AGGREGATION OF INEQUALITIES IN INTEGER PROGRAMMING

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Aggregation of Inequalities in Integer Programming

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Abstract

Given an $m \times n$ zero-one matrix A we ask whether there is a single linear inequality $ax \leq b$ whose zero-one solutions are precisely the zero-one solutions of $Ax \leq e$. We develop an algorithm for answering this question in $O(mn^2)$ steps and investigate other related problems. Our results may be interpreted in terms of graph theory and threshold logic.

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1. Introduction.

Given a set of linear equations

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \qquad (i = 1,2,\ldots,m)\ , \tag{1.1}$$

one may ask whether there is a single linear equation

$$\sum_{j=1}^{n} a_j x_j = b \tag{1.2}$$

such that (1.1) and (1.2) have precisely the same set of zero-one solutions. As shown by Bradley [2], the answer is always affirmative. (Actually, Bradley's results are more general. Some of them have been generalized further by Rosenberg [10].) In this paper, we shall consider a related question: given a set of linear inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1,2,\ldots,m)\ , \tag{1.3}$$

we shall ask whether there is a single linear inequality

$$\sum_{j=1}^{n} a_j x_j \le b \tag{1.4}$$

such that (1.3) and (1.4) have precisely the same set of zero-one solutions. In a sense, which we are about to outline, this problem has been solved long ago.

First, a few definitions. A function

$$f\colon \{0,1\}^n \to \{0,1\}$$

is called a <u>switching function</u>. If there are real numbers $a_1, a_2, \ldots, a_n$ and b such that

$$f(x_1, x_2, \ldots, x_n) = 0 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_j x_j \le b$$

then f is called a <u>threshold function</u>. If there are (not necessarily distinct) zero-one vectors $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_k$ and $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_k$ such that

$$f(y_i) = 0 \;,\; f(z_i) = 1 \quad \text{for all } i = 1,2,\ldots,k$$

and

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} z_i$$

then, for each integer m with $m \geq k$, the function f is called *m-summable.* If f is not m-summable then f is called *m-assumable.* It is well-known [3],[6] that a switching function is threshold if and only if it is m-assumable for every m . (The proof is quite easy: denote by S_i the set of all the zero-one vectors x with f(x) = i . By definition, f is threshold if and only if there-is a hyperplane separating S_0 from S_1 . Such a hyperplane exists if and only if the convex hulls-of S_0 and S_1 are disjoint. Clearly, these convex hulls are disjoint if and only if f is m-assumable for every m .)

Coming back to our problem, we may associate with (1.3) a switching function f defined by

$$f(x_1,x_2,\ldots,x_n) = 0 \quad \text{if and only if (1.3) holds.}$$

Then the desired inequality (1.4) exists if and only if f is m-assumable for every m . However, such an answer to our question is unsatisfactory on several counts. Above all, it does not provide an *efficient* algorithm for deciding whether (1.4) exists. We shall develop such an algorithm in the special case when all the coefficients a_{ij} and b_i in (1.3) are are zeroes and ones.

An $m \times n$ zero-one matrix $A = (a_{ij})$ will be called *threshold* if, and only if, there is a single linear inequality

$$\sum_{j=1}^{n} a_j x_j \leq b$$

whose zero-one solutions are precisely the zero-one solutions of the system

$$\sum_{j=1}^{n} a_{ij} x_j \leq 1 \qquad (i = 1,2,\ldots,m) \;. \tag{1.5}$$

Note that the zero-one solutions of (1.5) are completely determined by the set of those pairs of columns of A which have a positive dot product. This information is conveniently described by means of a graph; in order to make our paper self-contained, we shall now present a few elementary definitions from graph theory.

A graph G is an ordered pair (V,E) such that V is a finite set and E is some set of two-element subsets of V. The elements of V are called the vertices of G, the elements of E are called the edges of G. Two vertices $u,v \in V$ are called adjacent if $\{u,v\} \in E$ and nonadjacent otherwise. For simplicity, we shall denote each edge $\{u,v\}$ by uv. A subset S of V is called stable in G if no two vertices from S are adjacent in G.

With each $m \times n$ zero-one matrix A, we shall associate its intersection graph G(A) defined as follows. The vertices of G(A) are in a one-to-one correspondence with the columns of A; two such vertices are adjacent if and only if the corresponding columns have a positive dot product. The motivation for introducing the concept is obvious: the zero-one solutions of (1.5) are precisely the characteristic vectors of stable sets in G(A). We shall call a graph G with vertices $u_1, u_2, \ldots, u_n$ threshold if there are real numbers $a_1, a_2, \ldots, a_n$ and b such that the zero-one solutions of

$$\sum_{j=1}^{n} a_j x_j \le b$$

are precisely the characteristic vectors of stable sets in G. Clearly, G(A) is threshold if and only if A is threshold; let us also note that G(A) can be constructed from A in $O(mn^2)$ steps. Thus the question "Is A threshold?" reduces into the question "Is G(A) threshold?".

2. The Main Result.

In this section, we develop an algorithm for deciding, within $O(n^2)$ steps, whether a graph G on n vertices is threshold. We shall begin by showing that certain small graphs are not threshold. These graphs are called $2K_2$, P_4 and C_4; they are shown in Figure 1.

