# MATROID PARTITIONING

BY

DONALD E. KNUTH

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COMPUTER SCIENCE DEPARTMENT

School of Humanities and Sciences

STANFORD UNIVERSITY

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Abstract

This report discusses a modified version of Edmonds's algorithm for partitioning of a set into subsets independent in various given matroids. If $\mathcal{M}_1,\ldots,\mathcal{M}_k$ are matroids defined on a finite set E , the algorithm yields a simple necessary and sufficient condition for whether or not the elements of E can be colored with k colors such that (i) all elements of color j are independent in $\mathcal{M}_j$ , and (ii) the number of elements of color j lies between given limits, $n_j \le \|E_j\| \le n'_j$ . The algorithm either finds such a coloring or it finds a proof that none exists, after making at most $n^3 + n^2 k$ tests of independence in the given matroids, where n is the number of elements in E .

Keywords: matroid, matching, combinatorial geometry.

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Let $\mathcal{M}_1,\ldots,\mathcal{M}_k$ be matroids (i.e., pregeometries) over the n-element set E . Edmonds [1] has given an efficient algorithm for determining whether or not the elements of E can be partitioned into k disjoint subsets, $E = E_1 \cup \cdots \cup E_k$ , such that $E_j$ is independent in $\mathcal{M}_j$ for all j . The purpose of this paper is to present his algorithm in a somewhat different way, which indicates how he might have discovered it in the first place; and to extend the algorithm slightly so that bounds are placed on the number of elements in the subsets $E_j$ .

In order to make this report somewhat colorful, we shall imagine that the elements of E are being painted with k colors, so that $E_j$ contains the elements of color j . The reader is assumed to know the basic definitions of matroid theory, since by now there are dozens of papers in which these definitions occupy the first two pages. Edmonds's paper [1] indicates the wide variety of applications for matroid partitioning.

## Derivation of an algorithm

The natural way to get the elements colored is to start with them all blank and successively to paint them. Many combinatorial algorithms have the following general form: "Starting with a certain configuration, try to find a better configuration by some reasonably straightforward method. If this succeeds, replace the initial configuration by the improved one, and start again. If this fails, prove that no better configuration exists." Of course it is not always possible to carry out the latter proof; but in many important cases, such a proof is possible, hence a rather simple algorithm emerges. Matroid partitioning is such a case.

Suppose we have painted certain elements and that $E_j$ is the set of elements having color j ; we assume that $E_j$ is independent in $\mathcal{M}_j$. Let $E_0 = E \setminus (E_1 \cup \cdots \cup E_k)$ be the unpainted elements. If x is some element not of color j , we could paint it with that color if $x \cup E_j$ were independent in $\mathcal{M}_j$ . On the other hand, if $x \cup E_j$ is dependent, there is a unique circuit $P \subset x \cup E_j$ , and we can paint x with color j if the color of any element y of $P \cap E_j$ is scraped off. Then perhaps we can paint y with some other color.

A sequence of such repaintings might be denoted by, say,

$$x \to y \to z \to 0_3$$

meaning "paint x with the current color of y , then repaint y with the current color of z , then repaint z with color 3 ." In general we may write

$$x \to y \Leftrightarrow x \cup E_j \setminus y \text{ is independent in } \mathcal{M}_j$$

when $y \in E_j$ and $x \notin E_j$ ; and

$$x \to 0_j \Leftrightarrow x \cup E_j \text{ is independent in } \mathcal{M}_j$$

where x is an element of $E \setminus E_j$ and $0_j$ is a special symbol distinct from the elements of E ; we may think of $0_j$ as a 'standard' element of color j , whose color never needs to be washed off. Note that if $x \to 0_j$ then $x \to y$ for all $y \in E_j$ .

