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Asymptotic Representation of the Average Number of Active Modules in an n-way Interleaved Memory

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ABSTRACT

In an n-way interleaved memory the effective bandwidth depends on the average number of concurrently active modules. Using a model for the memory which does not permit queueing on busy modules and which assumes an infinite stream of calls on the modules, where the elements in the stream occur with equal probability, the average number is a combinatorial quantity. Hellerman has previously approximated this quantity by $n^{0.56}$.

We show in this paper that the average number is asymptotically equal to $\sqrt{\frac{\pi n}{2}} - \frac{1}{3}$. The method is due to Knuth and expresses the combinatorial quantity in terms of the incomplete gamma function and its derivatives.

Key words and key phrases

Interleaved memory systems, modular memory systems, memory performance analysis, memory accessing, incomplete gamma function.

CR Categories

6.34, 5.39

1. Introduction

1.1 n-way interleaved storage

To achieve a given storage capacity in a digital computer, it is possible to supply all storage in a single module or to spread it over several modules so that more than one access can take place at a time. Since in a 'typical' program, successive accesses tend to be to successive addresses, successive addresses are assigned to successive modules in a cyclic fashion. Such a memory with n modules is said to be n-way interleaved.

1.2 Average number of active modules

The number of modules that operate concurrently is the number of active modules and is a measure of the speed-improvement over a single-module system. Assume that we have an n-way interleaved memory.

Assume further:

1) an input stream of calls on the modules where each element of the stream is an integer in the range 1 through n.
2) the elements in the stream occur with equal probability.
3) the input stream is infinite so that the memory never idles for lack of work.
4) the time to inspect the input stream is zero or can be overlapped with the access to the modules.
5) queueing of requests on busy modules is not permitted.

With these assumptions Hellerman [1] has shown that the average number of active modules N_{avg} is

$$N_{avg} = \sum_{k=1}^{n} \frac{k^2(n-1)!}{(n-k)!\,n^k}$$

Hellerman [1] has also suggested that a good approximation for N_{avg} is $N_{avg} = n^{0.56}$ $1 \le n \le 45$ to within about 4%. The rest of this paper is concerned with finding an asymptotic formula for N_{avg}. We show that $N_{avg} = \sqrt{\frac{\pi n}{2}} - \frac{1}{3} + O(n^{-\frac{1}{2}})$.

2. Asymptotic representation for N_{avg}

2.1 Notation

For simplicity of presentation, let

$$Q(n) = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,n^k} \tag{1}$$

and

$$Q^{[P(n,k)]}(n) = \sum_{k=1}^{n} \frac{n!\,P(n,k)}{(n-k)!\,n^k} \tag{2}$$

i.e., sum of a series where each term of Q(n) is multiplied by P(n,k).

e.g.

$$Q^{[(n-k+1)^2]}(n) = \sum_{k=1}^{n} \frac{(n-k+1)^2\,n!}{n^k\,(n-k)!}$$

Trivially $Q^{[1]}(n) = Q(n)$

Also $$N_{avg} = \frac{1}{n} Q^{[k^2]}(n) \qquad (3)$$

The procedure used is to find a relation between $Q^{[k^2]}(n)$ and $Q(n)$ and to use the asymptotic representation developed by Knuth [2] for Q(n). Before doing that we have to compute $Q^{[k^2]}(n)$ and other related quantities in terms of the 'incomplete gamma function' and its derivatives (see below).

2.2 Computation of $Q^{[k^2]}(n)$

Sections 2.3 and 2.4 below show how to compute $Q^{[(n-k+1)^2]}(n)$ and $Q^{[n-k+1]}(n)$ in terms of the incomplete gamma function and its derivatives. Then the formula for

$$Q^{[k]}(n) = \sum_{k=1}^{n} \frac{kn!}{(n-k)!\,n^k}$$ can be found from

$$Q^{[k]}(n) + Q^{[n-k+1]}(n) = (n+1)Q^{[1]}(n) = (n+1)Q(n) \qquad (4)$$

Next, using formulas for $Q^{[(n-k+1)^2]}(n)$ and $Q^{[k]}(n)$, the formula for $Q^{[k^2]}$ in terms of the incomplete gamma function and its derivatives can be found from

$$Q^{[(n-k+1)^2]}(n) = (n^2+2n+1)Q(n) - 2(n+1)Q^{[k]}(n) + Q^{[k^2]}(n) \qquad (5)$$

