 2b , n small), the values of f(n,b,d) have already been obtained by Hanson [6]. His proof technique is based on the alternating path method. We will adopt a different approach, related to linear programming. This technique simplifies the proofs and enables us to complete the evaluation of f(n,b,d) without much additional effort. The result goes as follows.

<u>THEOREM</u>. Let n,b , d be positive integers with $n > 2b+1$.

A. If $d \le 2b$ and $n < 2b + \left[\frac{b}{[\frac{d+1}{2}]}\right]$ then

$$f(n,b,d) = \begin{cases} \min\{[\frac{nd}{2}],\ bd + [\frac{2(n-b)}{d+3}] \cdot \frac{d-1}{2}\} & \text{if } d \text{ is odd,} \\ \frac{nd}{2} & \text{if } d \text{ is even.} \end{cases}$$

B. If $d \le 2b$ and $n \ge 2b + \left[\frac{b}{[\frac{d+1}{2}]}\right]$ then

$$f(n,b,d) = bd + \left[\frac{b}{[\frac{d+1}{2}]}\right] \cdot [\frac{d}{2}]$$

C. If $d \ge 2b+1$ then

$$f(n,b,d) = \begin{cases} \max\{\binom{2b+1}{2},\ [\frac{b(n+d-b)}{2}]\} & \text{if } n \le b+d, \\ bd & \text{if } n > b+d . \end{cases}$$

In proving that f(n,b,d) cannot exceed the values given by our Theorem, we shall make use of Berge's matching formula [2]

$$\beta_1(G) = \min \frac{1}{2}\,(p(G) + |S| - k_0(G-S)) . \qquad (2)$$

Here $\beta_1(G)$ is the maximum number of independent edges in G , $P(G)$ is the number of vertices of G , S runs through all the subsets of the vertex-set of G and finally, $k_0(G-S)$ is the number of odd components (i.e., components with odd number of vertices) of the S-deleted graph G-S .

On the other hand, we shall construct (n,b,d)-graphs having f(n,b,d) edges. Then we shall use the following simple proposition: given any nonnegative integers n_1, n_2, d with $1 < d < n_1+n_2$ and $(d-1)n_1 + dn_2$ even, there is a graph G with n_1+n_2 vertices, n_1 of them of degree d-1 and the remaining n_2 of degree d . Actually, this statement is a corollary of a general existence theorem due to Erdös and Gallai [4]: Let $d_1 \geq d_2 > . . . \geq d_n$ be nonnegative integers. A necessary and sufficient condition for the existence of a graph G with n vertices $u_1, u_2, \ldots, u_n$, each u_i of degree d_i , is that $\sum_{i=1}^{n} d_i$ be even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}$$

for each $k = 1,2,\ldots,n-1$.

We conclude this section with two observations made by Hanson [6]. Firstly, Sauer's formula (1) appears to be a corollary of the theorem. Indeed, one has

$$\varphi(2,k) = \max_n f(n,k-1,k-1) = \lim_{n\to\infty} f(n,k-1,k-1) = (k-1)^2 + \left[\frac{k-1}{[\frac{k}{2}]}\right]\cdot[\frac{k-1}{2}] .$$

Similarly, the theorem implies that a graph with n vertices and at most b independent edges can have at most

$$f(n,b,n-1) = \max\{\binom{2b+1}{2}, \ b(n-b) + \binom{b}{2}\}$$

edges. This has been proved by Moon [8]. As noticed by J. A. Bondy, Moon's result follows instantly from Berge's matching formula (2).

2. Upper Bounds

LEMMA 1.

$$f(n,b,d) \leq \max\left(\min\left\{dn_0, \left[\frac{n_0(d+n-n_0)}{2}\right]\right\} + \sum_{i=1}^{m} \min\left\{\binom{n_i}{2}, \left[\frac{dn_i}{2}\right]\right\}\right)$$

where the maximum runs over all partitions

$$n = n_0 + n_1 + n_2 + \ldots + n_m$$

into nonnegative integers with $m = n+n_0-2b$ and all n_i $(1 \leq i \leq m)$ odd.

