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ABSTRACT

Let S be a set of linear inequalities that determine a bounded polyhedron P. The closure of S is the smallest set of inequalities that contains S and is closed under two operations: (i) taking linear combinations of inequalities, (ii) replacing an inequality $\sum a_j x_j \leq a_0$, where $a_1, a_2, \ldots, a_n$ are integers, by the inequality $\sum a_j x_j \leq a$ with $a \geq [a_0]$. Obviously, if integers $x_1, x_2, \ldots, x_n$ satisfy all the inequalities in S then they satisfy also all the inequalities in the closure of S. Conversely, let $\sum c_j x_j \leq c_0$ hold for all choices of integers $x_1, x_2, \ldots, x_n$, that satisfy-all the inequalities in S. Then we prove that $\sum c_j x_j \leq c_0$ belongs to the closure of S. To each integer linear programming problem, we assign a nonnegative integer, called its rank. (The rank is the minimum number of iterations of the operation (ii) that are required in order to eliminate the integrality constraint.) We prove that there is no upper bound on the rank of problems arising from the search far largest independent sets in graphs.

1. Characterizations and good characterizations.

Let us examine the formal structure of the two following theorems.

THEOREM A (Tutte [17]). Let G be a (finite undirected) graph. Then the two following conditions are equivalent.

(i) G has a perfect matching (that is, a set of pairwise disjoint edges that cover all the vertices of G),

(ii) if an arbitrary set S of vectors is deleted from G, then the number $k_0(G-S)$ of odd components (that is, components having an odd number of vertices each) of the resulting graph G does not exceed $|S|$.

THEOREM B (Gallai [11]). Let G be a (finite undirected) graph. Then the two following conditions are equivalent.

(i) G is k-colorable,

(ii) the edges of G can be directed in such a way that the resulting directed graph contains no (simple directed) path having k edges.

Both of these theorems, asserting the equivalence of (i) and (ii), are characterizations? Yet there is a considerable formal difference between the two. Theorem A gives necessary and sufficient conditions for the existence of a certain structure (perfect matching in G) in terms of the absence of another structure (a set S with $k_0(G-S) > |S|$). On the other hand, Theorem B gives necessary and sufficient conditions for the existence of a certain structure (k-coloring of G) in terms of the existence of another structure (the directions of the edges of G). Another aspect of this difference can be illuminated as follows. It is easy to convince one's supervisor that G has a perfect matching. To do this, one only has to exhibit the matching. (The question of the difficulty of finding the matching is irrelevant for our discussion.) It is equally easy (with help of Theorem A) to convince the supervisor that G has no perfect matching -- one has to exhibit a set S with $k_0(G-S) > |S|$. On the other hand, while it is easy to convince the supervisor that G has a k-coloring, Theorem B gives no easy way of showing that G has no k-coloring.

Apparently Edmonds [6] has been the first to turn attention to this feature of characterizations; he introduced the term "good characterizations" for the theorems of the first type. Hence Tutte's theorem is a good characterization while Gallai's theorem is not. Needless to say, the words "good characterization" form a nonseparable entity without any reference to the emotional charge of the adjective "good". The statement "Gallai's theorem is not a good characterization" asserts nothing whatsoever about the quality and depth of the theorem.

In our further considerations, the duality theorem of linear programming will play an important role, It expresses the maximum of a linear form $\sum c_i x_i$ subject to a set of constraints (primal problem) as a minimum of another form $\sum b_i y_i$ subject to other constraints (dual problem). Hence to show that a feasible primal solution $(x_1, x_2, \ldots, x_n)$ is optimal, one only has to exhibit a feasible dual solution $(y_1, y_2, \ldots, y_m)$ with $\sum c_i x_i = \sum b_i y_i$. In a way, the duality theorem of linear programming is a prototype of a good characterization.

Our last sentence has more into it than meets the eye. Actually, Edmonds [7] has shown how to relate Theorem A to the duality theorem and made it clear that his approach can be adopted in many different settings. It is the purpose of this paper to study various questions related to Edmonds' technique.

2. Edmonds polyhedra.

Let G be a graph with vertices $v_1, v_2, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_n$; for each $j = 1, 2, \ldots, m$ we set $S(j) = \{i : v_j$ is an endpoint of $e_i\}$. The problem of finding a perfect matching in G can be formulated as the following integer linear programming problem. Maximize

$$\sum_{i=1}^{n} x_i \tag{2.1}$$

subject to the constraints

$$x_i \geq 0 \qquad (i = 1, 2, \ldots, n), \tag{2.2}$$

$$\sum_{i \in S(j)} x_i \leq 1 \qquad (j = 1, 2, \ldots, m), \tag{2.3}$$

$$x_i = \text{integer} \qquad (i = 1, 2, \ldots, n). \tag{2.4}$$

Obviously, every characteristic vector $(x_1, x_2, \ldots, x_n)$ of a set of pairwise disjoint edges satisfies (2.2), (2.3) and (2.4). Vice versa, every vector $(x_1, x_2, \ldots, x_n)$ that satisfies (2.2), (2.3), (2.4) turns out to be a characteristic vector of a set of pairwise disjoint edges of G. Hence G has a perfect matching if and only if the maximum of (2.1) subject to (2.2), (2.3), (2.4) equals $\frac{1}{2}m$.

Because of the integrality constraint (2.4), we cannot express the maximum of (2.1) in terms of the minimum of a dual problem. Besides, if (2.4) is dropped then the maximum of (2.1) can increase. For instance, if G is a triangle then $x_1 = x_2 = x_3 = \frac{1}{2}$ satisfies (2.2), (2.3) and yields $\sum x_i = \frac{3}{2}$. However, the maximum of (2.1) subject to (2.2), (2.3), (2.4) equals one in this case. Nevertheless, there is a standard way of getting around the inequality constraint. One can think of the polyhedron P (in the n-dimensional Euclidean space) determined by (2.2), (2.3). The set F of lattice points inside P is finite and its convex hull E(P) is another polyhedron. A moment's reflection shows that the maximum of (2.1) over F equals the maximum of (2.1) over E(P) -- indeed, F is a subset of E(P) while the extremum points of E(P) come from F. More generally, for any polyhedron P and any linear form $\sum c_i x_i$, the problem of maximizing $\sum c_i x_i$ over the lattice points inside P reduces into the problem of maximizing $\sum c_i x_i$ over E(P). 'The latter is an ordinary (non-integer) linear programming problem that offers the advantage of using the duality theorem as long as the list of faces of F'(P) is known,

In general, it seems extremely difficult to determine all faces of E(P) from those of P. However, in the above case -- when P is defined by (2.2), (2.3) -- the list of faces of E(P) is available. Indeed, Edmonds [7] proved that all the inequalities that determine E(P) are (2.2), (2.3), and

$$\sum_{e_i \subset S} x_i \leq k \qquad (2.5)$$

Here S runs through all sets of 2k + 1 vertices (k arbitrary) and each edge is interpreted as a two-point set. Now, the maximum of

(2.1) subject to (2.2), (2.3), (2.4) equals the maximum of (2.1) subject to (2.2), (2.3), (2.5), which is, in turn, equal to the minimum of the corresponding dual program. Therefore Edmonds' theorem (combined with the duality theorem) yields instantly a good characterization of graphs without a perfect matching as follows. A graph G has no perfect matching if and only if there are non-negative real numbers $a_1, a_2, \ldots, a_m$ and b(S), where S ranges through all odd-cardinality sets of vertices, such that for each edge e with endpoints v_j, v_k the inequality

$$a_j + a_k + \sum_{e \subset S} b(S) \geq 1$$

is satisfied and

$$\sum_{i=1}^{m} a_i + \sum_{S} \frac{1}{2} (|S| - 1)\, b(S) < \frac{1}{2} m .$$

Besides, Edmonds [8] proved that the a_i and b(S) can be chosen to be zero or one. Under this added assumption, the above characterization reduces into Tutte's theorem. Of course, Edmonds' theorem is more general; it provides a max-min formula for any <u>weighted</u> matching problem. Edmonds also generalized these results to the case of optimum-weighted degree-constrained subgraphs of a given graph (see Section VIII of [7] and also [9]). Since these are the only cases when E(P) is a proper subset of P but the description of the faces of E(P) is known, we call E(P) the <u>Edmonds polyhedron</u> of P.