In effect, this arrow notation defines a directed graph on the n+k vertices $E \cup \{0_1,\ldots,0_k\}$ , and $x \to y \to z \to 0_3$ is an oriented path from x to $0_3$ . We shall denote oriented paths, as usual, by writing

$$x \xrightarrow{+} y \Leftrightarrow \text{there is a path } x = x_0 \to x_1 \to \cdots \to x_m = y\,, \quad m \geq 1\,.$$

If $x$ is uncolored and there is a path $x \overset{+}{\rightarrow} O_r$, this path specifies a repainting which results in a net increase of one more element painted the r-th color. This would give us a way to decrease the number of unpainted elements. However, we have overlooked an important consideration: All the "$\rightarrow$" relationships have been calculated with respect to a particular choice of the $E_j$, and some repaintings may invalidate future ones. In fact there do exist paths $x \overset{+}{\rightarrow} O_r$ which correspond to no correct repainting.

Fortunately this problem does not arise when we consider <u>shortest</u> paths instead of arbitrary paths.

<u>Lemma</u>. <u>In terms of the above notation, let</u> $x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_m = O_r$, $x_i \not\rightarrow x_j$ <u>for</u> $j > i+1$. <u>Then if</u> $x_i$ <u>is painted the color of</u> $x_{i+1}$, <u>for</u> $0 \le i < m$, <u>the resulting elements of color</u> $j$ <u>are independent in</u> $\mathfrak{M}_j$, <u>for</u> $1 \le j \le k$.

<u>Proof</u>. The result is trivial when $m = 1$. If $m > 1$, consider what happens after making just the m-th step of the repainting: Let $x_{m-1}$ have color $s$, and let

$$E'_r = E_r \cup x_{m-1}$$

$$E'_s = E_s \setminus x_{m-1}$$

$$E'_j = E_j \,, \quad \text{for } j \ne r,s \,.$$

Let $\rightarrow'$ denote relations in the directed graph corresponding to these $E'_j$; and let $x'_i = x_i$ for $0 \le i < m$, $x'_{m-1} = O_s$. The lemma will follow by induction, if we prove that $x'_0 \rightarrow' x'_1 \rightarrow' \cdots \rightarrow' x'_{m-1}$ and $x'_i \not\rightarrow' x'_j$ for $j < i+1$.

To prove that $x_i' \to' x_{i+1}'$, the only nontrivial case occurs when $x_{i+1}'$ has color $r$. In this case, $i+1 < m-1$ and we must show that the set $I = E_r' \setminus x_{i+1} \cup x_i$ is independent in $\mathcal{M}_r$. If it were dependent, it would contain a unique cycle $P$; and $P$ must contain both $x_i$ and $x_{m-1}$ since $I \setminus x_i$ and $I \setminus x_{m-1}$ are independent. But this would imply that $x_i \to x_{m-1}$, contrary to hypothesis.

On the other hand if $x_i' \to' x_j'$ for $j > i+1$, we reach an immediate contradiction unless $x_j'$ has color $s$ and $j < m-1$. Otherwise we find that $E_s' \setminus x_j \cup x_i$ is independent but $E_s \setminus x_j \cup x_i$ is dependent; thus there is a unique cycle $P \subseteq E_s \setminus x_j \cup x_i$, and $x_{m-1}$ and $x_i$ are both in this cycle, so $\tilde{x}_i \to x_{m-1}$. This contradiction completes the proof. □

This lemma tells us that an existing coloration can be improved (i.e., the number of unpainted elements reduced by one) if we can find a path from an uncolored element to $0_r$ for some $r$. This would give us an algorithm if we could show conversely that a better coloration exists only when there is such a path. Indeed, it isn't hard to convince oneself that this is true: Consider any painting $E_0, E_1, \ldots, E_k$ where each $E_j$ has $\|E_j\| = n_j$ elements, and suppose there is another one $E_0', E_1', \ldots, E_k'$ where $E_j'$ has $n_j + \delta_{jr} - \delta_{j0}$ elements. (Thus, the second coloration has one more element of color $r$.) Then there is some element $x$ in $E_r'$ which is independent of $E_r$, because $E_r$ has rank $n_r$ in $\mathcal{M}_r$ and it could not span all of the $n_r+1$ elements in $E_r'$. We can repaint $x$ with color $r$; then if $x$ was painted color $s$, we can find some $y$ in $E_s'$ which is independent of $E_s \setminus x$, etc. Each repainting brings the $E_j$ closer to the $E_j'$, so the process eventually terminates by finding an uncolored element to paint.