Proof. Let $G = (V,E)$ be an arbitrary (n,b,d)-graph. By Berge's formula (2), there is a set $S \subset V$ with $k_0(G-S) \geq n + |S| - 2b$. Let the odd components of G be $G_1, G_2, \ldots, G_M$. Then $M > m = n + |S| - 2b$. Let us denote $|S|$ by n_0 and the number of vertices of each G_i $(1 \leq i < m)$ by n_i; let us also set

$$n_m = n - \sum_{i=0}^{m-1} n_i .$$

Then n_m has the parity of $n-n_0-(m-1) = 2b-2n_0+1$ and so all n_i's with $1 \leq i \leq m$ are odd. We denote by x the number of edges of G having both endpoints in S, by y the number of edges of G having exactly one endpoint in S . For each $i = 1,2,\ldots,m-1$ we denote by z_i the number of edges of G_i and finally we denote by z_m the number of the remaining edges in G . Obviously, we have

$$2x + y \leq dn_0 \tag{3}$$

$$y \leq n_0(n-n_0) \tag{4}$$

$$z_i \leq \min\left\{\binom{n_i}{2}, \left[\frac{dn_i}{2}\right]\right\} \qquad (1 \leq i \leq m) \ . \tag{5}$$

Summing (3) and (4) and using the integrality of $x+y$ we obtain

$$x + y \leq \left[\frac{n_0(d+n-n_0)}{2}\right] \ . \tag{6}$$

Besides, (3) itself implies

$$x+y \leq dn \ . \tag{7}$$

Now, the desired conclusion follows from (5),(6),(7) and the fact that G has exactly $x + y + z_1 + z_2 + \ldots + z_m$ edges.

LEMMA 2.

$$f(n,b,d) \leq bd + \left[\frac{\frac{b}{d+1}}{2}\right] \cdot \left[\frac{d}{2}\right] \ . \tag{8}$$

(In particular, $f(n,b,d) \leq bd$ whenever $d \geq 2b+1$.) Besides, if d is odd then

$$f(n,b,d) < bd + \left[\frac{2(n-b)}{d+3}\right] \qquad y \ . \tag{9}$$

Proof: Let n , b , d be given. For each positive integer s , we set

$$g(s) = \min\{\binom{s}{2}, [\frac{ds}{2}]\} = \begin{cases} \binom{s}{2} & \text{if } s \le d+1 , \\ [\frac{ds}{2}] & \text{if } s \ge d+1 . \end{cases}$$

To each partition

$$n = n_0 + n_1 + \ldots + n_m \qquad (10)$$

with $n_1 \ge n_2 > \ldots > n_m$ and all the n_i's $(i = 1,2,\ldots,m)$ odd, we assign a positive integer -- the smallest subscript $k \ge 1$ such that $n_i = 1$ for all $i > k$. Among all the partitions (10) which maximize

$$\min\left\{dn_0 , \left[\frac{n_0(d+n-n_0)}{2}\right]\right) + \sum_{i=1}^{m} g(n_i) , \qquad (11)$$

we choose one with minimum k .

If $k > 1$ then necessarily $n_i \ge d+1$ for all i with $1 \le i \le k$. Indeed, it is not difficult to check that

$$s \le d \Rightarrow g(s) + g(t) \le g(s+t-1) .$$

Now, if $n_k \le d$ then set $n_k^* = 1$, $n_{k-1}^* = n_{k-1} + n_k - 1$ and $n_i^* = n_i$ $(i \ne k-1,k)$. Then

$$\sum_{i=1}^{m} g(n_i) \le \sum_{i=1}^{m} g(n_i^*)$$

and so the partition $n = n_0^* + n_1^* + \ldots + n_m^*$ maximizes (11). However, we have

$$|\{i: i > 1, n_i^* = 1\}| > |\{i : i > 1, n_i = 1\}|$$

contradicting the minimality of k .

Now, we shall distinguish three cases.

<u>Case 1</u>. $n_1 \le d$. Then necessarily $k = 1$ and so $n_1 = n-n_0-(m-1) = 2b-2n_0+1$. Since $1 \le n_1 < d$, we have

$$b - \frac{d-1}{2} \le n_0 \le b \quad . \tag{12}$$

Lemma 1 yields

$$f(n,b,d) \le dn_0 + \sum_{i=1}^{m} g(n_i) = dn_0 + \binom{n_1}{2} = dn_0 + \binom{2b-2n_0+1}{2} \quad .$$

Since $F(n_0) = dn_0 + \binom{2b-2n_0+1}{2}$ is a convex function with $F(b - \frac{d-1}{2}) = F(b) = bd$ and n_0 satisfies the constraints (12), we have $f(n,b,d) \le bd$. Hence in this case both inequalities (8),(9) are satisfied.