We have seen that the knowledge of the faces of E(P) yields immediately a max-min formula for the corresponding integer linear programming problem. Next, we will study the relations between the faces of E(P) and those of P.

3. <u>The main theorem.</u>

It is easy to see how (2.3) and (2.4) imply (2.5). Indeed, let S be any set of 2k+1 vertices of G. Summing the inequalities (2.3) for all j with $v_j \in S$ we obtain

$$2 \sum_{e_i \subset S} x_i \leq 2k+1 \quad ,$$

or

$$\sum_{e_i \subset S} x_i \leq k + \frac{1}{2} \quad .$$

By (2.4), the left-handside of the last inequality is an integer and so (2.5) follows. This observation leads us to the definition of a closure of a set S of linear inequalities. We shall say that an inequality $\sum a_j x_j \leq b$ belongs to the <u>elementary closure</u> of S if there are inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m)$$

in S and positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\sum_{i=1}^{m} \lambda_i a_{ij} = a_j = \text{integer} \qquad (j = 1, 2, \ldots, n) ,$$

$$\left[\sum_{i=1}^{m} \lambda_i b_i \right] \leq b$$

(here [x] denotes the integer part of x). The set of all inequalities belonging to the elementary closure of S will be denoted by $e^1(S)$; for any integer $k > 1$ we define $e^k(S)$ recursively by $e^k(S) = e(e^{k-1}(S))$. Finally, we set

$$d(S) = \bigcup_{k=1}^{\infty} e^k(S) \ ;$$

the set $c(S)$ will be called the <u>closure</u> of S. Evidently, all vectors $(x_1, x_2, \ldots, x_n)$ satisfying all the inequalities in S plus the integrality constraint

$$x_1 = \text{integer} \qquad (i = 1, 2, \ldots, n)$$

satisfy also all the inequalities in $c(S)$. A converse is given-by our next result.

<u>THEOREM 1</u>. Let the inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m) \qquad (3.1)$$

(where a_{ij}, b_i are real numbers) determine a bounded polyhedron in the n-dimensional Euclidean space. Let $c_0, c_1, \ldots, c_n$ be integers such that

$$\sum_{j=1}^{n} c_j x_j \leq c_0 \qquad (3.2)$$

holds for any choice of integers $x_1, x_2, \ldots, x_n$ satisfying (3.1). Then (3.2) belongs to the closure of (3.1).

In the proof, we will use the following auxiliary result.

LEMMA 1. Let (3.1) and (3.2) be as in Theorem 1; let c be an integer such that $\sum c_j x_j < c+1$ for every choice of reals $x_1, x_2, \ldots, x_n$ satisfying (3.1). Then the inequality $\sum c_j x_j \le c$ belongs to the closure of (3.1).

PROOF of Lemma 1. Let c^* be the maximum of $\sum c_j x_j$ subject to the constraints (3.1). By one of the versions of the duality theorem (see [13], Theorem 8.3.1), there are nonnegative reals $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\sum \lambda_i a_{ij} = c_j$ $(j = 1, 2, \ldots, m)$ and $\sum \lambda_i b_i = c^*$. Since $c^* < c+1$, we have $[c^*] \le c$ and the conclusion follows.

PROOF of Theorem 1. Let c^* be the maximum of $\sum c_i x_i$ subject to (3.1); set $c = [c^*]$. By Lemma 1, the inequality $\sum c_i x_i < c$ belongs to the closure of (3.1). If $c \le c_0$ then we are done. Next, we will assume $c > c_0$ and prove that the inequality $\sum c_i x_i \le c-1$ belongs to the closure of (3.1). Repeating this process $c-c_0$ times, we arrive at the desired conclusion.

Since (3.1) determines a bounded polyhedron, there is an integer M with $|x_i| \le M$ whenever (3.1) is satisfied. By Lemma 1, the inequalities

$$-x_i \le M \qquad (i = 1, 2, \ldots, n)$$

$$x_i \le M \qquad (i = 1, 2, \ldots, n)$$

belong to the closure of (3.1). Given a vector $(s_1, s_2, \ldots, s_k)$ where $0 \le k \le n$ and $s_i \in \{-M, -M+1, \ldots, M\}$ we construct a linear form (in the x_i's) $L(s_1, s_2, \ldots, s_k)$ and a number $R(s_1, s_2, \ldots, s_k)$ recursively as follows. Firstly, for the zero-length vector $\emptyset$, we set

$$L(\emptyset) = \sum c_i x_i, \qquad R(\emptyset) = c-1 .$$

Secondly, we set

$$L(s_1, s_2, \ldots, s_k) = (M+1+s_k)\, L(s_1, s_2, \ldots, s_{k-1}) - x_k ,$$

$$R(s_1, s_2, \ldots, s_k) = (M+1+s_k)\, R(s_1, s_2, \ldots, s_{k-1}) + M .$$

It follows directly from the definition that

$$L(s_1, s_2, \ldots, s_k) = L(s_1, s_2, \ldots, s_{k-1}) + L(s_1, s_2, \ldots, s_k - 1) ,$$

$$R(s_1, s_2, \ldots, s_k) = R(s_1, s_2, \ldots, s_{k-1}) + R(s_1, s_2, \ldots, s_k - 1)$$

whenever $s_k > -M$. Now, it is easy to establish (by induction on k) that

$$
\left.
\begin{aligned}
&L(s_1, s_2, \ldots, s_k) \\
&\quad = \sum c_i x_i + \sum_{s_i \neq -M} L(s_1, s_2, \ldots, s_i - 1) - \sum_{s_i = -M} x_i , \\
&R(s_1, s_2, \ldots, s_k) \\
&\quad = (c-1) + \sum_{s_i \neq -M} R(s_1, s_2, \ldots, s_i - 1) + \sum_{s_i = -M} M .
\end{aligned}
\right\} \quad (3.3)
$$

Our next observation is essential for the proof.

CLAIM: Let $(s_1, s_2, \ldots, s_k)$ be any vector with $0 \leq k \leq n$ and $s_i \in \{-M, -M+1, \ldots, M\}$. If

$$
\left.
\begin{aligned}
L(s_1, s_2, \ldots, s_i-1) &= R(s_1, s_2, \ldots, s_i-1) \qquad (s_i \neq -M) \\
-x_i &= M \qquad (s_i = -M)
\end{aligned}
\right\} \quad (3.4)
$$

$$
\sum c_i x_i = c
$$

then $x_i = s_i$ for all $i = 1, 2, \ldots, k$.