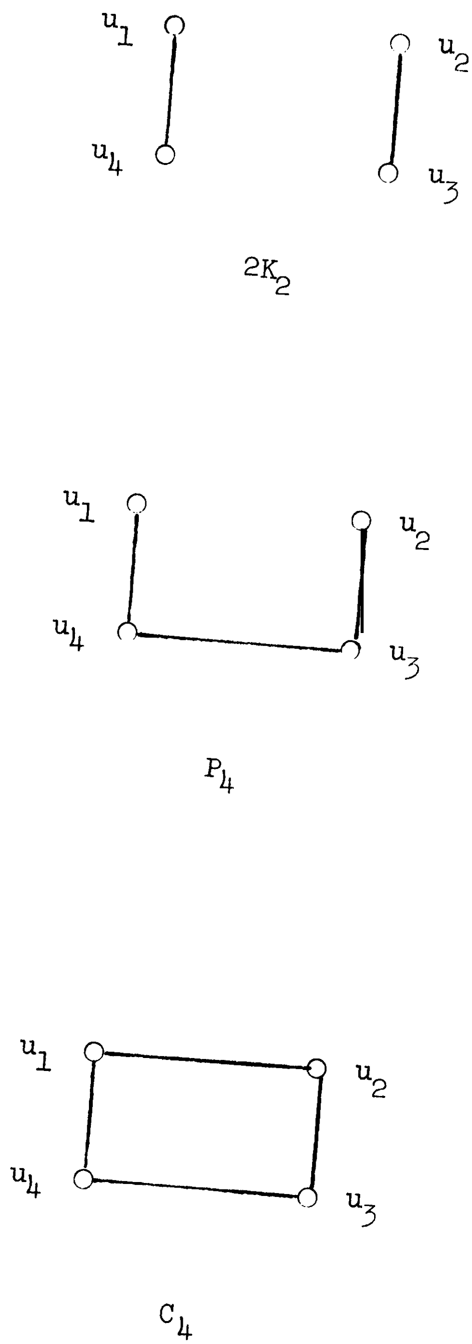


Figure 1

Fact 1. If G is $2K_2$, P_4 or C_4 then G is not threshold.

Proof. Assume that one of the above graphs G is threshold. Then there is a linear inequality

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \le b$$

whose zero-one solutions are precisely the characteristic vectors of stable sets in G. In particular, we have

$$a_1 + a_4 > b \ , \quad a_2 + a_3 > b \ , \quad a_1 + a_3 \le b \ , \quad a_2 + a_4 \le b \ ;$$

clearly, these four inequalities are inconsistent. $\square$

In order to make our next observation about threshold graphs, we need the notion of an "induced subgraph". Let $G = (V,E)$ be a graph and let S be a subset of V. The subgraph of G induced by S is the graph H whose set of vertices is S; two such vertices are adjacent in H if and only if they are adjacent in G.

Fact 2. If G is a threshold graph then every induced subgraph of G is threshold.

Proof. Let the zero-one solutions of

$$\sum_{j=1}^{n} a_j x_j \le b$$

be precisely the characteristic vectors of stable sets in G. Let H be a subgraph of G induced by S. Denote by Σ^* the summation over all the subscripts j with $u_j \in S$. Then the zero-one solutions of

$$\Sigma^* a_j x_j \le b$$

are precisely the characteristic vectors of stable sets in H. $\square$

Now, we have an easy way of showing that certain graphs are not threshold (simply by pointing out an induced subgraph isomorphic to $2K_2$, P_4 or C_4). On the other hand, we are about to develop a way of showing that certain graphs <u>are</u> threshold. Let G be a graph with vertices $u_1, u_2, \ldots, u_n$. G will be called <u>strongly threshold</u> if there are <u>positive integers</u> $a_1, a_2, \ldots, a_n$ and b such that the zero-one solutions of

$$\sum_{j=1}^{n} a_j x_j \leq b$$

are precisely the characteristic vectors of stable sets in G. (It will turn out later, and may be proved directly, that every threshold graph is strongly threshold.) We shall show that the property of being strongly threshold is preserved under two simple operations. Let G be a graph with vertices $u_1, u_2, \ldots, u_n$. By $G + K_1$, we shall denote the graph obtained from G by adding a new vertex u_{n+1} and all the edges $u_i u_{n+1}$ with $1 \leq i \leq n$. By $G \cup K_1$, we shall denote the graph obtained from G by adding a new vertex u_{n+1} and no edges at all.

<u>Fact 3</u>. If G is strongly threshold then $G + K_1$ and $G \cup K_1$ are strongly threshold.

<u>Proof</u>. Let $a_1, a_2, \ldots, a_n$ and b be positive integers such that the zero-one solutions of

$$\sum_{j=1}^{n} a_j x_j \leq b$$

are precisely the characteristic vectors of stable sets in G. Then the zero-one solutions of

$$\sum_{j=1}^{n} a_j x_j + b x_{n+1} \leq b$$

are precisely the characteristic vectors of stable sets in $G + K_1$. Similarly, the zero-one solutions of

$$2\sum_{j=1}^{n} a_j x_j + x_{n+1} \le 2b+1$$

are precisely the characteristic vectors of stable sets in $G \cup K_1$. □

Now, we are ready for the theorem.

