# CHROMATIC AUTOMORPHISMS OF GRAPHS

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V. CHVATAL AND J. SICHLER

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V. Chvátal and J. Sichler
Computer Science Dept., Stanford University, Stanford, Calif. 94305
Dept. of Mathematics, University of Manitoba, Winnipeg, Canada

Abstract

The coloring group and the full automorphism group of an n-chromatic graph are independent if and only if n is an integer $\geq 3$ .

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## 1. Introduction.

When coloring highly symmetric graphs, one often finds that the symmetries of a given graph determine to a certain extent the symmetries of its minimal colorings. We will say that an automorphism a and a coloring

$$c : V \longrightarrow R \qquad (1)$$

of a graph $H = (V,E)$ are compatible if there is a bijection $p : R \longrightarrow R$ with $c(a(v)) = p(c(v))$ for all $v \in V$. One might expect that a graph H having at least one non-identical automorphism always admits a non-identical automorphism a compatible with some minimal coloring of H (a minimal coloring of H is a coloring (1) with $|R|$ equal to the chromatic number $\chi(H)$ of H ). However, this is not always the case. The 3-chromatic graph H in Fig.1 admits 30 distinct 3-colorings and four distinct non-identical automorphisms but none of the 120 pairs are compatible.

(Fig.1 )

In discussions with Dr. Jarik Nešetřil of Charles University, we were led to the concept of a chromatic automorphism of H : this is an automorphism compatible with every minimal coloring of H . Obviously, the chromatic automorphisms form a subgroup C(H) of the full automorphism group A(H) of H . Besides, C(H) is always a normal subgroup of A(H) . To see this, let f be an arbitrary automorphism of H and let $a \in C(H)$ . If c is a minimal coloring of H , then $c \circ f^{-1}$ is another such coloring and there is a $p : R \longrightarrow R$ such that $c \circ (\circ f^{-1} a \circ f) = (c \circ f^{-1}) \circ a \circ f =$ $= p \circ (c \circ f^{-1}) \circ f = p \circ c$ , that is, $f^{-1} a \circ f \in C(H)$ .

It is well-known that any group G is isomorphic to the full automorphism group of some graph H (Frucht [1] has been the first to prove this). Now, it is natural to ask which pairs (G,N) - where G is a group and N a normal subgroup of G - are representable as (A(H),C(H)) of some graph H . The answer is given in the next section.

## 2. The main result.

**THEOREM.** Let G be a group and let N be a normal subgroup of G . Let $n \geq 3$ be an integer. Then there exists an n-chromatic graph H with $A(H) \cong G$ and $C(H) \cong N$ .

**Proof.** If G is the one-element group, then the statement follows immediately from the main result of [2] . From now on we shall assume that $|G| > 1$ .

A graph H with the required properties will be constructed. To help the reader, we give first an informal description of the construction with $n = 3$ and then proceed in a more precise manner. Let e be the unit element of G and let $<$ be an arbitrary well-ordering of the set $G - \{e\}$ . For each pair $(x,y) \in (G - \{e\})^2$ with $x < y$ we take a copy of the graph in Fig.3, for each pair $(x,y) \in G^2$ we take a copy of the graph in Fig.2. Identifying all the vertices with equal labels we obtain the desired 3-chromatic graph H.

(Fig.2)

(Fig.3)