PROOF of the Claim: We proceed by induction on k. The Claim is trivially true for $k = 0$. By the induction assumption, the Claim holds for the vector $(s_1, s_2, \ldots, s_{k-1})$ and so the equations (3.4) with $i \neq k$ imply $x_i = s_i$ for all $i = 1, 2, \ldots, k-1$. If

$s_k = -M$ then we are done. If $s_k \neq -M$ then we argue as follows. The equations (3.3) and (3.4) imply that

$$L(s_1, s_2, \ldots, s_{k-1}) = R(s_1, s_2, \ldots, s_{k-1}) + 1 .$$

By definition, we have

$$L(s_1, s_2, \ldots, s_k - 1) = (M+s_k)\, L(s_1, s_2, \ldots, s_{k-1}) - x_k$$

$$R(s_1, s_2, \ldots, s_k - 1) = (M+s_k)\, R(s_1, s_2, \ldots, s_{k-1}) - M .$$

Using the last three equations and (3.4) with $i = k$ we deduce $x_k = s_k$ which is the derived result. Thus the Claim is proved,

Now, we are ready for the final coup de grace. Inductively, we shall sweep through the entire set of inequalities

$$L(s_1, s_2, \ldots, s_k) \leq R(s_1, s_2, \ldots, s_k) \qquad (3.5)$$

in a specified order, and prove that each of these belongs to the closure of (3.1). (In particular, the inequality $L(\emptyset) \leq R(\emptyset)$ -- which comes last in our ordering -- is the one we want.) The linear order a lexicographic one with each blank -- corresponding to $s_{k+1}, s_{k+2}, \ldots, s_n$ -- interpreted as $M+1$. More precisely, we say that (3.5) precedes the inequality $L(t_1, t_2, \ldots, t_r) \leq R(t_1, t_2, \ldots, t_r)$ if and only if, either $s_j < t_j$ or $r < j \leq k$ where j is the largest subscript with $s_i = t_i$ for all $i < j$.

CASE 1. $k = n$. (This case includes the very first inequality in our set, one with $s_1 = s_2 = \cdots = s_n = -M$.) By the induction assumption, all the inequalities

$$L(s_1,s_2,\ldots,s_i-1) \leq R(s_1,s_2,\ldots,s_i-1) \qquad (s_i \neq -M) \qquad (3.6)$$

belong to the closure of (3.1). Moreover, the inequalities

$$-x_i \leq M \qquad (s_i = -M) \qquad (3.7)$$

$$\sum c_i x_i \leq c \qquad (3.8)$$

belong to the closure of (3.1). Summing up (3.6), (3.7), (3.8) and using (3.3) we arrive at the inequality

$$L(s_1,s_2,\ldots,s_n) \leq R(s_1,s_2,\ldots,s_n) + 1 . \qquad (3.9)$$

This inequality holds for every choice of reals $x_1, x_2, \ldots, x_n$ that obey (3.6), (3.7) and (3.8). Besides, our Claim implies that equality in (3.9) can occur only if $x_i = s_i$ $(i = 1,2,\ldots,n)$ and $\sum c_i x_i = c$. However, these $n+1$ equations are inconsistent with at least one of the constraints (3.1) -- otherwise the assumption of our Theorem is violated. Therefore

$$L(s_1,s_2,\ldots,s_n) < R(s_1,s_2,\ldots,s_n) + 1$$

holds for any choice of reals $x_1, x_2, \ldots, x_n$ satisfying (3.1), (3.6), (3.7), (3.8). By Lemma 1, the inequality

$$L(s_1, s_2, \ldots, s_n) \leq R(s_1, s_2, \ldots, s_n) \tag{3.10}$$

belongs to the closure of (3.1), (3.6), (3.7), (3.8). As (3.6), (3.7) and (3.8) belong to the closure of (3.1) themselves, we conclude that (3.10) belongs to the closure of (3.1).

CASE 2. $k < n$. By the induction assumption, the inequality

$$L(s_1, s_2, \ldots, s_k, M) \leq R(s_1, s_2, \ldots, s_k, M)$$

belongs to the closure of (3.1). This inequality can be written as

$$(2M+1)\, L(s_1, s_2, \ldots s_k) - x_k \leq (2M+1)\, R(s_1, s_2, \ldots, s_k) + M .$$

Besides, the inequality

$$x_k \leq M$$

belongs to the closure of (3.1). Adding the last two inequalities and dividing by $2M+1$ we obtain

$$L(s_1, s_2, \ldots, s_k) < R(s_1, s_2, \ldots, s_k) + \frac{2M}{2M+1} .$$

Therefore

$$L(s_1, s_2, \dots, s_k) \leq R(s_1, s_2, \dots, s_k)$$

belongs to the closure of (3.1).

Now, we have proved that all the inequalities (3.5), including $\sum c_i x_i \leq c-1$, belong to the closure of (3.1). Repeating this argument $c-c_0$ times (as mentioned above) we prove that (3.2) belongs to the closure of (3.1) and finish thus the proof of Theorem 1.

One more remark. It is easy to see that the Edmonds polyhedron of P can be described by inequalities

$$\sum_{j=1}^{n} a_{ij}^* x_j \leq b_i^* \qquad (i=1, 2, \dots, m)$$

where all the a_{ij}^*'s and b_i^*'s are integers. Hence Theorem 1 can be restated as follows.

COROLLARY 1A. If (3.1) defines a bounded polyhedron P then the closure of (3.1) determines $E(P)$.

4. The Boolean case and Branch-and-Bound method.

Among the integer linear programing problems, those with the constraints

$$x_i = 0 \text{ or } 1 \qquad (i = 1, 2, \ldots, n)$$

are particularly important. The problems arising from combinatorial considerations have nearly always this form; the x_i's usually represent the characteristic vector of a set satisfying specified conditions. In this section, we turn our attention to these problems. We shall consider polyhedra defined by inequalities

$$\left.\begin{array}{ll} 0 \le x_i \le 1 & (i = 1, 2, \ldots, n) \\ \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, 2, \ldots, m) \end{array}\right\} \qquad (4.1)$$

and present an alternative proof of Theorem 1 within this restricted class. The proof may be found to be more direct and transparent than the one given above; besides, it is related in an amusing way to the branch and bound method. As in the preceding section, we only have to prove the following statement.

Let $c_1, c_2, \ldots, c_n$ and c be integers such that

(i) the inequality $\sum c_i x_i \le c$ belongs to the closure of (4.1),

(ii) there are no integers $x_1, x_2, \ldots, x_n$ satisfying (4.1) and $\sum c_i x_i = c$.

Then the inequality $\sum c_i x_i \le c-1$ belongs to the closure of (4.1).

Actually, we are going to prove that all the inequalities

$$\sum_{i=1}^{n} c_i x_i + \sum_{i \in A} x_i - \sum_{i \in B} x_i \leq c-1 + |A| \qquad (4.2)$$

where A, B are disjoint subsets of $\{1, 2, \ldots, n\}$ belong to the closure of (4.1). The proof goes by backward induction on $|A| + |B|$; the inequality (4.2) with $A = B = \emptyset$ is the one we want. The induction step is easy. If $|A| + |B| < n$ then there is a subscript $k \notin A \cup B$ and, by the induction assumption, both inequalities

$$x_k + \sum c_i x_i + \sum_A x_i - \sum_B x_i \leq c-1 + |A| + 1$$

$$- x_k + \sum c_i x_i + \sum_A x_i - \sum_B x_i \leq c-1 + |A|$$

belong to the closure of (4.1). Adding them and dividing by two we obtain

$$\sum c_i x_i + \sum_A x_i - \sum_B x_i \leq c-1 + |A| + \frac{1}{2}$$

and conclude that (4.2) belongs to the closure of (4.1). It remains to verify that all the inequalities (4.2) with $|A| + |B| = n$ belong to the closure of (4.1). Here, we distinguish two cases.