Derivation of good characteristics

So we know that the above path method will indeed lead to a good algorithm for matroid partitioning. However, experience with other algorithms; for which matroid partitioning provides a generalization, encourages us to look for more: We would like to find a "simple reason" that the painting cannot be extended, so that a person who doesn't necessarily believe that our computer program is correct can see for himself that the best painting it has found is optimum. This is far more desirable than if we merely said "the computer has made an exhaustive search and found nothing better." A simple reason that improvements are impossible is what Edmonds has called a good characterization. The programmer can present his supervisor with a convincing answer, whether the algorithm succeeds or not.

Therefore let us try to find a good characterization. Suppose there is no oriented path $x \stackrel{+}{\to} 0_r$ satisfying the conditions of the lemma, for any uncolored $x \in E_0$ and for some fixed value of $r$. Let

$$B_j = \{x \mid x \in E_j \text{ and } x \stackrel{+}{\to} 0_r\} ,$$

$$A_j = E_j \setminus B_j ,$$

for $0 \le j \le k$. Then $B_0$ is empty, for if $x \in B_0$ the shortest path $x \stackrel{+}{\to} 0_r$ would satisfy the conditions of the lemma. Let

$$A = A_0 \cup A_1 \cup \cdots \cup A_k , \quad B = B_1 \cup \cdots \cup B_k ,$$

so that we have partitioned $E$ into two disjoint sets $A$ and $B$. Experience with other algorithms suggests that we might be able to use these sets $A$ and $B$ to obtain a "good characterization".

If $x$ is independent of $A_j$ in $\mathcal{M}_j$, then either $x$ is independent of $E_j$ in $\mathcal{M}_j$, or $x \in B_j$, or $x \to y$ for some $y \in B_j$. These three cases imply that either $x \in B$ or $B_j$ is empty. In other words, the following statement holds for $1 \leq j \leq k$ :

if $x \in A$, and either $B_j \neq \emptyset$ or $j = r$, then $x$ depends on $A_j$ in $\mathcal{M}_j$ .

A little fiddling around with this condition, and simplifying, leads to the good characterization that is desired:

<u>Theorem 1</u>. <u>Let</u> $\mathcal{M}_1, \ldots, \mathcal{M}_k$ <u>be matroids on a set</u> $E$ . <u>It is possible to find disjoint subsets</u> $E_1, \ldots, E_k$ <u>of</u> $E$ , <u>such that</u> $E_j$ <u>is independent in</u> $\mathcal{M}_j$ <u>and</u> $\|E_j\| = n_j$ , <u>if and only if</u>

$$\|A\| \leq \|E\| - \sum_{j=1}^{k} \max(n_j - r_j(A), 0)$$

<u>for all</u> $A \subseteq E$ , <u>where</u> $r_j$ <u>is the rank function in</u> $\mathcal{M}_j$ .

<u>Proof</u>. The condition is necessary, for if $E_1, \ldots, E_k$ is such a collection of subsets and $A \subseteq E$ then $\|E_j \cap A\| \leq r_j(A)$ , hence

$$\|E_j \cap (E \setminus A)\| \geq n_j - r_j(A) .$$

Also clearly $\|E_j \cap (E \setminus A)\| \geq 0$ . Summing over $j$ gives

$$\|E \setminus A\| \geq \sum_{j=1}^{k} \max(n_j - r_j(A), 0)$$

which is the condition of the theorem.