More generally and more precisely, we set

$$\mathrm{ess}G^2 = \{(x,y) : x,y \in G \ , \ x \neq j\},$$

$$R = \{(x,y) : x,y \in G - \{e\} \ , \ x < y\}.$$

The vertex-set of H will be $V = V_1 \cup V_2 \cup \ldots \cup V_6$ , where

$$V_1 = G \times \{1\} ,$$

$$V_2 = (G - \{e\}) \times \{2\} ,$$

$$V_3 = (G - \{e\}) \times \{3\} ,$$

$$V_4 = \mathrm{ess}G^2 \times \{1,2,\ldots,2n-1\} ,$$

$$V_5 = G/N \times (G - \{e\}) \times \{1,2,\ldots,n-1\} ,$$

$$V_6 = \{1,2,3\} \times R .$$

The edges of H will be all the two-point sets

$$\{((x,y),j),((x,y),k)\} \quad (0 < |j-k| < n),$$

$$\{(x,1),((x,y),j)\} \quad (0 < j < n) ,$$

$$\{(y,1),((x,y),j)\} \quad (n < j < 2n) ,$$

$$\{(x,1),(xN,z,j)\} ,$$

$$\{((x,y),j),(xN,x^{-1}y,k)\} \quad (0 < j < n, \ j \neq k) ,$$

$$\{(z,2),(z,3)\} ,$$

$$\{(z,2),(xN,z,j)\} ,$$

$\{(x,3),(j,(x,y))\}$ $(j \in \{1,2,3\})$ ,

$\{(y,3),(j,(x,y))\}$ $(j \in \{2,3\})$ ,

$\{(2,(x,y)),(j,(x,y))\}$ $(j \in \{1,3\})$

and no other ones. Now, we will show that the graph described above has all the desired properties.

Let a be an arbitrary automorphism of G . First of all, we note that the elements of $V_2$ are the only vertices of H not contained in any triangle of H . Therefore $a(V_2) = V_2$ . When $V_2$ is removed, the resulting graph has just two components: the component induced by $V_3 \cup V_6$ , which contains vertices of degree two in H , while the other component , induced by $V_1 \cup V_4 \cup V_5$ , contains no such vertices. Thus $a(V_3 \cup V_6) = V_3 \cup V_6$ and $a(V_1 \cup V_4 \cup V_5) = V_1 \cup V_4 \cup V_5$ . The elements of $V_3$ are the only vertices of the first component which are adjacent to the elements of $V_2$ , so that $a(V_3) = V_3$ and $a(V_6) = V_6$ . A similar argument applied to $V_1 \cup V_4 \cup V_5$ yields $a(V_5) = V_5$ and $a(V_1 \cup V_4) = V_1 \cup V_4$ . Since the group G is non-trivial, the degrees of the elements of $V_1$ are not smaller than 3n-3 , while $V_4$ contains only vertices whose degrees do not exceed 3n-4 . Thus $a(V_1) = V_1$ and $a(V_4) = V_4$ . Altogether, $a(V_i) = V_i$ for i = 1,2,...,6 .

So we are in position enabling us to define bijections $a': G - \{e\} \longrightarrow G - \{e\}$ and $a^*: G \longrightarrow G$ by

$$a(x,2) = (a'(x),2) , \quad a(x,1) = (a^*(x),1) .$$

Since $(x,3)$ is the only element of $V_3$ adjacent to $(x,2)$, we have $a(x,3) = (a'(x),3)$. Moreover, it is easy to see that $x < y$ if and only if H has a vertex v of degree two whose distance from $(y,3)$ is two and which is adjacent to $(x,3)$. Consequently,

$$x < y \text{ if and only if } a'(x) < a'(y). \qquad (2).$$

A well-ordered set, however, is a rigid structure: the only bijective transformation a' satisfying (2) is the identity mapping. Hence $a'(x) = x$ for all $x \in G - \{e\}$; we conclude that $a(u) = u$ for all $u \in V_2 \cup V_3$, which yields $a(u) = u$ for all $u \in V_6$ as an easy consequence.

The vertex $((x,y),n-1)$ is the only vertex in $V_4$ of degree $3n-4$ which is adjacent to $(x,1)$ and has distance two from $(y,1)$. Hence $a((x,y),n-1) = (a^*(x),a^*(y),n-1)$. Now, by a series of similar easy arguments, there it follows that $a((x,y),j) = (a^*(x),a^*(y),j)$ for all $j = 1,2,\dots,2n-1$. Since $(xN,x^{-1}y,k)$ is the only vertex in $V_5$ adjacent to all $((x,y),j)$ with $0 < j < n$, $j \neq k$, the equality $a(xN,x^{-1}y, j) = (a^*(x)N,a^*(x)^{-1}a^*(y),j)$ must hold. Finally, $(x^{-1}y,2)$ is the only vertex in $V_2$ adjacent to each $(xN,x^{-1}y,j)$; hence $a(x^{-1}y,2) = (x^{-1}y,2)$ must also be adjacent to $(a^*(x)N,a^*(x)^{-1}a^*(y),j)$ for all $j$. Consequently,