CASE 1. $\sum_{j \in A} a_{ij} \leq b_i$ for all $i = 1, 2, \ldots, m$. In this case, we have

$$\sum_{i \in A} c_i < c$$

for otherwise (ii) is violated by

$$x_1 = \begin{cases} 1, & i \in A \\ 0, & i \in B \end{cases}$$

Setting $M = \max|c_i|$ we have

$$(M+c_i)x_i \leq M+c_1 \qquad (i \in A)$$

$$(M-c_i)(-x_i) \leq 0 , \qquad (i \in B)$$

$$(M-1) \sum c_i x_i \leq (M-1)c$$

Adding these inequalities and dividing by M we obtain

$$\sum c_i x_i + \sum_A x_i - \sum_B x_i \leq c + |A| + \frac{1}{M}(-c + \sum_A c_i)$$

and conclude that (4.2) belongs to the closure of (4.1).

CASE 2. $\sum_{j \in A} a_{ij} > b_i$ for some i. Setting $M = \max|a_{ij}|$ we have

$$(M-a_{ij})x_j \leq M-a_{ij} , \qquad (j \in A)$$

$$(M+a_{ij})(-x_j) \leq 0, \qquad (j \in B)$$

$$\sum a_{ij} x_j \leq b_i .$$

Adding these inequalities and dividing by M we obtain

$$\sum_{j\in A} x_j - \sum_{j\in B} x_j \leq |A| + \frac{1}{M}(b_i - \sum_{j\in A} a_{ij})$$

and conclude that

$$\sum_A x_j - \sum_B x_j \leq |A| -1$$

belongs to the closure of (4.1). Therefore (4.2) also belongs to the closure of (4.1).

The proof is finished. The reader may have noticed that not all the inequalities (4.2) are required for the induction leading to $\sum c_i x_i \leq c-l$. Indeed, we can restrict ourselves only to those with

$$A \cup B = \{1,2, \ldots, |A| + |B|\} .$$

Then the induction is performed along a binary tree with n+1 levels. All the 2^k vertices of the k-th level are labelled by distinct zero-one vectors $(z_1, z_2, \ldots, z_k)$ and associated with inequalities

$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{k} (2z_i-1)x_i \leq c-l + \sum_{i=1}^{k} zi .$$

Each vertex labelled $(z_1, z_2, \ldots, z_k)$ with $k < n$ has two successors labelled $(z_1, z_2, \ldots, z_k, 0)$ and $(z_1, z_2, \ldots, z_k, 1)$. The inequality assigned to a parent vertex is obtained by adding the inequalities at its two successors, dividing by two and rounding the right-hand side down to the nearest integer. (Since the right-hand sides of the successors differ in parity, the rounding always cuts down exactly one half.) The inequalities at the terminal vertices are obtained in one of two different ways, according to whether $(z_1, z_2, \ldots, z_n)$ is feasible with respect to (4.1) or not.

The whole picture rather resembles a binary search (in vain) for a feasible vector $(z_1, z_2, \ldots, z_n)$ that would satisfy $\sum c_i z_i = c$. Actually, it turns out that our method is a translation of the branch-and-bound method [1] into the language of linear inequalities, During the search, we are after the inequality $\sum c_i x_i \leq c-1$. Therefore we split all possible choices of integers $x_1, x_2, \ldots, x_n$ into two classes (corresponding to $x_1 = 0$ and $x_1 = 1$) and proceed to prove the inequality in each class separately. The two classes correspond to the two first-level inequalities. Indeed, the inequality

$$-x_1 + \sum c_i x_i \leq c-1$$

is just another way of saying "if $x_1 = 0$ then $\sum c_i x_i \leq c-1$; if $x_1 = 1$ then possibly $\sum c_i x_i = c$". Similarly, the inequality

$$x_1 + \sum c_i x_i \leq c \quad ,$$

reads "if $x_1 = 0$ then possibly $\sum c_i x_i = c$; if $x_1 = 1$ then necessarily $\sum c_i x_i \leq c-1$." The dichotomy between $x_1 = 0$ and $x_1 = 1$ is taken care of by the rounding device, We go on like this, step by step, and require one more x_i at each step to be fixed at a specified value (zero or one) until we hit the level where all the x_i's are fixed. If they are feasible (Case 1) then they cannot satisfy $\sum c_i x_i = c$; if they are not feasible (Case 2) then they cannot be reached at all.

Often, it happens that a k-th level inequality ($k < n$) belongs to the elementary closure of the inequalities (4.1) and $\sum c_i x_i \leq c$. In that case, we can stop branching out from the corresponding vertex and simplify the proof considerably. In the following section, we illustrate this situation (Example 1).

5. Combinatorial applications: independent sets in hypergraphs.

Many extremal combinatorial problems can be formulated as problems of finding the largest independent set in a hypergraph. A hypergraph H is an ordered pair (V,E) where V is a set and E a collection of subsets of V (see [2]). A set $X \subset V$ is called independent (in H) if there is no $A \in E$ with $A \subset X$. If V is finite then the problem of finding the largest set X independent in H is the following zero-one linear programming problem. Maximize $\sum_{i \in V} x_i$ subject to

$$\left.\begin{array}{ll} 0 \leq x_i \leq 1 & (i \in V)\,, \\ \sum_{i \in A} x_i \leq |A| -1 & (A \in E)\,, \end{array}\right\} \qquad (5.1)$$

$$x_i = \text{integer} \qquad (i \in V)\,. \qquad (5.2)$$

Theorem 1 guarantees that each inequality $\sum_{i \in V} x_i \leq x_0$ valid under constraints (5.1), (5.2) belongs to the closure of (5.1). Therefore a solution to the above problem can be always obtained through a series of elementary closure operations,

EXAMPLE 1: Hamiltonian circuits in graphs.

The problem of determining whether a given graph $G^* = (V^*, E^*)$ has a Hamiltonian circuit is one of the above kind. The corresponding hypergraph $H = (V,E)$ has $V = E^*$ and the collection E includes two different kinds of sets A:

(i) those consisting of three distinct edges of G^* that have all one vertex in common;

(ii) those consisting of circuits in G^* having less than $|V^*|$ edges.

It is quite easy to see that each independent set of size $|V^*|$ in H constitutes a Hamiltonian circuit in G^* and vice versa, Thus G^* has a Hamiltonian circuit if and only if the corresponding zero-one linear programming problem has a feasible solution with

$$\sum_{i \in E^*} x_i = |V^*| \quad .$$

As an example, we consider the Petersen graph with edges enumerated as in Fig. 1. Setting up the linear programming problem, we arrive at constraints

$$0 \leq x_i \leq 1 \qquad (i = 1, 2, \ldots, 15) \;, \tag{5.3}$$

ten "star constraints" of the type (i), that is

$$\left.\begin{array}{l} x_1 + x_2 + x_7 \leq 2 \;, \\ \\ x_2 + x_3 + x_8 \leq 2 \;, \\ \\ \text{etc} \end{array}\right\} \tag{5.4}$$

and many "circuit constraints" of the type (ii), that is

$$x_1 + x_2 + x_3 + x_4 + x_5 \le 4 \quad ,$$

$$x_1 + x_2 + x_8 + x_{14} + x_{10} + x_5 \le 5 \quad , \tag{5.5}$$

etc.