<u>Theorem 1</u>. For every graph G , the following three conditions are equivalent:

(i) G is threshold,

(ii) G has no induced subgraph isomorphic to $2K_2$, P_4 or C_4 ,

(iii) there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G and a partition of $\{v_2, v_3, \ldots, v_n\}$ into disjoint subsets P and Q such that

(*) every $v_j \in P$ is adjacent to all the vertices v_i with $i < j$,

(*) every $v_j \in Q$ is adjacent to none of the vertices v_i with $i < j$.

<u>Proof</u>. The implication (i) ⇒ (ii) follows from Fact 1 and Fact 2. The implication (iii) ⇒ (i) may be deduced from Fact 3. Indeed, let G_t denote the subgraph of G induced by $\{v_1, v_2, \ldots, v_t\}$. If $v_{t+1} \in P$ then $G_{t+1} = G + K_1$; if $v_{t+1} \in Q$ then $G_{t+1} = G_t \cup K_1$. Hence, by induction on t , every G_t is strongly threshold.

It remains to be proved that (ii) ⇒ (iii) . We shall accomplish this by means of an algorithm which finds, for every graph G , either one of the three forbidden induced subgraphs or the ordering and partition described in (iii). If G has n vertices then the algorithm takes $O(n^2)$ steps.

Before the description of the algorithm, a few preliminary remarks may be in order. It will be convenient to introduce the notion of the <u>degree</u> $d_G(u)$ of a vertex u in a graph G ; this quantity is simply the number of vertices of G which are adjacent to u . At each stage of the algorithm, we shall deal with some sequence S of k vertices of G ; the remaining vertices will already be enumerated as

$v_{k+1}, v_{k+2}, \dots, v_n$ and partitioned into sets P and Q . Furthermore, each $w \in S$ will be adjacent to all the vertices from P and to no vertices from Q , hence it will be adjacent to exactly $d_G(w) - |P|$ vertices from S . The algorithm is fairly straightforward; only Step 4 may require justification. Executing that step, we shall first find vertices $u_1, u_2, u_3 \in S$ such that $d_G(u_1) \geq d_G(u_2)$ and such that u_3 is adjacent to u_2 but not to u_1 . It follows easily that there must be a fourth vertex $u_4 \in S$ which is adjacent to u_1 but not to u_2 . The algorithm goes as follows.

<u>Step 0</u>. For each vertex w of G , evaluate $d_G(w)$. (This may take as many as $O(n^2)$ steps-) Then arrange the vertices of G into a sequence $w_1, w_2, \dots, w_n$ such that

$$d_G(w_1) \geq d_G(w_2) \geq \dots \geq d_G(w_n)$$

call this sequence S . (This can be done in O(n log n) steps; the rest of the algorithm takes only O(n) steps.) Set $k = n$ and $P = Q = \emptyset$.

<u>Step 1</u>. If $k = 1$ then S has only one term; call that vertex v_1, and stop. If $k > 1$ then let u be the first term of S and let v be the last term of S ; note that

$$|P|+k-1 \geq d_G(u) \geq d_G(w) \geq d_G(v) \geq |P|$$

for every $w \in S$. If $d_G(u) = |P|+k-1$, go to Step 2. If $d_G(v) = |P|$, go to Step 3. If $|P| < d_G(v) \leq d_G(u) < |P|+k-1$, go to Step 4.

<u>Step 2</u>. Set $v_k = u$, delete u from S , replace P by $P \cup \{v_k\}$, replace k by k-1 and return to Step 1.

<u>Step 3</u>. Let $v_k = v$, delete v from S , replace Q by $Q \cup \{v_k\}$, replace k by k-1 and return to Step 1.

<u>Step 4</u>. Let $u_1 = u$. Find a vertex $u_3 \in S$ which is not adjacent to u_1 . Find a vertex $u_2 \in S$ which is adjacent to u_3 . Find a vertex $u_4 \in S$ which is adjacent to u_1 but not to u_2 . Then stop (the vertices u_1, u_2, u_3, u_4 induce $2K_2$ or P_4 or C_4 in G). □

In the rest of this section, we shall present several consequences of Theorem 1.

Remark 1. For every graph $G = (V,E)$, we may define a binary relation $<$ on V by writing $u < v$ if, and only if,

$$uw \in E,\ w \neq v \Rightarrow wv \in E .$$

By this definition, $<$ is reflexive and transitive but not necessarily antisymmetric. From Theorem 1, we conclude the following.

Corollary 1A. A graph G is threshold if and only if for every two distinct vertices u, v of G, at least one of $u < v$ and $v < u$ holds.

Remark2. For every graph $G = (V,E)$ and for every vertex u of G, we define

$$N(u) = \{v \in V: v \text{ is adjacent to } u\} .$$

From Theorem 1, we conclude the following.

Corollary 1B. A graph G is threshold if and only if there is a partition of V into disjoint sets A, B and an ordering $u_1, u_2, \ldots, u_k$ of B such that

(*) every two vertices in A are adjacent,

(*) no two vertices in B are adjacent,

(*) $N(u_1) \supseteq N(u_2) \supseteq \cdots \supseteq N(u_k)$.