<u>Case 2</u>. $n_1 > d+1$, d even. Here Lemma 1 gives

$$f(n,b,d) < dn_0 + \sum_{i=1}^{m} g(n_i) = dn_0 + \sum_{i=1}^{k} \frac{dn_i}{2} = dn_0 + \frac{d}{2} \sum_{i=1}^{k} n_i =$$

$$= dn_0 + \frac{d}{2}(n-n_0-(m-k)) = bd + k$$ [illegible] $.

Besides, we have $k(d+1) < \sum_{i=1}^{k} n_i = n-n_0-(m-k) = 2b+k-2n_0 > 2b+k$ and so $k \le [\frac{2b}{d}]$ .But then

$$f(n,b,d) \le bd + k \cdot \frac{d}{2} \le bd + [\frac{2b}{d}] \cdot \frac{d}{2}$$

which is the desired inequality (8).

Case 3. $n_1 \geq d+1$, d odd. Again, Lemma 1 yields

$$f(n,b,d) \leq dn_0 + \sum_{i=1}^{m} g(n_i) = dn_0 + \sum_{i=1}^{k} \frac{dn_i - 1}{2} = dn_0 + \frac{d}{2}\sum_{i=1}^{k} n_i - \frac{k}{2} =$$

$$= dn_0 + \frac{d}{2}(n - n_0 - (m-k)) - \frac{k}{2} = bd + k \cdot \frac{d-1}{2}$$

We have $n_i \geq d+1$ whenever $1 \leq i \leq k$. Moreover, each n_i $(1 \leq i \leq k)$ is odd while $d+1$ is even. Hence we have $n_i \geq d+2$ whenever $1 \leq i \leq k$.

Besides, we have $k(d+2) \leq \sum_{i=1}^{k} n_i = n - n_0 - (m-k) = 2b - 2n_0 + k$ and so

$$k \leq \left[\frac{2b-2n_0}{d+1}\right]$$

If $n_0 > 2b - n + \frac{2n-2b}{d+3}$ then

$$k \leq \left[\frac{2b-2n_0}{d+1}\right] \leq \left[\frac{2n-2b}{d+3}\right];$$

if $n_0 \leq 2b - n + \frac{2n-2b}{d+3}$ then

$$k \leq n + n_0 - 2b \leq \left[\frac{2n-2b}{d+3}\right].$$

Moreover, since $n_0 \geq 0$, we have $k \leq \left[\frac{2b-2n_0}{d+1}\right] \leq \left[\frac{2b}{d+1}\right]$. The inequalities

$$k \leq \left[\frac{2(n-b)}{d+3}\right], \quad k \leq \left[\frac{2b}{d+1}\right], \quad f(n,b,d) \leq bd+k \cdot \frac{d-1}{2}$$

yield the desired results (8), (9).

LEMMA 3.

$$f(n,b,d) \le \max\{\binom{2b+1}{2}, [\frac{b(d+n-b)}{2}]\}.$$

Proof: Let n, b, d be given and let (10) be a partition which maximizes (11). Then Lemma 1 yields

$$f(n,b,d) \le \frac{n_0(d+n-n_0)}{2} + \sum_{i=1}^{m} \binom{n_i}{2} \le \frac{n_0(d+n-n_0)}{2} + \binom{n-n_0-(m-1)}{2}.$$

This can be written as $f(n,b,d) \le H(n_0)$ where

$$H(n_0) = \frac{n_0(d+n-n_0)}{2} + \binom{2b-2n_0+1}{2}$$

is a convex function of n_0. Since $n_0 = n - \sum_{i=1}^{m} n_i \le n-m = 2b-n_0$, we have $0 \le n_0 < b$. Therefore

$$f(n,b,d) \le \max\{H(0),H(b)\} = \max\left\{\binom{2b+1}{2}, \frac{b(d+n-b)}{2}\right\}$$

and the desired result follows by integrality of $f(n,b,d)$.

3. Constructions

LEMMA 4. If d is odd, $n > 2b$ and

$$[\frac{2(n-b)}{d+3}](d-1) > (n-2b)d \qquad (13)$$

then $f(n,b,d) \ge [\frac{nd}{2}]$.