2.3 Computation of $Q^{[(n-k+1)^2]}(n)$

2.3.1 Consider the series

$$S_2 = 1 \cdot 1^2 + n\cdot 2^2 + \frac{n^2}{2!}\cdot 3^2 + \ldots + \frac{n^{n-1}}{(n-1)!}\cdot n^2 + \frac{n^n}{n!}\cdot(n+1)^2 + \ldots$$

We know that

$$xe^x = x + x^2 + \frac{x^3}{2!} + \ldots + \frac{x^{n+1}}{n!} + \ldots$$

Therefore
$$\frac{d}{dx}(xe^x) = 1 + 2x + \frac{3}{2!}x^2 + \ldots \tag{6}$$

and
$$\frac{d}{dx}\left[x\cdot\frac{d}{dx}(xe^x)\right] = 1^2 + 2^2 \cdot \frac{3^2}{2!} \ldots + \frac{(n+1)^2}{n!}x^n + \ldots$$

Therefore the above series

$$S_2 = \frac{d}{dx}\left[x\frac{d}{dx}(xe^x)\right]_{x=n}$$

$$= e^n(1+3n+n^2) \tag{7}$$

Next, consider

$$\frac{n!}{n^n}e^n(1+3n+n^2) = \frac{n!}{n^n}\cdot S_2 \text{ from (7)}$$

$$= \frac{n!}{n^n}\left[(1\cdot 1^2 + n\cdot 2^2 + \ldots \frac{n^{n-1}}{(n-1)!}n^2) + (\frac{n^n}{n!}(n+1)^2 + \ldots)\right]$$

$$= Q^{[(n-k+1)^2]}(n) + R_2(n) \tag{8}$$

where $R_2(n) = \sum_{k\geq 0} \frac{n!\, n^k (n+k+1)^2}{(n+k)!}$

2.3.2 Computation of $R_2(n)$

Let $\gamma(a,x)$ denote the 'incomplete gamma function' ; i.e.,

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt \quad (9)$$

Knuth [2] has shown that

$$e^x \gamma(a,x) = \frac{x^a}{a} + \frac{x^{a+1}}{a(a+1)} + \frac{x^{a+2}}{a(a+1)(a+2)} + \dots \quad (10)$$

Let $G(a,x) = xe^x\gamma(a,x)$

$$= \frac{x^{a+1}}{a} + \frac{x^{a+2}}{a(a+1)} + \frac{x^{a+3}}{a(a+1)(a+2)} + \dots \quad (11)$$

Then $\frac{\partial}{\partial x}G(a,x) = \frac{(a+1)x^a}{a} + \frac{(a+2)x^{a+1}}{a(a+1)} + \dots \quad (12)$

and $\frac{\partial}{\partial x}\left[x\frac{\partial G}{\partial x}\right] = \frac{(a+1)^2}{a} x^a + \frac{(a+2)^2 x^{a+1}}{a(a+1)} \dots$

Let $H_2(a,x) = \frac{\partial}{\partial x}\left[x\frac{\partial G}{\partial x}\right] = \sum_{k\geq 0} \frac{(a+k+1)^2 x^{a+k}}{a(a+1)\dots(a+k)}$

Then $H_2(n,n) = \sum_{k\geq 0} \frac{(n+k+1)^2 n^{n+k}}{n(n+1)\dots(n+k)}$

$$= n^n(n-1)!\sum_{k\geq 0}\frac{(n+k+1)^2 n^k}{(n+k)!}$$ so that

$$R_2(n) = \frac{n!\ H_2(n,n)}{n^n\ (n-1)!} \quad (13)$$

From (11) we can find $H_2(a,x)$ to be

$$H_2(a,x) = \gamma(a,x)e^x\ (1+3x+x^2) + (3x+2x^2)e^x \frac{\partial}{\partial x}\gamma(a,x)$$
$$+ x^2 e^x \frac{\partial^2}{\partial x^2}\gamma(a,x) \quad (14)$$