$$a^*(x)^{-1}a^*(y) = x^{-1}y \qquad (3)$$

whenever $(x,y) \in \mathrm{ess}G^2$. Setting $x = e$ in (3) and writing $z = a^*(e)$ we obtain

$$a^*(y) = zy \tag{4}$$

for all $y \neq e$ ; $a^*(e) = z$ by definition. Our findings can be summarized as follows. Given any $a \in A(H)$ there is a $z_a = z \in G$ such that

$$\left.\begin{aligned} &a(x,1) = (zx,1) \ , \\ &a((x,y),j) = ((zx,zy),j) \ , \\ &a(xN,w,j) = (zxN,w,j) \ , \\ &a(u) = u \quad \text{for all } u \in V_2 \cup V_3 \cup V_6 \ . \end{aligned}\right\} \tag{5}$$

Conversely, it is easy to verify that the formulas (5) define an automorphism of $H$ for an arbitrary $z \in G$ . It is clear that the assignment $a \longmapsto z_a$ is a group isomorphism of $A(H)$ onto $G$ .

It is quite obvious that $H$ is n-chromatic. Given any two vertices $u$, $v$ of $H$ , $u \sim v$ will mean that $c(u) = c(v)$ for each n-coloring $c$ of $H$ . It is not difficult to see that

$$(x,1) \sim ((x,y),n) \sim (y,1) \sim (x^{-1}y,2) \ ,$$

$$((x,y),j) \sim ((x,y),j+n) \sim (xN,x^{-1}y,j) \qquad (0 < j < n) \ .$$

If $z = z_a \in N$ , then $zxN = xN$ for all $x \in G$ ; the corresponding automorphism $a$ (defined by (5)) satisfies

$$a(u) = u \quad \text{for all } u \in V_2 \cup V_3 \cup V_5 \cup V_6 \ ,$$

$$a(u) \sim u \quad \text{whenever } u \in V_1 \text{ or } u = ((x,y),n) \ ,$$

$$a((x,y),j) = ((zx,zy),j) \sim (zxN,(zx)^{-1}(zy),j) = (xN,x^{-1}y,j) \sim ((x,y),j) \ ,$$

$$a((x,y),j+n) = ((zx,zy),j+n) \sim ((zx,zy),j) \sim$$
$$\sim ((x,y),j) \sim ((x,y),j+n) ,$$

whenever $0 < j < n$ . Altogether, we have $a(u) \sim u$ for all $u \in V$ ; a is compatible with every minimal coloring, i.e., $a \in C(H)$ .

Conversely, let $z = z_a \in G - N$ . Set $p(1) = 2$ , $p(2) = 1$ , $p(j) = j$ for $j = 3,\ldots,n$ and define a mapping $c : V \longrightarrow \{1,\ldots,n\}$ by

$$c(u) = n \quad (u \in V_1 \cup V_2) ,$$
$$c((x,y),n) = n ,$$
$$c(u) = 1 \quad (u \in V_3) ,$$
$$c(2,(x,y)) = 2 ,$$
$$c(1,(x,y)) = c(3,(x,y)) = 3 ,$$
$$c(N,w,j) = c((x,y),j) = c((x,y),j+n) = j \ (0 < j < n) \quad \text{if } x \in N ,$$
$$c(xN,w,j) = c((x,y),j) = c((x,y),j+n) = p(j) \quad (0 < j < n) \text{ if } x \notin N .$$

It is easy to verify that c is a coloring of H . Let us note that $c(N,w,1) = 1 = c(x,3)$ ; however, $c(a(N,w,1)) = c(zN,w,1) = p(1) = 2$ , while $c(a(x,3)) = c(x,3) = 1$ . Hence a is not compatible with c , $a \notin C(H)$ . We have shown that an automorphism a is chromatic if and only if $z_a \in N$ . Thus $C(H) \cong N$ - which finishes the proof.