Let us sketch the proof. If G has the structure described by Corollary 1B then G cannot possibly have an induced subgraph isomorphic to $2K_2$, P_4 or C_4; hence G is threshold. On the other hand, if G is threshold then G has the structure described by (iii) of Theorem 1. In that case, we may set $A = V-Q$, $B = Q$ and order B consistently with $v_1, v_2, \ldots, v_n$.

Remark 3. For every graph G, we define the complement $\bar{G}$ of G to be a graph with the same set of vertices as G; two distinct vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. From the equivalence of (i) and (ii) in Theorem 1, we conclude the following.

Corollary 1C. A graph is threshold if and only if its complement is threshold.

Let us point out that this fact does not seem to follow directly from the definition.

Remark 4. In order to decide whether a graph G (with vertices $u_1, u_2, \ldots, u_n$) is threshold, it suffices to know only the degrees $d_G(u_1), d_G(u_2), \ldots, d_G(u_n)$ of its vertices. Indeed, executing Steps 1, 2 and 3 of the algorithm, we manipulate only these quantities. On the other hand, if we are about to execute Step 4 then we already know that G is not threshold.

Remark 5. Theorem 1 implies that threshold graphs are very rare. Indeed, from (iii) of Theorem 1, we conclude that the number of distinct threshold graphs with vertices $u_1, u_2, \ldots, u_n$ does not exceed

$$n!\, 2^{n-1} .$$

On the other hand, the number of all distinct graphs with the same set of vertices is

$$2^{n(n-1)/2} .$$

Hence a randomly chosen graph will almost certainly be not threshold.

Remark 6. With each graph G on vertices $u_1, u_2, \ldots, u_n$, we may associate a switching function

$$f: \{0,1\}^n \to \{0,1\}$$

by setting $f(x_1, x_2, \ldots, x_n) = 0$ if and only if $(x_1, x_2, \ldots, x_n)$ is the characteristic vector of some stable set in G. A switching function arising in this way will be called graphic. From Theorem 1, we conclude the following.

Corollary I.D. A graphic switching function is threshold if and only if it is 2-assumable.

Let us point out that for switching functions that are not graphic, the "if" part of Corollary 1D is no longer true. Indeed, for every m with $m \geq 2$, there are switching functions which are m-assumable but not $(m+1)$-assumable. Ingenious examples of such functions have been constructed by Winder [12].

Remark 7. When $\underset{\sim}{A} = (a_{ij})$ is an $m \times n$ zero-one matrix, we shall consider the following zero-one linear programming problem:

$$\left.\begin{array}{lll} \text{maximize } \sum_{j=1}^{n} c_j x_j & \text{subject to the constraints} & \\ \sum_{j=1}^{n} a_{ij} x_j \leq 1 & (1 \leq i \leq m) & , \\ x_j = 0, 1 & (1 \leq j \leq n) & . \end{array}\right\} \tag{2.1}$$

Defining $c(u_j) = c_j$ for every vertex u_j of $G(A)$, we reduce (2.1) to the following problem:

$$\begin{array}{l} \text{in } G(A), \text{ find a stable set } S \\ \text{maximizing } c(S) = \sum_{u \in S} c(u) . \end{array} \tag{2.2}$$

In general, (2.2) is hard; one may ask whether it becomes any easier when $\underset{\sim}{A}$ is threshold. The answer is affirmative. Indeed, if $G(\underset{\sim}{A})$ is threshold then we can find the ordering $v_1, v_2, \ldots, v_n$ and the partition $P \cup Q$ described in (iii), Theorem 1; this takes only $O(mn^2)$ steps. Then we define

$$S_1 = \begin{cases} \emptyset & \text{if } c(v_1) < 0 \\ \{v_1\} & \text{if } c(v_1) \geq 0 \end{cases}$$

and, for each t with $2 < t < n$,

$$S_t = \begin{cases} S_{t-1} & \text{if } v_t \in Q, \text{ and } c(v_t) < 0 \\ S_{t-1} \cup \{v_t\} & \text{if } v_t \in Q \text{ and } c(v_t) \geq 0 \\ S_{t-1} & \text{if } v_t \in P \text{ and } c(v_t) < c(S_{t-1}) \\ \{v_t\} & \text{if } v_t \in P \text{ and } c(v_t) \geq c(S_{t-1}) \end{cases}$$

Clearly, S_n is a solution of (2.2).

3. Variations.

Let $\underset{\sim}{A} = (a_{ij})$ be an $m \times n$ zero-one matrix. We shall denote by $t(\underset{\sim}{A})$ the smallest t for which there exists a system of linear inequalities

$$\sum_{j=1}^{n} c_{ij} x_j \leq d_i \qquad (1 \leq i < t) \qquad (3.1)$$

such that (3-l) and

$$\sum_{j=1}^{n} a_{ij} x_j \leq 1 \qquad (1 \leq i \leq m) \qquad (3.2)$$

have the same set of zero-one solutions. Theorem 1 characterizes matrices A with $t(A) = 1$; in this section, we shall discuss the problem of-finding $t(A)$ for every matrix A .

Again, the language of graph theory will be useful. For every graph $G = (V,E)$, we shall denote by $t(G)$ the smallest t such that there are threshold graphs $G_1 = (V,E_1)$, $G_2 = (V,E_2)$,..., $G_t = (V,E_t)$ with $E_1 \cup E_2 \cup \ldots \cup E_t = E$. Our next result may not sound too surprising. Note, however, that Theorem 1 is used in its proof.