Conversely, if we have disjoint subsets $E_1, \ldots, E_k$ with $E_j$ independent in $\mathcal{M}_j$ and $\|E_j\| \leq n_j$ and $\|E_r\| < n_r$, the algorithm sketched above will be able to increase $\|E_r\|$ without changing the number of elements in the other sets $E_j$. This must be so, for if the algorithm fails, the set $A$ constructed above satisfies the condition $r_j(A) = \|A_j\|$ or $(A_j = E_j$ and $j \neq r)$ , for all $j$ . Therefore

$\|B_j\| = \|E_j\| - \|A_j\| \leq \max(n_j - r_j(A), 0)$ ; and $\|B_r\| < n_r - r_r(A)$ . Hence $\| B \| = \| E \| - \| A \| < \sum \max(n_j - r_j\ A), 0)$ contradicts the condition of the theorem.

□

The special case of this theorem in which all $\mathcal{M}_j$ are identical and all $n_j = r_j(E)$ was proved by Edmonds [2].

A similar characterization applies when we ask whether or not all elements can be painted.

Theorem 2. Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on a set E . It is possible to find disjoint subsets $E_1, \ldots, E_k$ of E, such that $E_j$ is independent in $\mathcal{M}_j$ and $\|E_j\| \leq n'_j$ and $E = E_1 \cup \ldots \cup E_k$ , if and only if

$$\| A \| = \sum_{j=1}^{k} \min(r_j(A), n'_j)$$

for all $A \subseteq E$ , where $r_j$ is the rank function in $\mathcal{M}_j$ .

Proof. The condition is necessary, since $\| A \| = \sum \|E_j \cap A\| < \sum \min(r_j(A), n'_j$ in any such partitioning.

Conversely, the condition is sufficient. Consider an algorithm which looks for paths $x \overset{+}{\rightarrow} 0_r$ where $x \in E_0$ and j/E,// $< n'_r$ , and which paint:: such x , until this is no longer possible. A construction like that preceding Theorem 1 can be used, but with

$$B_j = \{x | x \in E_j, \text{ and } x \overset{+}{\rightarrow} 0_r \text{ for some } r \text{ with } \|E_r\| < n'_r\} .$$

Then we find $\|A_j\| = r_j(A)$ or $\|A_j\| = n'_j$ for $1 \leq j < k$ . Hence either all elements are painted, or $A_0$ is nonempty and $\| A \| = \|A_0\| + \ldots + \|A_k\| > \sum \min(r_j(A), n'_j)$ .

□

Theorem 2 is implicit in the paper of Edmonds [1], who proved it when all the $n'_j$ are infinite. To get the general case, simply "truncate" $\mathcal{M}_j$ by saying that a set is dependent in $\mathcal{M}_j$ whenever it contains more than $n'_j$ elements. Furthermore we can derive Theorem 1 from Theorem 2, by setting $n'_j = n_j$ and introducing a new matroid $\mathcal{M}_0$ with all sets independent and $n'_0 = \|E\| - (n_1 + \ldots + n_k)$. However, the following theorem seems to be a mild generalization of Edmonds's theorem, not so readily deducible from it:

Theorem 3. $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on a set $E$, and let $(n_j, n'_j)$ be pairs of numbers with $n_j \le n'_j$ for $1 \le j \le k$. It is possible to find disjoint subsets $E_1, \ldots, E_k$ of $E$, such that $E_j$ is independent in $\mathcal{M}_j$ and $n_j \le \|E_j\| \le n'_j$ and $E = E_1 \cup \cdots \cup E_k$, if and only if both the conditions of Theorems 1 and 2 hold for all $A \subseteq E$.

Proof. Consider an algorithm which first looks for a painting satisfying Theorem 1; if it fails, it finds a set A which violates the first condition. If it succeeds, it continues to extend the painting as in Theorem 2. If this fails, it finds a set A which violates the second condition. ☐