## 3. Concluding remarks.

Our theorem is best possible in the sense that the range of the chromatic number n of the representing graph cannot be extended without imposing additional restriction on the choice of N and G . The case n = 1 is trivial: every graph $H = (V,E)$ with $\chi(H) = 1$ has $A(H) = C(H) \cong Sym_{|V|}$ . The smallest pair (G,N) which is not realizable as (A(H),C(H)) of a 2-colorable H is $(C_3,\{e\})$ . Indeed, if $H = (V,E)$ is a 2-chromatic graph with $A(H) \cong C_3$ and $C(H) \cong \{e\}$ , then H must be disconnected (otherwise H is uniquely colorable and every automorphism is chromatic). No two different components of H are isomorphic - if there were isomorphic components, A(H) would have an element of order two. Exactly one component has a non-trivial automorphism (otherwise $|A(H)| \geq 4$); denote this component by $H_0$ and the rest of the graph by $H_1$ . Let a be one of the two non-trivial automorphisms of H ; a is not chromatic. Let c be a 2-coloring of H which is not compatible with a . Since $H_0$ is uniquely colorable, $c(u) = c(v)$ is equivalent to $c(a(u)) = c(a(v))$ for all $u,v \in H_0$ . As a is not compatible with c , $c(a(u)) = 2$ if $c(u) = 1$ and $c(a(v)) = 1$ if $c(u) = 2$ for all $u \in H_0$ . But then $c(a^3(u)) \neq c(u)$ , which is a contradiction as $a^3$ is the identity mapping.

Finally, we will show that $(C_3,C_3)$ is not realizable as

(A(H),C(H)) of a graph H with infinite chromatic number n . Assume that there is such a graph H . It contains at most one vertex adjacent to all other vertices (if there were two such vertices u, v , then the mapping $a : V \longrightarrow V$ defined by a(u) = v , a(v) = u , a(w) = w for all the other vertices, would be an automorphism of H ). V contains three distinct vertices u,v,w with a(u) = v , a(v) = w , a(w) = u ; at least one of them - say u - is not related to some other vertex u* . But then $\{a(u),a(u^*)\} \neq \{u,u^*\}$ ; since n+1 = n , there is a minimal coloring c of H with c(u) = c(u*) and $c(a(u)) \neq c(a(u^*))$ . a is not chromatic, $C(H) \neq A(H)$ , which is a contradiction.

## R e f e r e n c e s.


[1] R. Frucht, Herstellung von Graphen mit vorgegebener abstrakten Gruppe, Compositio Math. 6(1938), 239 - 250 .

[2] P. Hell, Full embeddings into some categories of graphs, to appear in Alg. Universalis .