It is notoriously well-known that the Petersen graph has no Hamiltonian circuit. Equivalently, one has

$$\sum_{i=1}^{15} x_i \le 9 \tag{5.6}$$

for every choice of integers $x_1, x_2, \ldots, x_{15}$ satisfying (5.3), (5.4), (5.5). The integrality constraint is essential here -- indeed, setting $x_i = \frac{2}{3}$ $(i = 1, 2, \ldots, 15)$ we satisfy (5.3), (5.4), (5.5) and violate (5.6). We are going to show that (5.6) belongs to the closure of (5.3), (5.4), and (5.5), giving thus a proof of the non-existence of a Hamiltonian circuit in the Petersen graph.

Let us consider the binary tree in Fig. 2. With each of its vertices (labelled A, B), we associate an inequality

$$\sum_{i=1}^{15} x_i + \sum_{i \in A} x_i - \sum_{i \in B} x_i \le 9 + |A|$$

It is not difficult to prove that the inequalities assigned to the terminal vertices belong to the closure of (5.3), (5.4), (5.5).

(Fig. 2)

For instance, the inequality corresponding to $A = \{9, 14, 1\}$, $B = \emptyset$ is obtained as the sum of the inequalities

$$x_3 + x_4 + x_9 \le 2 , \qquad x_8 + x_{12} + x_{14} \le 2 ,$$
$$x_9 + x_{13} + x_{15} \le 2 , \qquad x_1 + x_5 + x_6 \le 2 ,$$
$$x_{10} + x_{11} + x_{14} \le 2 , \qquad x_1 + x_2 + x_7 \le 2$$

The inequality corresponding to $A = \{1, 9\}$, $B = \{4, 13, 14\}$ is a sum of the inequalities

$$x_{10} + x_{11} + x_{14} \le 2 ,$$
$$x_1 + x_5 + x_6 \le 2 ,$$
$$x_1 + x_2 + x_7 \le 2$$
$$x_9 \le 1 ,$$
$$- x_{14} \le 0 ,$$
$$x_3 + x_8 + x_{12} + x_{15} + x_9 \le 4 .$$

Similarly, every other inequality corresponding to a terminal vertex of our tree can be obtained as a sum of a subset of (5.3), (5.4), (5.5). As in Section 4, the inequality at each parental vertex (labelled A,B) can be obtained by taking the sum of the two inequalities assigned to its descendants (labelled $A \cup \{k\}$, B and A, $B \cup \{k\}$), dividing by two and rounding the right-hand side down. Thus we conclude that (5.6) belongs to $e^5(S)$ where S is the set of inequalities (5.3), (5.4), (5.5).

The application of this technique to the problems of existence of Hamiltonian circuits is discussed in detail in [5]. In particular, [5] contains the following "one-two-three theorem". Given any graph $G = (V,E)$ consider the S of inequalities

$$0 \le x_i \le 1 \qquad (i \in E) ,$$

$$\sum_{N \in i} x_i \le 2 \qquad (v \in V) ,$$

$$\sum_{i \subset W} x_i \le |W| - 1 \qquad (W \subset V,\ 0 < |W| < |V|) .$$

If the maximum of $\sum_{i \in E} x_i$ subject to the constraints S and so called "comb inequalities" (which belong to $e^1(S)$) equals $|V|$ then G has the following properties:

(i) deletion of k vertices from G always results in a graph with at most k components (in other words, G is 1-tough),

(ii) V can be covered by pairwise disjoint circuits (in other words, G has a 2-factor),

(iii) given any $u, v, w \in V$ there is a circuit in G that passes through all three u, v, w (in other words, G is 3-cyclable).

EXAMPLE 2: Moser's cube problem.

Let us consider the three-dimensional tick-tack-toe cube with 27 points $(0,0,0), (0,0,1), \ldots, (2,2,2)$. Our objective is to select as many of these 27 points as possible without choosing three collinear ones. Assigning to each point (a,b,c) a variable x_i with $i = 9a + 3b + c + 1$ (see Fig. 3) we arrive at the following integer programming formulation of the problem.

(Fig. 3)

Maximize $\sum_{i=1}^{27} x_i$ subject to

$$\left.\begin{array}{ll}
0 \le x_i \le 1 & (i = 1, 2, \ldots, 27) \\
x_1 + x_2 + x_3 < 2 & \\
x_4 + x_5 + x_6 \le 2 & \\
\quad \cdots & \\
x_1 + x_5 + x_9 \le 2 & \\
\quad \cdots & \\
x_1 + x_{14} + x_{27} \le 2 & \\
\quad \cdots & \\
x_i = \text{integer} & (i = 1, 2, \ldots, 27)
\end{array}\right\} \quad (5.7)$$

(Altogether, we have 49 constraints of the form $x_i + x_j + x_k \le 2$, corresponding to 49 collinear triples.) Setting $x_i = 2/3$ $(i = 1,2,\ldots,27)$ we satisfy all the inequalities (5.7) and obtain $\sum x_i = 18$. However, it can be shown that every choice of 17 points out of our 27 always contains a collinear triple. Equivalently, the inequality $\sum x_i \leq 16$ belongs to the closure of (5.7). This can be shown as follows. We have

$$\frac{5}{6}(x_1 + x_2 + x_3) \leq \frac{5}{3},$$

$$\frac{5}{6}(x_1 + x_4 + x_7) \leq \frac{5}{3},$$

$$\frac{5}{6}(x_3 + x_6 + x_9) \leq \frac{5}{3},$$

$$\frac{5}{6}(x_7 + x_8 + x_9) \leq \frac{5}{3},$$

$$\frac{1}{3}(x_1 + x_5 + x_9) \leq \frac{2}{3},$$

$$\frac{1}{3}(x_3 + x_5 + x_7) \leq \frac{2}{3},$$

$$\frac{1}{6}(x_2 + x_5 + x_8) \leq \frac{1}{3},$$

$$\frac{1}{6}(x_4 + x_5 + x_6) \leq \frac{1}{3}.$$

Adding these inequalities up we conclude that

$$2(x_1 + x_3 + x_7 + x_9) + (x_2 + x_4 + x_6 + x_8) + x_5 \leq [\frac{26}{3}] = 8$$

belongs to the closure of (5.7). Multiplying the last inequality by $\frac{4}{5}$ and adding the inequalities

$$\frac{2}{5}\,(x_1 + x_5 + x_9) \leq \frac{4}{5} \quad ,$$

$$\frac{2}{5}\,(x_3 + x_5 + x_7) \leq \frac{4}{5} \quad ,$$

$$\frac{1}{5}\,(x_2 + x_5 + x_8) \leq \frac{\;}{5}\; 2 \quad ,$$

$$\frac{1}{5}\,(x_4 + x_5 + x_6) \leq \frac{2}{5}$$

we find that

$$2(x_1 + x_3 + x_7 + x_9) + (x_2 + x_4 + x_6 + x_8) + 2x_5 \leq [\tfrac{44}{5}] = 8 \qquad (5.8)$$

belongs to the closure of (5.7). Now, we set

$$A = x_1 + x_3 + x_7 + x_9 + x_{19} + x_{21} + x_{25} + x_{27} \,,$$

$$B = x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{16} + x_{18} + x_{20} + x_{22} + x_{24} + x_{26} \,,$$

$$C = x_5 + x_{11} + x_{13} + x_{15} + x_{17} + x_{23} \,,$$

$$D = x_{14} \,.$$

Hence A is the sum of variables assigned to the corners of the cube, B corresponds to edges, C to faces and D to the center of the cube. The inequality (5.8) applies to the points in the bottom horizontal plane. Adding up nine inequalities of this sort (corresponding to nine planes perpendicular to one of the coordinate axes) we obtain

$$6A + 4B + 4C + 6D < 72 \ .$$

Adding up all the 12 contraints corresponding to lines that join centers of edges via centers of faces we obtain

$$2B + 2C \leq 24 \ .$$

Dividing the sum of the last two inequalities by six we arrive at

$$\sum_{i=1}^{27} x_i = A + B + C + D \leq 16$$

which is the desired result.