so that from (13) and (14)

$$R_2(n) = \frac{n!}{n^n} e^n (1+3n+n^2)\left[\frac{\gamma(n,n)}{(n-1)!}\right] +$$

$$\frac{n!}{n^n} e^n \frac{(3n+2n^2)}{(n-1)!}\left[\frac{\partial}{\partial x}\gamma(a,x)\right](n,n) +$$

$$\frac{n!}{n^n} e^n \frac{n^2}{(n-1)!}\left[\frac{\partial^2}{\partial x^2}\gamma(a,x)\right](n,n) \qquad (15)$$

2.4 Computation of $Q^{[n-k+1]}(n) = \sum_{k=1}^{n} \frac{(n-k+1)n!}{(n-k)!n^k}$

2.4.1 Let $S_1 = 1\cdot 1 + n\cdot 2 + \frac{n^2}{2!}\cdot 3 + \ldots + \frac{n^{n-1}}{(n-1)!}\cdot n + \ldots$ (16)

From (6) above,

$$S_1 = \left[\frac{d}{dx}(xe^x)\right]_{x=n} = e^n(1+n) \qquad (17)$$

From (17),

$$\frac{n!}{n^n} e^n (1+n) =$$

$$= \frac{n!}{n^n}\left[\left(1\cdot 1 + n\cdot 2 + \ldots \frac{n^{n-1}}{(n-1)!}\right) + \left(\frac{n^n}{n!}(n+1) \ldots\right)\right]$$

$$= Q^{[n-k+1]}(n) + R_1(n) \qquad (18)$$

where

$$R_1(n) = \sum_{k\geq 0} \frac{n!\, n^k (n+k+1)}{(n+k)!}$$

2.4.2 Computation of $R_1(n)$

Let $H_1(a,x) = \frac{\partial}{\partial x} G(a,x) = \frac{\partial}{\partial x} [xe^x \gamma(a,x)]$

Then from (12),

$$H_1(a,x) = \frac{a+1}{a} x^a + \frac{(a+2)}{a(a+1)} x \quad \ldots\ldots$$

$$= \sum_{k\geq 0} \frac{(a+k+1)\, x^{a+k}}{a(a+1)\ldots(a+k)}$$

so that

$$H_1(n,n) = \sum_{k\geq 0} \frac{(n+k+1)n^{n+k}}{n(n+1)\ldots(n+k)}$$

$$= n^n (n-1)! \cdot \sum_{k\geq 0} \frac{(n+k+1)\, n^k}{(n+k)!}$$

Therefore,

$$R_1(n) = \frac{n!\, H_1(n,n)}{n^n\, (n-1)!} \qquad (19)$$

From (11), we can find $H_1(a,x)$ to be

$$H_1(a,x) = e^x(1+x)[\gamma(a,x)] + xe^x \frac{\partial}{\partial x} \gamma(a,x) \qquad (20)$$

so that

$$R_1(n) = \frac{n!\, e^n(1+n)}{n^n} \left[\frac{\gamma(n,n)}{(n-1)!}\right] + \frac{n!\, e^n\, n}{n^n\, (n-1)!} \frac{\partial}{\partial x}\left[\gamma(a,x)\right]_{(n,n)} \qquad (21)$$

2.3 Computation of $\frac{\partial}{\partial x}\gamma(a,x)$ and $\frac{\partial^2}{\partial x^2}\gamma(a,x)$

From (9),

$$\frac{\partial}{\partial x}\gamma(a,x) = x^{a-1}e^{-x} \tag{22}$$

and

$$\frac{\partial^2}{\partial x^2}\gamma(a,x) = -x^{a-1}e^{-x} + (a-1)x^{a-2}e^{-x} \tag{23}$$

Therefore

$$\frac{\partial}{\partial x}\gamma(a,x)\Big|_{(n,n)} = n^{n-1}e^{-n} \tag{24}$$

and

$$\frac{\partial^2}{\partial x^2}\gamma(a,x)\Big]_{(n,n)} = (n-1)n^{n-2}e^{-n} - n^{n-1}e^{-n} \tag{25}$$