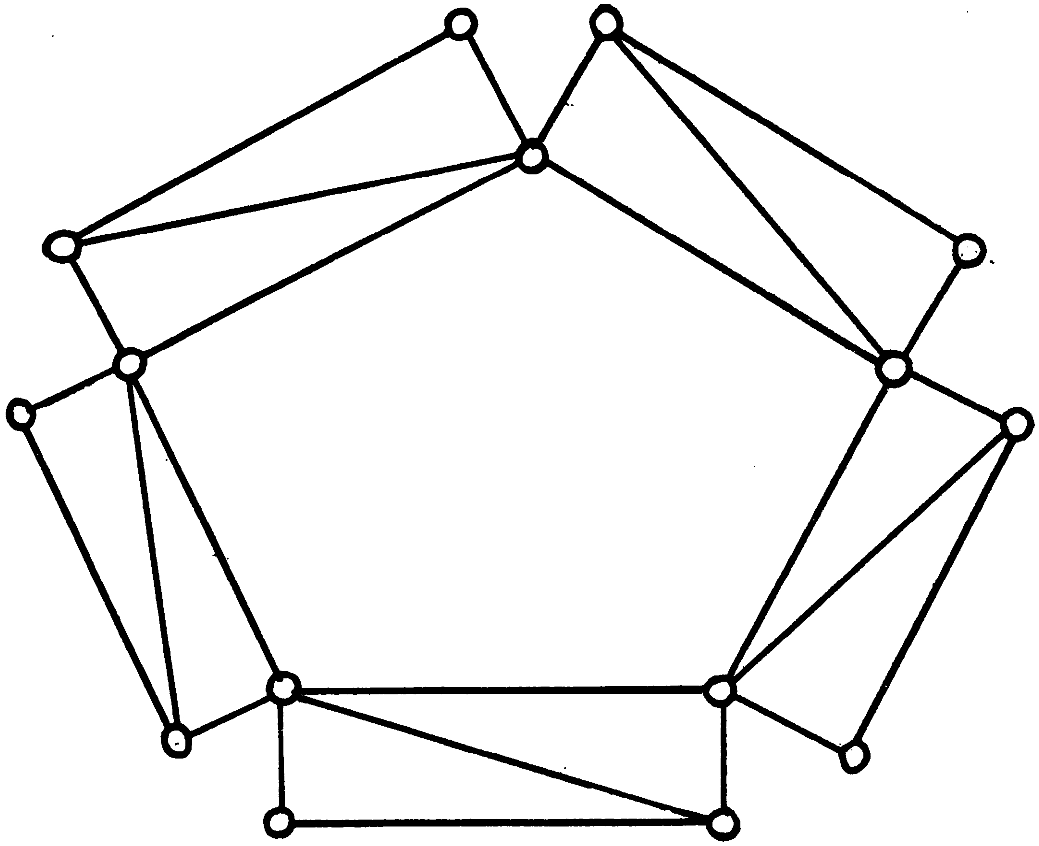

Fig. 1

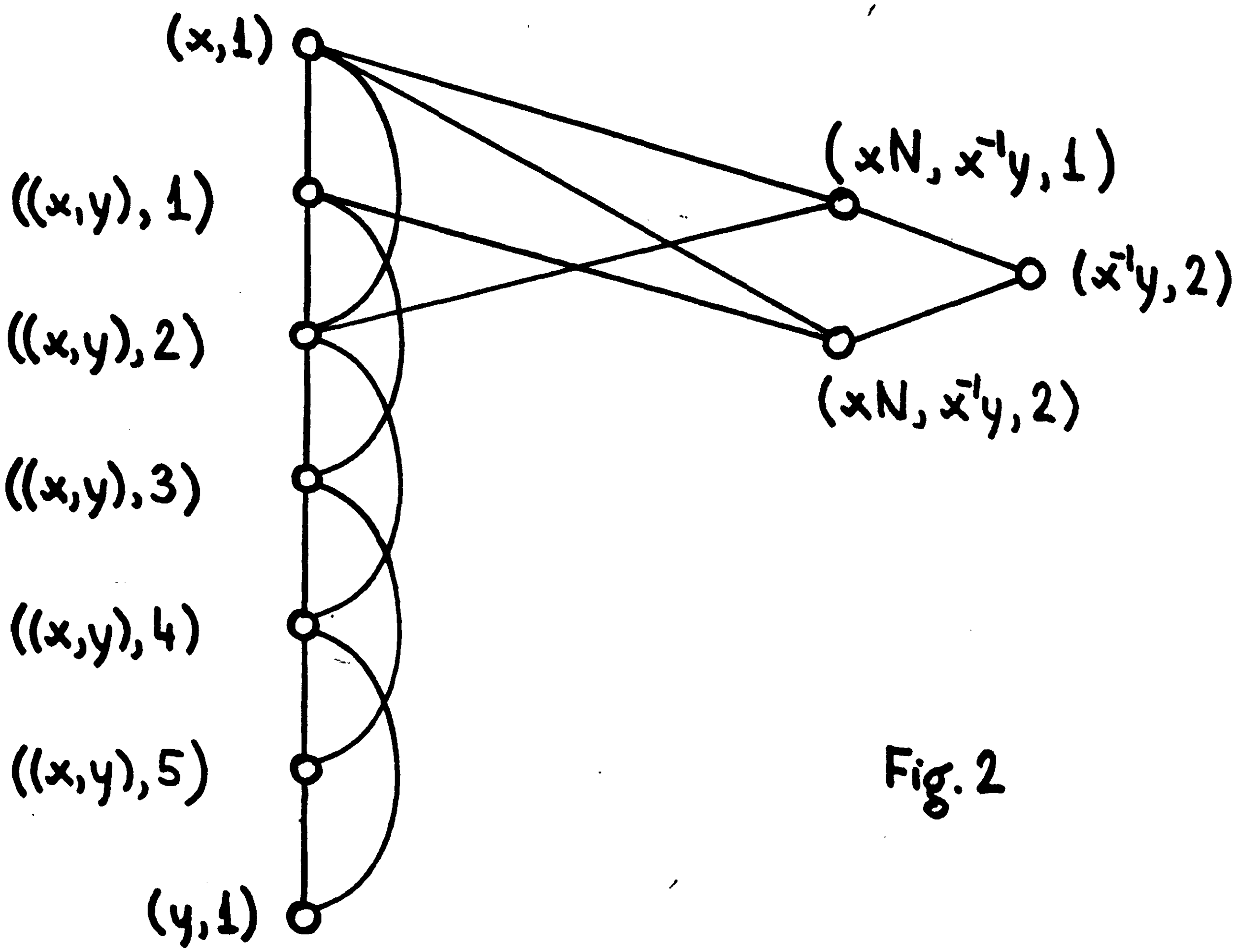
(x,1)
((x,y),1)
((x,y),2)
((x,y),3)
((x,y),4)
((x,y),5)
(y,1)
(xN, $x^{-1}$y, 1)
($x^{-1}$y, 2)
(xN, $x^{-1}$y, 2)