## The algorithm

The proof of Theorem 3 leads essentially to the following algorithm, which either finds a partition $E_1, \ldots, E_k$ as specified in that theorem, or finds a set A which proves that no such partition is possible. For ease in description, the algorithm is not "optimized" here.

```
begin E0 :=E; for j :=1 until k do Ej :=∅; for x∈E do color(x) :=0;
   for j :=1 until k do for i :=1 until nj do augment(j);
   while E0 ≠ ∅ do augment(0);
   for j :=1 until k do output Ej;
exit:   end.

procedure augment (integer value r);
  begin for x∈E do succ(x) :=none;
      A:=E;
      B :=if r > 0 then {Or] else {Oj | ||Ej|| < nj'};
    comment  later succ(x) will be set to y if there is a shortest
             path x → y →* Oj for some Oj now in B. Also
             A = {x|succ(x) = none];
    while B ≠ ∅ do
    begin C :=∅;
      for y∈B do for x∈A do
        begin j :=color(y);
          if x∪Ej\y independent in 𝔐j
          then begin succ(x) :=y; A :=A\x; C :=C ∪x;
                  if color(x) = 0 then go to repaint
               end
        end;
      B :=C
    end;
    output A; output "This set A violates the condition of Theorem";
        output if r > 0 then 1 else 2; go to exit;
repaint:  while x∈E do
          begin y :=succ(x); j :=color(x); Ej :=Ej\x; j :=color(y);
             Ej :=Ej ∪x; color(x) :=j; x :=y;
          end
  end.
```

The innermost loop of this algorithm is the test whether $x \cup E_j \backslash y$ is independent in $\mathcal{M}_j$, and it is performed at most $O(n^2 + nk)$ times per call of augment, where $n = \|E\|$. Hence it is performed at most $O(n^3 + n^2 k)$ times in all. However in practice this estimate is probably much too high since the loop will terminate quickly. (The loop "for $x \in A$" should consider those $x \in E_0$ before the other $x$'s.) It is an open question whether this $O(n^3)$ upper bound can be reduced.

Discussion

Consider a very special case of this algorithm, namely the "bipartite matching" or "distinct representatives" problem. Given an $n \times k$ matrix of 0's and 1's, it is desired to encircle exactly one 1 in every row and at most one 1 in every column. Here $\mathcal{M}_j$ corresponds to column $j$ and element $x$ to row $x$, and $n_j = 0$, $n_j' = 1$ for all $j$. A set $E_j$ of rows is independent in $\mathcal{M}_j$ if and only if $E_j = \emptyset$ or $E_j = \{x\}$ where row $x$ contains a 1 in column $j$. In this case the test for independence is, of course, extremely simple, and the algorithm runs in $O(n^3 + n^2 k)$ units of time. Hopcroft and Karp have shown how to reduce this to $O(n^{2.5})$ when $n = k$.

If this example is slightly generalized so that a set $E_j$ is independent in $\mathcal{M}_j$ iff row $x$ contains a 1 in column $j$ for all $x \in E_j$, and if we allow arbitrary $n_j$ and $n_j'$, we have the problem of encircling exactly one 1 in each row, and between $n_j$ and $n_j'$ of them in column $j$. The algorithm works in $O(n^3 + n^2 k)$ units of time for this case also. Ford and Fulkerson [4, 5] call this the "system of restricted representatives" ,(SRR) problem, and they proved Theorem 3 in this case. The conditions in both Theorems 1 and 2 can be simplified in the SRR problem, to

$$\|A\| \leq \|E\| - \sum \{n_j \mid r_j(A) = 0\}$$

and $$\|A\| \leq \sum \{n'_j \mid r_j(A) > 0\}$$

respectively, by altering the set $A$ whenever $0 < r_j(A) < n_j$ or $n'_j$.

Another important case of the algorithm occurs when $k = 2$ and $\mathcal{M}_2$ is taken as the orthogonal complement (or dual) to some matroid $\mathcal{M}$. Then this algorithm can be used to find maximum-cardinality intersections of $\mathcal{M}_1$ and $\mathcal{M}$. (See Edmonds [3, p. 82].)

The algorithm can also probably be generalized to allow the $E_j$'s to overlap, with each $x$ appearing at least $n_x$ and at most $n'_x$ times, and where the set $\{j \mid x \in E_j\}$ is independent in some given matroid $\mathcal{M}_x$. Edmonds [3, p. 83] shows essentially that matroid intersection would give such an algorithm if all the lower bounds are zero.

## References


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