Theorem 2. Let $\underset{\sim}{A}$ be a zero-one matrix and let G be $G(A)$. Then $t(A) = t(G)$.

Proof. The inequality $t(\tilde{A}) \leq t(G)$ is fairly routine. Indeed, there are t threshold graphs $G_i = (V,E_i)$ with $\cup E_i = E$ and $t = t(G)$. For each i , there is an inequality

$$\sum_{j=1}^{n} c_{ij}x_j \leq d_i$$

whose zero-one solutions are precisely the characteristic vectors of stable sets in G_i . A subset of V is stable in G if and only if it is stable in every G_i .Hence the zero-one solutions of the system

$$\sum_{j=1}^{n} c_{ij}x_j \leq d_i \qquad (1 \leq i \leq t) \tag{3.3}$$

are precisely the characteristic vectors of stable sets in G . Since $G = G(A)$, thecharacteristic vectors of stable sets in G are precisely the zero-on solutions of (3.1). Hence $t(\tilde{A}) \leq t = t(G)$.

In order to prove the reversed inequality, we shall use Theorem 1. There is a system (3.2) with $t = t(A)$ such that (3.1) and (3.2) have the same set of zero-one solutions. "Set $V = \{u_1,u_2,\ldots,u_n\}$ for each i , define

$$E_i = \{u_ru_s: r \neq s \text{ and } c_{ir}+c_{is} > d_i\}$$

and $G_i = (V,E_i)$. Since (3.1) and (3.2) have the same set of zero-one solutions, we have

$$\bigcup_{i=1}^{t} E_i = \{u_ru_s: a_{ir}+a_{is} > 1 \text{ for some } i = 1,2,\ldots,m\} .$$

Hence $G = (V, \cup E_i)$ is $G(\tilde{A})$; it remains to be proved that each G_i is threshold. Assume the contrary. Then, by part (ii) of Theorem 1, there are vertices u_r , u_s , u_p , u_q such that

$$u_ru_q \in E_i, \quad u_su_p \in E_i$$

$$u_ru_p \notin E_i \ , \quad u_su_q \notin E_i \quad .$$

Hence by the definition of E_i, we have

$$c_{ir} + c_{iq} > d_i \ , \quad c_{is} + c_{ip} > d_i \ ,$$
$$c_{ir} + c_{ip} \leq d_i \ , \quad c_{is} + c_{iq} \leq d_i \ ,$$

clearly, these four inequalities are inconsistent. □

Next, we shall establish an upper bound on $t(G)$. In order to do that, we shall need a few more graph-theoretical concepts. A triangle is a graph consisting of the pairwise adjacent vertices; a star (centered at u) is a graph all of whose edges contain the same vertex u. The stability number $\alpha(G)$ of a graph G is the size of the largest stable set in G.

Theorem 3. For every graph G on n vertices, we have $t(G) \leq n-\alpha(G)$. Furthermore, if G contains no triangle then $t(G) = n-\alpha(G)$.

Proof. Write $G = (V,E)$ and $k = n-\alpha(G)$. Let S be a largest stable set in G; enumerate the vertices in $V-S$ as $u_1, u_2, \ldots, u_k$. For each i with $1 \leq i \leq k$, let E_i consist of all the edges of G which contain u_i. Then each $G_i = (V,E_i)$ is a star and therefore a threshold graph. Since S is stable, we have $\cup E_i = E$. Hence $t(G) \leq k$.

Secondly, let us assume that G contains no triangle. There are t threshold graphs $G_i = (V,E_i)$ with $1 < i < t$, $t = t(G)$ and $\cup E_i = E$. It follows easily from Theorem 1 that each G_i, being threshold and containing no triangle, must be a star. Hence there are vertices $u_1, u_2, \ldots, u_t$ such that every edge of every G_i contains u_i. Since $\cup E_i = E$, the set

$$V - \{u_1, u_2, \ldots, u_t\}$$

is stable in G. Hence $\alpha(G) \geq n-t(G)$. □

Let us note that we may have $t(G) = n-\alpha(G)$ even when G does contain a triangle. For example, see the graph in Figure 2.

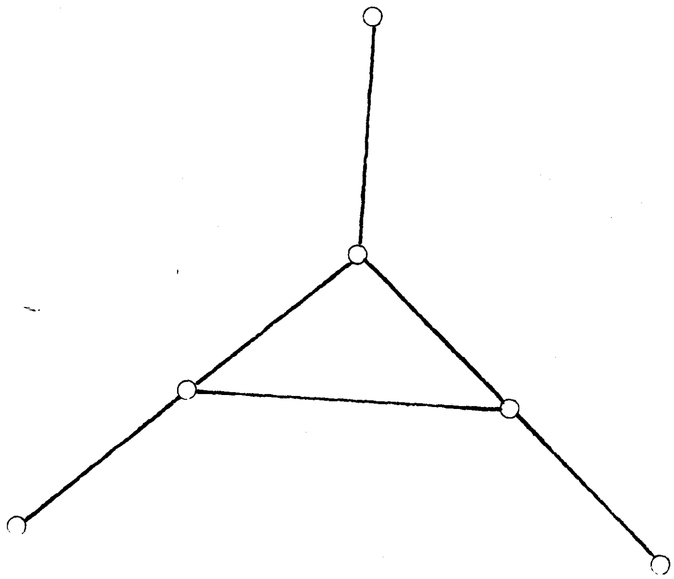


Figure 2

When $a(G)$ is very large, the upper bound on $t(G)$ given by Theorem 3 is much smaller than n. On the other hand, if $a(G)$ is very small then $t(G)$ is often very small. (In particular, if $\alpha(G) = 1$ then $t(G) = 1$.) Thus one might hope that, say, $t(G) \leq n/2$ for every graph on n vertices. Our next result shows such hopes to be very much unjustified.