Fig. 2

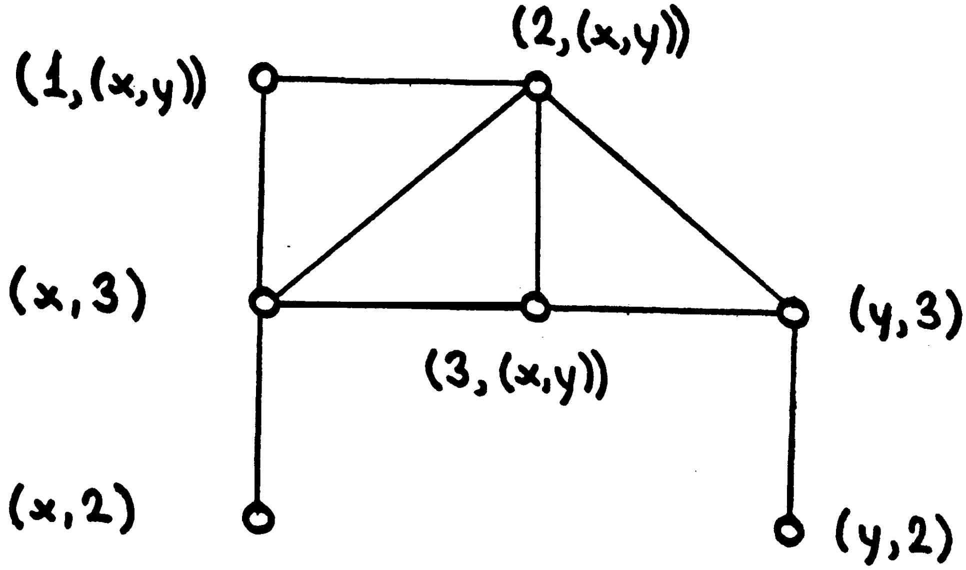
(1,(x,y))
(2,(x,y))
(x,3)
(3,(x,y))
(y,3)
(x,2)
(y,2)

Fig. 3