More generally, one can consider the 3^n-cube and ask for the largest size $f(n)$ of its subset containing no three collinear points. It is easy to show that $f(1) = 2$, $f(2) = 6$, $f(3) = 16$; recently Chandra proved that $f(4) = 43$. It is not difficult to show that $f(n) \leq c \cdot 3^n / \sqrt{n}$, see [4]. Moser [16] conjectured that $f(n) = o(3^n)$; this, apparently difficult, problem is still unsettled. Perhaps the technique indicated here could help to solve Moser's conjecture.

6. Combinatorial applications: coloring of hypergraphs.

A k-coloring of a hypergraph $H = (V,E)$ is a partition

$$V = C_1 \cup C_2 \cup \cdots \cup C_k$$

such that each C_i is independent in H. In a coloring problem, one asks for the smallest k such that H admits a k-coloring. The coloring problems include the celebrated four-color conjecture as well as the problems of Ramsey's type [3]. At first, it seems that the coloring problems are different from those considered in the previous section. Yet there is an easy way of reducing them to the previous type. Given a hypergraph $H = (V,E)$ and a positive integer k we consider the hypergraph $H^* = (V^*, E^*)$ where $V^* = V \times \{1,2,\ldots,k\}$ and E^* includes two kinds of sets A^*:

(i) all the sets $A^* = A \times \{j\}$ where $A \in E$, $1 \leq j \leq k$,

(ii) all the couples $A^* = \{(v,i),(v,j)\}$ where $v \in V$, $i \neq j$.

A moment's reflection shows that H is k-colorable if and only if H^* contains an independent set of size $|V|$. An amusing consequence of this trick goes as follows. Every algorithm that finds the largest independent set in a graph (with n vertices and m edges) within $f(n,m)$ steps can be used to check the k-colorability of a graph within $f(nk, mk + n\binom{k}{2})$ steps.

EXAMPLE 3: Ramsey's theorem.

It is well-known that, whenever one colors the 15 edges of a complete graph with six vertices by two colors (customarily, red and blue are used), a monochromatic triangle is bound to pop out. Guided by the philosophy explained above, we can formulate this statement as follows. The maximum of

$$T = \sum_{1 \le i < j \le 6} (x_{ij} + y_{ij})$$

subject to

$$\left.\begin{array}{ll} x_{ij} + x_{jk} + x_{ik} \le 2 & \\ & (1 \le i < j < k \le 6) \\ y_{ij} + y_{jk} + y_{ik} \le 2 & \\ & \\ -x_{ij} \le 0 & \\ & \\ -y_{ij} \le 0 & (1 \le i < j \le 6) \\ & \\ x_{ij} + y_{ij} \le 1 & \end{array}\right\} \quad (6.1)$$

and

$$x_{ij} = \text{integer}, \qquad y_{ij} = \text{integer}$$

does not exceed 14. (Here $x_{ij} = 1$ corresponds to the edge $\{i,j\}$ colored day-glow orange [19] and $y_{ij} = 1$ corresponds to $\{i,j\}$ colored vermilion.)

We proceed to show that $T \leq 14$ belongs to the closure of (6.1). We easily find that $T \leq 15$ does so. Indeed, this is just the sum of all the inequalities $x_{ij} + y_{ij} \leq 1$. Actually, the maximum of T subject to (6.1) equals 15 and can be attained by setting $x_{ij} = y_{ij} = \frac{1}{2}$.

Now, adding up the inequalities

$$x_{12} + x_{13} + x_{23} \leq 2$$

$$x_{12} + x_{14} + x_{24} \leq 2$$

$$x_{13} + x14 + x34 \leq 2$$

$$y_{23} + y_{24} + y_{34} \leq 2$$

$$x_{ij} + y_{ij} \leq 1 \qquad (ij \neq 23,\ 24,\ 34)$$

$$T \leq 15$$

we obtain the inequality

$$2(T + x_{12} + x_{13} + x_{14}) \leq 35.$$

Hence

$$T + x_{12} + x_{13} + x_{14} \leq 17$$

belongs to the closure of (6.1), In the same way, we deduce

$$T + x_{12} + x_{13} + x_{15} \leq 17\ ,$$

$$T + x_{12} + x_{14} + x_{15} \leq 17\ ,$$

$$T + x_{13} + x_{14} + x_{15} \leq 17\ .$$

Adding up the last four inequalities and $2T \leq 30$ on the top, we obtain

$$3(2T + x_{12} + x_{13} + x_{14} + x_{15}) \leq 98$$

Therefore

$$2T + x_{12} + x_{13} + x_{14} + x_{15} < 32$$

belongs to the closure of (6.1). Similarly, we obtain

$$2T + x_{12} + x_{13} + x_{14} + x_{16} \leq 32 \ ,$$

$$2T + x_{12} + x_{13} + x_{15} + x_{16} < 32 \ ,$$

$$2T + x_{12} + x_{14} + x_{15} + x_{16} < 32 \ ,$$

$$2T + x_{13} + x_{14} + x_{15} + x_{16} \leq 32 \ .$$

Adding up the last five inequalities and $2T \leq 30$, we arrive at

$$4(3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16}) \leq 190 \ ,$$

so that

$$3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} \leq 47$$

belongs to the closure of (6.1). By the same series of arguments, the inequality

$$3T + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \leq 47$$

belongs to the closure of (6.1). Adding up these two inequalities and all

$$x_{ij} + y_{ij} \leq 1 \qquad (2 \leq i < j < 6)$$

we arrive at $7T \leq 104$. Therefore $T \leq 14$ belongs to the closure of (6.1).

The astute reader has noticed that our proof simulates the standard one. We investigated colorings where some of the edges can be left uncolored but no monochromatic triangle occurs; the total number of colored edges is T. We start by observing that $x_{12} = x_{13} = x_{14} = 1$ is incompatible with T = 15 (in other words, if in a full coloring all three edges $\{1,2\}, \{1,3\}, \{1,4\}$ are colored day-glow orange then we run into a contradiction -- either one of the triangles 123, 124, 134 is day-glow orange or else 234 is vermilion). Equivalently, $T + x_{12} + x_{13} + x_{14} \leq 17$. Thus only two of the three edges $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ can be colored day-glow orange. Now, symmetry and common sense show that only two out of the five edges $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$ can be colored day-glow orange ($3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} \leq 47$). However, the process of getting this inequality from $T + x_{1i} + x_{1j} + x_{1k} \leq 17$ is painfully slow. Similarly, only two out of the five edges $\{1,i\}$ can be colored vermilion ($3T + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \leq 47$) and so the coloring can never be full ($T \leq 14$). Along the same lines, one can translate the entire proof of Ramsey's theorem into the closure operation language.

7. A hierarchy of combinatorial problems.

A T.V. commerical for our main theorem would read

$$\boxed{\text{COMBINATORICS = NUMBER THEORY + LINEAR PROGRAMMING}} \quad (7.1)$$

Indeed, a host of combinatorial problems can be formulated as integer linear programming problems, Then the process of solving them can be split into two parts, that is

(i) determining enough new inequalities that belong to the closure of the original ones,

(ii) solving the resulting ordinary (non-integer) linear programming problem.