2.6 Computation of $Q^{[k]}(n)$

From (21) and (24) we have,

$$R_1(n) = \frac{n!\,e^n}{n^n}(1+n)\left[\frac{\gamma(n,n)}{(n-1)!}\right] + n \tag{26}$$

Then, from (18),

$$Q^{[n-k+1]}(n) = \frac{n!\,e^n(1+n)}{n^n} - R_1(n)$$

From (26),

$$Q^{[n-k+1]}(n) = \frac{n!\,e^n}{n^n}\left[1+n - \frac{\gamma(n,n)}{(n-1)!}(1-tn)\right] - n \tag{27}$$

Further, Knuth [2] has shown that,

$$Q(n) = \frac{n!}{n^n} e^n \left[1 - \frac{\gamma(n,n)}{(n-1)!}\right] \tag{28}$$

From (27), (28) and (4) we get,

$$Q^{[k]}(n) = \frac{n!\ e^n}{n^n} \left[1+n - \frac{\gamma(n,n)}{(n-1)!}(1+n) - (1+n) + \frac{\gamma(n,n)}{(n-1)!}(1+n)\right]$$

$$+ \quad n$$

Therefore $$Q^{[k]}(n) = n \tag{29}$$

2.7 Computation of N_{avg}

From (24), (25) and (15) we get

$$R_2(n) = \frac{n!\ e^n}{n^n}(1+3n+n^2)\left[\frac{\gamma(n,n)}{(n-1)!}\right] + 2n(1+n) \tag{30}$$

Therefore from (8),

$$Q^{[(n-k+1)^2]}(n) = \frac{n!\,e^n}{n^n}(1+3n+n^2) - R_2(n)$$

$$= \frac{n!\ e^n}{n^n}(1+3n+n^2)\left[1 - \frac{\gamma(n,n)}{(n-1)!}\right] -$$

$$2n(1+n) \tag{31}$$

From (5), 31), (29), and (28) we obtain

$$Q^{[k^2]}(n) = Q^{[(n-k+1)^2]}(n) - (n^2 + 2n + 1)\, Q(n) + 2(n+1)\, Q^{[k]}(n)$$

$$= \frac{n!\ e^n}{n^n}\quad 1 + 3n + n^2 \quad n^2 \quad 2n \quad 1) \left[1 \quad \frac{\gamma(n,n)}{(n-1)!}\right]$$

$$2n\ (1+n) + 2n(1+n)$$

$$= \frac{n!\ e^n}{n^n}\left[1 \quad \frac{\gamma(n,n)}{(n-1)!}\right] \quad n$$

$$= nQ\ n) \tag{32}$$

From (3) and (32) finally we obtain

$$N_{avg} = Q(n) \quad — \tag{33}$$

2.8 <u>Asymptotic formula for N_{avg}</u>

Knuth [2] has given the asymptotic formula for Q(n) as

$$Q(n) = \sqrt{\frac{\pi n}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} - \frac{4}{135n} + \frac{1}{288}\sqrt{\frac{\pi}{2 \cdot n^3}} + O(n^{-2}) \qquad (34)$$

Hence we obtain the desired asymptotic formula for N_{avg} as:

$$N_{avg} = \sqrt{\frac{\pi n}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} - \frac{4}{135n} + \frac{1}{288}\sqrt{\frac{\pi}{2n^3}} + O(n^{-2}) \qquad (35)$$

3.0 Comparison of N_{avg}, $n^{0.56}$ and the result computed by the asymptotic formula

N_{avg}, $n^{0.56}$, and $\sqrt{\frac{\pi n}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} - \frac{4}{135n} + \frac{1}{288}\sqrt{\frac{\pi}{2n^3}}$

were calculated for values of n 2 through 50 on the HP 9100 programmable calculator. The results are summarized below:

1. The maximum percent error in the asymptotic result was 0.0713 at n=2.
2. The minimum percent error in the asymptotic result was 0.00015 at n=50.
3. The percent error in the asymptotic result decreased continuously with n.
4. The percent error and absolute error in the asymptotic result were always less than the percent error and absolute error in the empirical formula for N_{avg}, respectively.

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