<u>Corollary 3A</u>. For every positive ε there is a graph G on n vertices such that $t(G) > (1-\varepsilon)n$.

<u>Proof</u>. Erdös [7] has proved that for every positive integer k there is a graph G on n vertices such that G contains no triangle, $\alpha(G) < k$ and, for some positive constant c (independent of k), $n > c(k/\log k)^2$. Given a positive ε, choose k large enough, so that $\varepsilon ck \geq (\log k)^2$, and consider the graph G with the above properties. We have

$$a(G) < k < \frac{n}{ck}(\log k)^2 \leq \varepsilon n$$

and so, by Theorem 3, $t(G) = n-\alpha(G) > (1-\varepsilon)n$. □

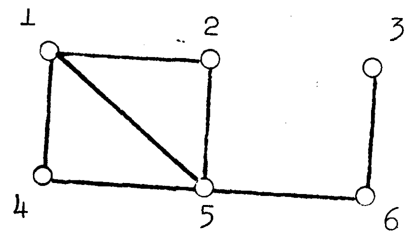
Finally, we shall show that the problem of finding $t(G)$ is very hard; more precisely, we shall show that it is "NP-hard". Perhaps a brief sketch of the meaning of this term is called for- There is a certain wide class of problems; this class is called NP. It includes some very hard problems such as the problem of deciding whether the vertices of a graph are colorable in k colors. An algorithm for solving a problem is called <u>good</u> if it terminates within a number of steps not exceeding some (fixed) polynomial in the length of the input [5]. A few years ago, Cook [4] proved that the existence of a good algorithm for finding the stability number of a graph would imply the existence of a good algorithm for <u>every</u> problem in NP. Such a conclusion, if true, is very strong. (For example, it implies the existence of a good algorithm for the celebrated traveling salesman problem.) A problem X is called <u>NP-hard</u> if the existence of a good algorithm for X would imply the existence of a good algorithm for every problem in NP. (For more information on the subject, the reader is referred to [1] and [8].)

Corollary 3B. The problem of finding $t(G)$ is NP-hard.

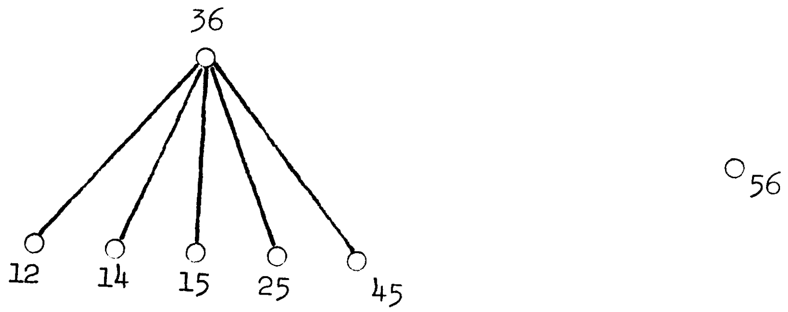
Proof. Poljak [9] proved that even for graphs G that contain no triangles, the problem of finding "a(G) is NP-hard. For such graphs, however, we have $\alpha(G) = n-t(G)$; hence the existence of a good algorithm for finding $t(G)$ would imply the existence of a good algorithm for Poljak's problem. Since Poljak's problem is NP-hard, our problem is NP-hard. •l

We shall close this section with two remarks on $t(G)$.

Remark 1. First of all, we shall present a simple lower bound on $t(G)$. For every graph $G = (V,E)$, let us define a new graph $G^* = (V^*,E^*)$ as follows. The vertices of G^* are the edges of G ; that is, $V^* = E$. Two vertices of G^* , say $\{u,v\} \in V^*$ and $\{w,z\} \in V^*$, are adjacent in G^* if and only if the set $\{u,v,w,z\}$ induces $2K_2$, P_4 or C_4 in G . Figure 3 shows an example of G and G^* .



G



G^*

Figure3

As usual, the chromatic number $\chi(H)$ of a graph $H = (V,E)$ is the smallest k such that V can be partitioned into k stable sets. We claim that

$$t(G) \geq \chi(G^*) \quad . \tag{3.4}$$

Indeed, there are threshold graphs $G_i = (V,E_i)$ with $1 \leq i \leq t$, $t = t(G)$ and $\cup E_i = E$. By (ii) of Theorem 1 and by our definition of G^*, each E_i is a stable set of vertices in G^*. Hence $\chi(G^*) \leq t$.

Note that the problem of finding the chromatic number of a graph is NP-hard; hence for large graphs G, the right-hand side of (3.4) may be very difficult to evaluate. For small graphs, however, (3.4) is quite useful and often precise. In fact, we know of no instance where it holds with the sharp inequality sign.