Proof: Set $m = [\frac{2(n-b)}{d+3}]$ and $n_0 = 2b-n+m$. As $2b \le n$, we have

$$2b(1 - \frac{1}{d+3}) \le n(1 + \frac{1}{d+1} - \frac{2}{d+3})$$

and so

$$n_0 = 2b - n + [\frac{2(n-b)}{d+3}] \le \frac{n}{d+1}$$

which can be written as

$$n_0 d < n - n_0 \quad .$$

Besides,, (13) yields $n_0 d \ge m$. Now, let us set

$$a = \begin{cases} dn_0 & \text{if } n \text{ is even ,} \\ dn_0 - 1 & \text{if } n \text{ is odd .} \end{cases}$$

We have

$$a \equiv dn_0+n \equiv n_0+n \equiv n_0+n-2b \equiv m \pmod 2 \tag{14}$$

and

$$n-n_0 \ge dn_0 \ge a \ge dn_0-1 \ge m-1 \quad . \tag{15}$$

Set $n_i = d+2$ for $i = 1,2,\ldots,m-1$ and

$$n_m = n - \sum_{i=0}^{m-1} n_i = 2n - 2b - m(d+3) + d + 2 > d + 2 \quad .$$

By (14) and (15), $a-m$ is an even nonnegative integer not exceeding $\sum_{i=1}^{m} (n_i-1)$. Let s be the greatest integer with $a-m \ge \sum_{i=1}^{s} (n_i-1)$; then $0 < s \le m$. Set

$$a_i = \begin{cases} n_i & \text{if } 1 \leq i \leq s \, , \\ a-m-\sum_{i=1}^{s}(n_i-1)+1 & \text{if } i = s+1 \, , \\ 1 & \text{if } s+1 < i < m \, . \end{cases}$$

Obviously, each a_i is odd and $\sum_{i=1}^{m} a_i = a$. Take disjoint graphs $G_1, G_2, \ldots G_m$ where each G_i has exactly n_i vertices, a_i of them of degree $d-1$ and the remaining $n_i - a_i$ of degree d. The a vertices of $G_1 \cup G_2 \cup \ldots \cup G_m$ having degree $d-1$ will be enumerated as $u_1, u_2, \ldots, u_a$. Add a new set S of n_0 vertices $v_1, v_2, \ldots,$ join each v_i to all the vertices u_j with $(i-1)d < j \leq \min(id, a)$ and call the resulting graph G.

If $a = dn_0$ (i.e., if n is even) then all the n vertices of G have degree d; if $a = dn_0 - 1$ (i.e., if n is odd) then $n-1$ vertices of G have degree d and the last one has degree $d-1$. In both cases, G has $[\frac{nd}{2}]$ edges. Since $k_0(G-S) > m$, G contains at most b independent edges.

<u>LEMMA 5</u>. If d is odd, $n > 2b$ and

$$[\frac{2(n-b)}{d+3}](d-1) < (n-2b)d \qquad (16)$$

then

$$f(n,b,d) > bd + [\frac{2(n-b)}{d+3}]$$

<u>Proof</u>: Set $m = [\frac{2(n-b)}{d+3}]$ and $n_0 = 2b-n+m$. Then (16) yields $n_0 d < m$. Set $n_i = d+2$ for $i = 1,2,\ldots,m-1$ and

$$n_m = n - \sum_{i=0}^{m-1} n_1 = 2n - 2b - m(d+3) + d + 2 > d + 2 \quad .$$

Take disjoint graphs $G_1, G_2, \ldots, G_m$ where each G_i has n_i-1 vertices of degree d and one vertex u_i of degree d-l . Add a new set S of n_0 vertices $v_1, v_2, \ldots,$ join each v_i to all the vertices u_j with $(i-l)d < j \leq id$ and call the resulting graph G . Obviously, all but $m-n_0 d$ vertices of G have degree d ; the remaining $m-n_0 d$ vertices have degree d-l . Hence G has exactly

$$\frac{1}{2}(nd - (m-n_0 d)) = bd + m \cdot \frac{d-1}{2}$$

edges. Since $k_0(G-S) = m = n - 2b + |S|$, G contains at most b independent edges.

To make this paper self-contained, we need three more lemmas; these are due to Hanson [6].

LEMMA 6. If $d \leq 2b$ and $n \geq 2b + \left\lceil \frac{b}{[\frac{d+1}{2}]} \right\rceil$ then

$$f(n,b,d) \geq bd \left\lceil \frac{b}{[\frac{d+1}{2}]} \right\rceil \cdot [\frac{d}{2}] \quad .$$

Proof: Case 1, d odd. Set $m = [\frac{2b}{d+1}]$, $n_i = d+2$ for $i = 1,2,\ldots,m-1$ and

$$n_m = 2b + m - (m-1)(d+2) = 2b - m(d+1) + d + 2 > d + 2 \quad .$$

Take disjoint graphs $G_1, G_2, \ldots, G_m$ where each G_i has n_i-1 vertices of degree d and one of degree d-l . Add n-(2b+m) isolated vertices and call the resulting graph G . Clearly, G has

$$\frac{1}{2}((2b+m)d-m) = bd + m \cdot \frac{d-1}{2}$$

edges and at most

$$\sum_{i=1}^{m}\left[\frac{n_i}{2}\right] = \sum_{i=1}^{m}\frac{n_i-1}{2} = \frac{1}{2}\left(\sum_{i=1}^{m} n_i - m\right) = b$$

independent edges.