The first phase depends heavily on the divisibilityproperties of the linear combinations of our original coefficients. Hence a justification of the slogan (7.1). (It is not exactly a strong one, but then again -- we all know how the T.V. commercials are.)

Now, we ask how vital a role the part (i) plays. It turns out that for certain, rather important and naturally arising combinatorial problems, all the vertices of the underlying polyhedra P are lattice points, that is E(P) = P. In these cases (that include network-flow problems, matchings in bipartite graphs, etc.), the phase (i) becomes void and the desired max-min formula turns out to'bejust a special case of the duality theorem. For a more detail, see [14], [18]. Next, we consider an arbitrary problem of maximizing a linear form $\sum c_i x_i$ subject to a set S of linear inequalities plus the integrality constraint

$$x_j = \text{integer} \qquad (j = 1, 2, \ldots, n).$$

Our Theorem 1, resp. Corollary 1A, guarantees that there is a positive integer k such that the desired maximum equals the maximum of $\sum c_i x_i$ subject to (a finite subset of) linear constraints $e^k(S)$. The smallest such k will be called the rank of the problem. Hence the problems of finding a maximal flow through a network have rank zero and so do the problems of finding a maximum-weighted matching in a bipartite 'graph. Edmonds' theorem shows that the problems of finding a maximum-weighted matchinginan arbitrary graph have rank one (the added inequalities (2.5) belong to $e^1(S)$). The higher the rank of a problem, the more involved the phase (i) of its solution. In a way, one may classify the difficulty of solving an integer linear programming problem by its rank. (However,there is no indication of a relationship between the rank of a problem and its computational complexity.)

Finally, we turn our attention to the search for a largest independent set in a graph $G = (V,E)$. The set of all cliques (that is, maximal complete subgraphs) in G will be denoted by $\mathcal{C}$; each $A \in \mathcal{C}$ will be seen as a subset of V. The problem is to maximixe $\sum_{i \in V} x_i$ subject to

$$\left.\begin{aligned} -x_i &\le 0 && (i \in V), \\ \sum_{i \in A} x_i &\le 1 && (A \in \mathcal{C}), \\ x_i &= \text{integer} && (i \in V). \end{aligned}\right\} \qquad (7.2)$$

We are going to show that these problems can have arbitrarily high rank.

An inequality $\sum a_j x_j \leq b$ will be called <u>positive regular</u> if $a_j \geq 0$ $(j = 1, 2, \ldots, n)$ and $b \geq \max(a_1, a_2, \ldots, a_n) > 0$. A <u>strength</u> of such an inequality is the ratio $(\sum a_j) b^{-1}$. A linear inequality will be called <u>negative regular</u> if it reads $-x_j \leq 0$.

LEMMA 2. Let S be a set of linear inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m)$$

where, for each $i = 1, 2, \ldots, n$, the ith inequality reads $-x_i \leq 0$ and, for each $i = n+1, n+2, \ldots, m$, the ith inequality is positive regular of strength $\leq s$. Let the inequality

$$\sum_{j=1}^{n} a_j x_j \leq b \tag{7.3}$$

belong to $e^1(S)$. Then (7.3) can be written as a linear combination of the negative regular inequalities in S and a positive regular inequality of strength $< 2s$ that belongs to $e^1(S)$.

PROOF. There are nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$a_j = \sum_{i=1}^{m} \lambda_i a_{ij} = -\lambda_j + \sum_{i=n+1}^{m} \lambda_i a_{ij} = \text{integer} \qquad (j = 1, 2, \ldots, n),$$

$$b = \left[\sum_{i=1}^{m} \lambda_i b_i\right] = \left[\sum_{i=n+1}^{m} \lambda_i b_i\right].$$

Set

$$\mu_i = \begin{cases} \lambda_i - [\lambda_i], & i = 1, 2, \ldots, n \\ \lambda_i, & i = n+1, n-l-2, \ldots, m . \end{cases}$$

Then all the μ_i's are nonnegative and (7.3) can be written as a sum of inequalities

$$[\lambda_i] (-x_i) \leq 0 , \tag{7.4}$$

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \mu_i a_{ij}\right) x_j \leq \left[\sum_{i=1}^{n} \mu_i b_i\right] . \tag{7.5}$$

For each $j = 1, 2, \ldots, n$ we have

$$\sum_{i=1}^{m} \mu_i a_{ij} = -(\lambda_j - [\lambda_j]) + \sum_{i=n+1}^{m} \lambda_i a_{ij} = \text{integer}$$

Besides, we have $\lambda_j - [\lambda_j] < 1$ and $\sum_{i=n+1}^{m} \lambda_i a_{ij} \geq 0$. Therefore each $c_j = \sum_{i=1}^{m} \mu_i a_{ij}$ is a nonnegative integer. If $c_j = 0$ for all $j = 1, 2, \ldots, n$ then (7.3) is- a sum of inequalities (7.4) and we are done. Next, we assume $c_k \geq 1$ for some k. Since $b_i \geq a_{ij}$ for all $i = n+1, n+2, \ldots, m$ and $j = 1, 2,$ C.. , n, we have

$$\sum_{i=n+1}^{m} \lambda_i b_i \geq \sum_{i=n+1}^{m} \lambda_i a_{ij} \geq c_J$$

for each j and so $b \geq c_j$ for each j. Hence (7.5) is positive regular and $b \geq 1$. Then $2b = 2[\sum \lambda_i b_i] > \sum \lambda_i b_i$ and so

$$2s \cdot b > \sum_{i=n+1}^{m} \lambda_i (sb_i) \geq \sum_{i=n+1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij}\right)$$

$$\geq \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{n} a_{ij}\right) = \sum_{j=1}^{n} c_j \ .$$

Hence (7.5) has strength < 2s and the proof is finished.

A repeated application of Lemma 2 yields

THEOREM 2. Let S be a set of regular inequalities; let k be a positive integer. Then each inequality that belongs to $e^k(S)$ can be written as a linear combination of negative regular inequalities that belong to S and a positive regular inequality that belongs to $e^k(S)$. Besides, if all positive regular inequalities in S have strength $\leq s$ then all positive regular inequalities in $e^k(S)$ have strength $< 2^k \cdot s$.

COROLLARY 2A. Given any N there is a graph G such that the problem of maximizing $\sum_{i \in V} x_1$ subject to (7.2) has rank greater than N.

PROOF. Erdős [10] has shown that given any n there is a graph G with more than $2^{n/2}$ vertices that contains neither a complete subgraph with n vertices nor an independent set of n vertices. Let k be the rank of the corresponding problem. Then the maximum of $\sum x_i$ subject to $e^k(S)$ does not exceed n-l and so, by Lemma 1, the inequality

$$\sum_{i \in V} x_i \leq n-1$$

belongs to $e^{k+1}(S)$. By Theorem 2, the strength of each positive regular inequality in $e^{k+1}(S)$ is smaller than $(n-1)\,2^{k+1}$. Hence we have

$$|V| . (n-1)^{-1} < (n-1) \; 2^{k+1}$$

and so

$$2^{k+1} > 2^{n/2} . (n-1)^{-2} \quad .$$

If n is sufficiently large (with respect to N) then the last inequality implies k > N which is the desired conclusion.