Problem. Is there a graph G such that $t(G) > \chi(G^*)$?

Remark 2. We shall outline a heuristic for finding a "small" (although not necessarily the smallest,) number of threshold graphs $G_i = (V,E_i)$ such that $\cup E_i = E$, thereby providing an upper bound on $t(G)$. The heuristic is based on a subroutine for finding a "large" threshold graph $G^0 = (V,E^0)$ with $E^0 \subset E$.

The subroutine goes as follows. Given a graph $G = (V,E)$, find a vertex v of the largest degree in G, let S be the set of all the vertices adjacent to v and let $H = (S,T)$ be the subgraph of G induced by S. Applying the subroutine recursively to H, find a "large" threshold graph $H^0 = (S,T^0)$ with $T^0 \subset T$. Then define

$$E^0 = T^0 \cup \{wv: w \in S\}$$

and $G^0 = (V,E^0)$.

The heuristic goes as follows. Given a graph $G = (V,E)$, use the subroutine to find a large threshold graph $G^0 = (V,E^0)$ with $E^0 \subset E$. Applying the heuristic recursively to the graph $(V, E-E^0)$, find threshold graphs $G_i = (V,E_i)$ with $\cup E_i = E$ and, say, $1 \leq i \leq k$. Then define $G_{k+1} = G^0$.

Clearly, the running time for this heuristic is $O(n^3)$.

4. Pseudothreshold Graphs.

A switching function $f: \{0,1\}^n \to \{0,1\}$ is called pseudothreshold [11] if there are real numbers $a_1, a_2, \ldots, a_n, b$ (not all of them zero), such that, for every zero-one vector $(x_1, x_2, \ldots, x_n)$, we have

$$\sum_{j=1}^{n} a_j x_j < b \Rightarrow f(x_1, x_2, \ldots, x_n) = 0 ,$$

$$\sum_{j=1}^{n} a_j x_j > b \Rightarrow f(x_1, x_2, \ldots, x_n) = 1 .$$

By analogy, we shall call a graph pseudothreshold if there are real numbers $a(u), b$ $(u \in V)$, not all of them zero, such that, for every subset S of V, we have

$$\begin{aligned} \sum_{u \in S} a(u) < b &\Rightarrow S \text{ is stable,} \\ \sum_{u \in S} a(u) > b &\Rightarrow S \text{ is not stable.} \end{aligned} \tag{4.1}$$

In this section, we shall investigate the pseudothreshold graphs. (We do so at the suggestion of the referee of an earlier version of this paper,) In fact, we shall develop an algorithm for deciding whether a graph is pseudothreshold. When G has n vertices, the algorithm terminates within $O(n^4)$ steps; it is not unlikely that this bound may be improved.

We shall begin by making our definition a little easier to work with.

Fact 1. A graph is pseudo-threshold if and only if there are real numbers $a(u), b$ $(u \in V)$ such that b is positive and, for every subset S of V, we have (4.1).

Proof. The "if" part is trivial; in order to prove the "only if" part, we shall consider a pseudothreshold graph $G = (V,E)$. We may assume $E \neq \emptyset$ (otherwise $a(u) \equiv 0$ and $b = 1$ does the job). Since the empty set is stable, (4.1) implies $b \geq 0$. In order to prove $b > 0$, we shall assume $b = 0$ and derive a contradiction. First of all, since every one-point set is stable, we have $a(u) \leq 0$ for every $u \in V$. Secondly, since not every $a(u)$ is zero, there is a vertex w with $a(w) < 0$. Finally, since $E \neq \emptyset$, there are adjacent vertices u and v. Setting $S = \{u,v,w\}$ we contradict (4.1). □

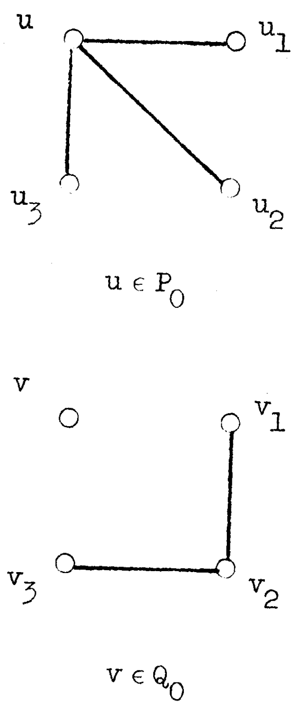
From now on, we shall assume $b > 0$. For every graph $G = (V,E)$ we shall define two subsets P_0, Q_0 of V. The set P_0 consists of all the vertices u for which there are three other vertices u_1, u_2, u_3 such that

$$uu_1, uu_2, uu_3 \in E \quad , \quad u_1u_2, u_1u_3, u_2u_3 \notin E \quad .$$

The set Q_0 consists of all the vertices v for which there are three other vertices v_1, v_2, v_3 such that

$$vv_1, vv_2, vv_3, v_1v_3 \notin E \quad , \quad v_1v_2, v_2v_3 \in E \quad .$$

These definitions are illustrated in Figure 4.