Case 2, d even. Set $m = [\frac{2b}{d}]$, $n_i = d+1$ for $i = 1,2,\ldots,m-1$ and

$$n_m = 2b + m - (m-1)(d+1) = 2b - md + d + 1 > d + 1 .$$

Take disjoint graphs $G_1, G_2, \ldots, G_m$ where each G_i has n_i vertices, all of degree d. Add $n-(2b+m)$ isolated vertices and call the resulting graph G. Clearly, G has

$$\frac{1}{2}(2b+m)\cdot d = bd+m$$

edges and at most

$$\sum_{i=1}^{m}\left[\frac{n_i}{2}\right] = \sum_{i=1}^{m}\frac{n_i-1}{2} = \frac{1}{2}\left(\sum_{i=1}^{m} n_i - m\right) = b$$

independent edges.

LEMMA 7. If d is even, $d \le 2b$ and $n \le 2b + [\frac{2b}{d}]$ then $f(n,b,d) \ge \frac{nd}{2}$.

Proof: Set $m = n-2b$; then $m(d+1) \le n$. For each $i = 1,2,\ldots,m-1$, set $n_i = d+1$; set also $n_m = n-(m-1)(d+1) > d+1$. Let G be a disjoint union of graphs $G_1, G_2, \ldots, G_m$ where each G_i has n_i vertices, all of degree d. Then G has exactly $\frac{1}{2}dn$ edges and at most

$$\sum_{i=1}^{m}\left[\frac{n_i}{2}\right] = \sum_{i=1}^{m}\frac{n_i-1}{2} = \frac{1}{2}(n-m) = b$$

independent edges.

LEMMA 8.

(i) If $d \geq 2b$, $n \geq 2b+1$ then $f(n,b,d) \geq \binom{2b+1}{2}$.

(ii) If $d > b$, $d+1 \leq n \leq d+b$ then $f(n,b,d) > [\frac{b(n+d-b)}{2}]$.

(iii) If $d > b$, $n \geq b+d$ then $f(n,b,d) \geq bd$.

Proof:

(i) Take a complete graph with $2b+1$ vertices, add $n-(2b+1)$ isolated vertices.

(ii) If b(d-n+b) is odd, take a graph G_0 with b-1 vertices of degree d-n+b and one of degree d-n+b-1 . If $b(d-n+b)$ is even, take a graph G_0 with b vertices of degree $d-n+b$. Add n-b new vertices, join each of them to all the vertices of G_0 and call the resulting graph G . Obviously, the degrees of vertices of G do not exceed $\max\{d,b\} = d$; since each edge of G has at least one endpoint in G_0 , we conclude that G has at most b independent edges. Finally, G has exactly

$$[\frac{b(d-n+b)}{2}] + b(n-b) = [\frac{b(n+d-b)}{2}]$$

edges.

(iii) Take a complete bipartite graph with b vertices in one part and d in the other; add $n-(b+d)$ isolated vertices.

References

[1] H. L. Abbott, "Some remarks on a combinatorial theorem of Erdös and Rado," Canad. Math. Bull. 9 (1966), 155-160.

[2] C. Berge, "Sur le couplage maximum d'un graphe," C. R. Acad. Sci. Paris 247(1958),258-259.

[3] V. Chvátal, "On finite delta-systems of Erdös and Rado," Acta Math. Acad. Sci. Hungar. 21 (1970), 341-355.

[4] P. Erdös and T. Gallai, "Graphs with prescribed degrees of vertices," (Hungarian), Mas. Lápok 11 (1960), 264-274 (see also [7], Chapter 6).

[5] P. Erdös and R. Rado, "Intersection theorems for systems of sets," J. London Math. Soc. 35 (1960), 85-90.

[6] D. Hanson, "An extremal problem in graph theory," to appear in J. Combinatorial Theory.

[7] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass. 1969.

[8] J. W. Moon, "On independent complete subgraphs in a graph," Canad. J. Math. 20 (1968), 95-102.