8. APPENDIX: Relations to Gomory's algorithm.

An alternative proof of Theorem 1 can be based on Gomory's integer programming algorithm [12]. Here we begin with a set of inequalities

$$\left.\begin{aligned} -x_j &\le 0 \\ \sum_{j=1}^{n} a_{ij} x_j &\le b_i \end{aligned}\right\} \tag{8.1}$$

$$\sum_{j=1}^{n} c_j x_j \le c_0 \tag{8.2}$$

where a_{ij}, b_i, c_j are <u>integers,</u> the polyhedron defined by (8.1) is bounded and (8.2) holds for every choice of integers $x_1, x_2, \ldots, x_n$ that satisfy (8.1). Gomory describes a way of generating new constraints, called <u>cuts,</u> that are satisfied by every choice of integers $x_1, x_2, \ldots, x_n$ satisfying (8.1) as well as all the previously generated cuts, It turns out that these cuts belong to the closure of (8.1); an account of this is given by Hu ([15], Section 13.3). Gomory proves that, after a finite number of cuts are generated in a systematic fashion, the maximum of $\sum$ ci xi, subject to (8.1) and the added cuts, can be attained by integers $x_1, x_2, \ldots, x_n$. Therefore, by the duality theorem, (8.2) belongs to the closure of (8.1). Now, to prove Theorem 1 in its full generality, one has to get rid of the inequalities $x_j \ge 0$ in (8.1) as well as to get around the integrality assumption placed upon a_{ij}, b_i. However, these modifications can be carried out in quite a routine manner.

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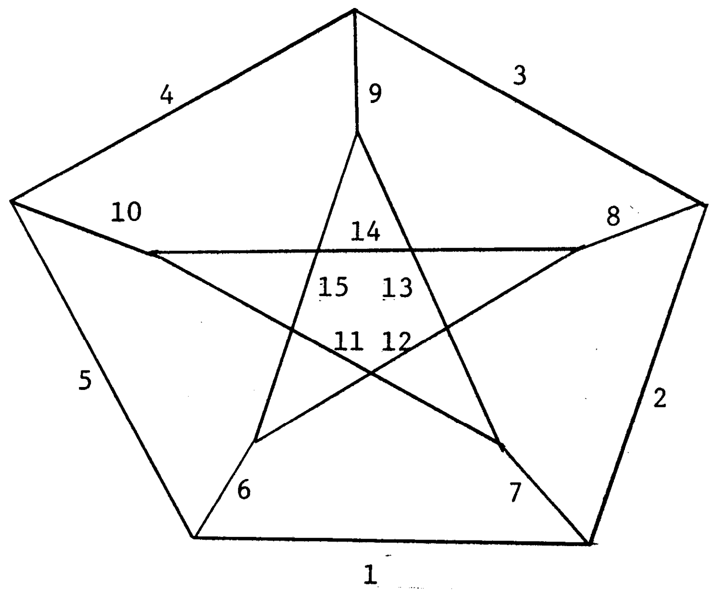

Fig. 1

7	8	9
4	5	6
12		3

BOTTOM PLANE

16	17	18
13	14	15
10	11	12

MIDDLE PLANE

25	26	27
22	23	24
19	20	21

TOP PLANE

Fig. 3

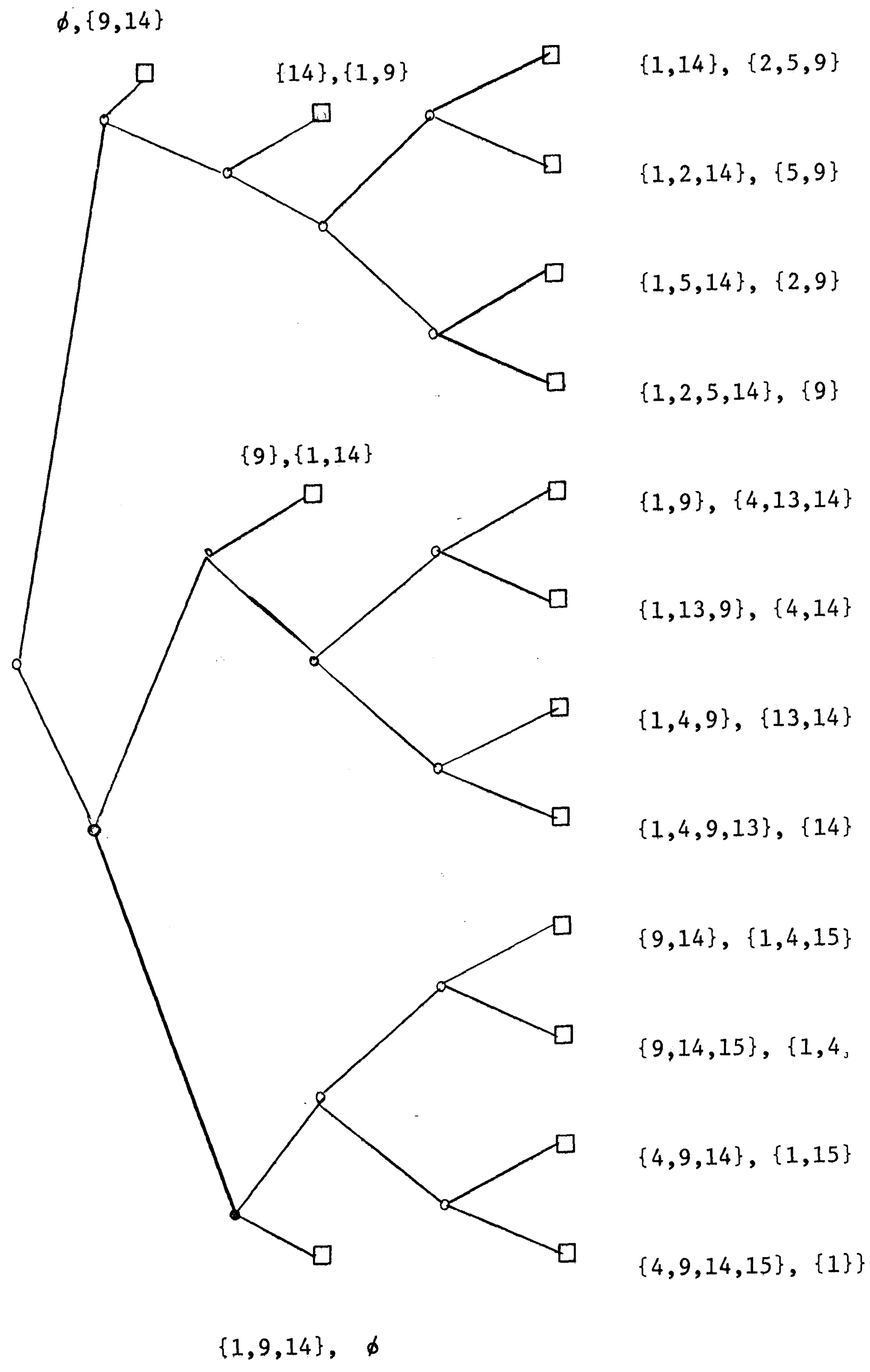
ϕ,{9,14}
{14},{1,9}
{1,14}, {2,5,9}
{1,2,14}, {5,9}
{1,5,14}, {2,9}
{1,2,5,14}, {9}
{9},{1,14}
{1,9}, {4,13,14}
{1,13,9}, {4,14}
{1,4,9}, {13,14}
{1,4,9,13}, {14}
{9,14}, {1,4,15}
{9,14,15}, {1,4,
{4,9,14}, {1,15}
{4,9,14,15}, {1}}
{1,9,14}, ϕ

Fig. 2