$u \in P_0$

$v \in Q_0$

Figure 4

Fact 2. Let $G = (V,E)$ be a pseudothreshold graph. Then

$$u \in P_0 \Rightarrow a(u) \geq 2b/3$$

$$v \in Q_0 \Rightarrow a(u) \leq b/3 \quad .$$

Proof. First of all, if $u \in P_0$ then

$$a(u_1) + a(u_2) + a(u_3) \leq b$$
$$a(u) + a(u_1) \geq b$$
$$a(u) + a(u_2) \geq b$$
$$a(u) + a(u_3) \geq b$$

and so $3a(u) \geq 2b$. Secondly, if $v \in Q_0$ then

$$a(v) + a(v_1) + a(v_3) < b ,$$
$$a(v) + a(v_2) \leq b ,$$
$$a(v_1) + a(v_2) \geq b ,$$
$$a(v_2) + a(v_3) \geq b$$

and so $3a(v) \leq b$. $\square$

Next, we shall define (by induction on t)

$$P_{t+1} = P_t \cup \{u \in V: uv \in E \text{ for some } v \in Q_t\} ,$$
$$Q_{t+1} = Q_t \cup \{v \in V: uv \notin E \text{ for some } u \in P_t\} ,$$

and

$$P^* = \bigcup_{t=0}^{\infty} P_t \quad , \quad Q^* = \bigcup_{t=0}^{\infty} Q_t \quad .$$

Bact . If G is a pseudothreshold graph then $P^* \cap Q^* = \emptyset$.

Proof. It suffices to prove that

$$u \in P^* \Rightarrow a(u) \geq 2b/3 ,$$

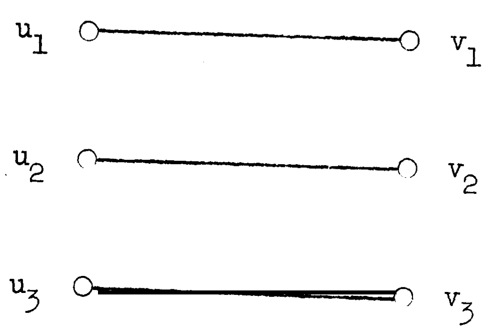
$$v \in Q^* \Rightarrow a(v) \leq b/3 ,$$

these implications follow easily (by induction on t) from Fact 2. □

From the definition of P^* and Q^* , we readily conclude the following.

Fact 4. If $P^* \cap Q^* = \emptyset$ then every two vertices in P^* are adjacent and no two vertices in Q^* are adjacent. □

Our next observation involves the graph $3K_2$ shown in Figure 5.



$3K_2$

Figure 5

Fact No pseudothreshold graph contains an induced subgraph isomorphic to $3K_2$.

proof. Assume the contrary. Then

$$a(u_1) + a(u_2) + a(u_3) \leq b ,$$
$$a(v_1) + a(v_2) + a(v_3) \leq b ,$$
$$a(u_1) + a(v_1) \geq b ,$$
$$a(u_2) + a(v_2) \geq b ,$$
$$a(u_3) + a(v_3) > b .$$

Trivially, these inequalities are inconsistent with $b > 0$. □

Theorem 4. For every graph $G = (V,E)$, the following three properties are equivalent:

(i) G is pseudo-threshold,

(ii) $P^* \cap Q^* = \emptyset$ and G has no induced subgraph isomorphic to $3K_2$,

(iii) there is a partition of V into pairwise disjoint subsets P, Q and R such that

(*) every vertex from P is adjacent to every vertex from $P \cup R$,

(*) no vertex from Q is adjacent to another vertex from $Q \cup R$,

(*) there are no three pairwise nonadjacent vertices in R.

Proof. The implication (i) ⇒ (ii) follows from Fact 3 and Fact 4. To see that (iii) ⇒ (i), simply set $b = 2$ and

$$a(u) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \in R \\ 2 & \text{if } u \in P . \end{cases}$$

It remains to be proved that (ii) ⇒ (iii). We shall do this by means of a very simple algorithm which terminates in $O(n^4)$ steps either by showing that (ii) does not hold or by constructing the partition described in (iii). The algorithm goes as follows.

First of all, find P^* and Q^*. (This can certainly be done in $O(n^4)$ steps.) Then find out whether $P^* \cap Q^* = \emptyset$. (If not, stop: (ii) does not hold.) Then set $S = V - (P^* \cup Q^*)$; note that by the definition of P^* and Q^*, every vertex from S is adjacent to all the vertices from P^* and to no vertex from Q^*. Let S_0 consist of all the vertices in S which are adjacent to no other vertex in S ; define

$$P = P^* , Q = Q^* \cup S_0 , \quad R = S - S_0 .$$

Find out whether there are three pairwise nonadjacent vertices in R. If not, stop: P, Q and R have all the properties described in (iii). If, on the other hand, there are three pairwise nonadjacent vertices $u_1, u_2, u_3 \in R$ then each u_i is adjacent to some $v_i \in R$. All three v_i's are distinct and pairwise nonadjacent (otherwise, as the reader can easily verify, we would have $R \cap (P_0 \cup Q_0) \neq \emptyset$.) Hence G has an induced subgraph isomorphic to $3K_2$ and so (ii) does not hold. □

Remark. It may be worth pointing out the following corollary of Theorem 1: If G is pseudothreshold then one can satisfy (4.1) with $b = 2$ and each $a(u) \in \{0,1,2\}$.

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