C^m CONVERGENCE OF TRIGONOMETRIC INTERPOLANTS

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ABSTRACT

For $m \geq 0$, we obtain sharp estimates of the uniform accuracy of the m-th derivative of the n-point trigonometric interpolant of a function for two classes of periodic functions on $\mathbb{R}$. As a corollary, the n-point interpolant of a function in C^k uniformly approximates the function to order $o(n^{1/2-k})$, improving the recent estimate of $\mathcal{O}(n^{1-k})$. These results remain valid if we replace the trigonometric interpolant by its K-th partial sum, replacing n by K in the estimates.

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1. Introduction and Notation

Using the concept of aliasing, Snider [6] obtains an $\mathcal{O}(n^{1-k})$ estimate of the uniform accuracy of the n-point trigonometric interpolants of periodic C^k functions for $k > 2$, improving the $\mathcal{O}(n^{-1/2})$ estimate for C^2 functions presented in Isaacson and Keller [2]. Kreiss and Oliger [4] use aliasing to show that if the Fourier coefficients $\hat{v}(\xi)$ of a periodic function $v(x)$ satisfy $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-\beta})$ with $\beta > 1$, then the trigonometric interpolants of v uniformly approximate v to order $\mathcal{O}(n^{1-\beta})$. This also gives an $\mathcal{O}(n^{1-k})$ estimate for C^k functions since the largest β we can use in general is $\beta = k$. We use aliasing and a different property of the Fourier coefficients of C^k functions--the fact that C^k is contained in the Sobolev space H^k-- to obtain an $o(n^{1/2-k})$ estimate for $k > 1$.

In [5], Kreiss and Oliger estimate the L^2 accuracy of trigonometric interpolants and their derivatives for functions in Sobolev spaces. This paper applies their approach and an extension of a theorem appearing in Zygmund [7] to obtain an $o(n^{1/2+m-s})$ estimate of the uniform accuracy of the m-th derivatives of trigonometric interpolants of functions in the Sobolev spaces H^s for $s > \frac{1}{2} + m$. By similar methods we obtain an $o(n^{m-k})$ estimate for functions in C^k whose k-th derivatives have absolutely converging Fourier series if $k > m$, and we show that these two estimates are sharp. We also obtain an $\mathcal{O}(n^{1/2+m-k-\alpha})$ estimate for functions in the Holder space $C^{k,\alpha}$ if $0 < \alpha < 1$ and $k + \alpha > \frac{1}{2} + m$. These results remain valid if we replace the trigonometric interpolant by its K-th partial sum, replacing n by

K in the estimates.

All functions considered will be assumed to be defined on $\mathbb{R}$ and one-periodic. We use the following notation.

$\|v\|_\infty$ denotes $\sup|v(x)|$.

L^2 is the set of complex-valued measurable functions $v(x)$ for which

$$\|v\|_2^2 = \int_0^1 |v(x)|^2 dx < \infty \quad .$$

The Fourier series of a function $v(x) \in L^2$ is

$$\sum_{\xi=-\infty}^{\infty} \hat{v}(\xi) e^{2\pi i \xi x}$$

where $$\hat{v}(\xi) = \int_0^1 v(x) e^{-2\pi i \xi x} dx \quad .$$

$D^k v$ denotes $d^k v/dx^k$. If we say that $D^k v \in B$ for some space of functions B, we mean that $D^{k-1} v$ is an indefinite integral of the function $D^k v$ in B. C^k is the set of functions with k continuous derivatives.

$$\|v\|_{C^k} = \sum_{j=0}^{k} \|D^j v\|_\infty$$

For a real number $s > 0$, H^s is the set of functions $v(x) \in L^2$ such that

$$\|v\|_{H^s}^2 = |\hat{v}(0)|^2 + \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 < \infty \quad .$$

A is the set of functions $v(x) \in L^2$ with absolutely converging Fourier series, i.e.,

$$\sum_{\xi=-\infty}^{\infty} |\hat{v}(\xi)| < \infty$$

For $0 < \alpha < 1$, let

$$[v]_\alpha = \sup_{x,y \in \mathbb{R}} \frac{|v(x)-v(y)|}{|x-y|^a}$$

For an integer $k > 0$, $C^{k,\alpha}$ is the set of functions $v(x) \in C^k$ such that $[D^k v]_\alpha < \infty$.

If $v \in A$, then v is equal a.e. to a continuous function. Since we are interested in interpolation, we will tacitly assume that $A \subset C^0$ and similarly that $H^s \subset C^0$ for $s > \frac{1}{2}$. For an integer $k > 1$, H^k is the set of functions $v(x)$ such that $D^k v \in L^2$ and thus $C^k \subset H^k$. See Agmon [1] for a discussion of L^2 derivatives.

2. Trigonometric Interpolation

We state some well known results on trigonometric interpolation. These appear in this form for odd n in Kreiss and Oliger [4]. See also Isaacson and Keller [2] and Zygmund [7].

A. n is odd. Let $N > 0$ be an integer and $h = \frac{1}{2N+1}$ and let $x_\nu = \nu h$ for $\nu = 0,1,2,\ldots,2N$. There is a unique trigonometric polynomial $I_N v(x)$ of order at most N which interpolates $v(x)$ at the points x_ν for $0 \le \nu \le 2N$ given by

(1) $$I_N v(x) = \sum_{\xi=-N}^{N} a(\xi) e^{2\pi i \xi x}$$

where

(2) $$a(\xi) = h \sum_{\nu=0}^{2N} v(x_\nu) e^{-2\pi i \xi x_\nu} .$$

The effect called aliasing is the fact that

(3) $$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}(\xi + j(2N+1)) \qquad |\xi| \le N$$

provided that the Fourier series for $v(x)$ converges at the points x_ν for $0 \le \nu \le 2N$.

Following the notation of Zygmund, define for $1 \le K \le N$

(4) $$I_{N,K} v(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i \xi x}$$

where $a(\xi)$ is given by (2). $I_{N,K}v$ is the K-th partial sum of $I_N v$, and $I_{N,N}v = I_N v$. If $v(x)$ is real-valued, so is $I_{N,K}v$.

B. N is even. Let $N > 0$ be an integer and $h = \frac{1}{2N}$ and let $x_\nu = \nu h$ for $0 \le \nu \le 2N-1$. There is a unique trigonometric polynomial $E_N v(x)$ of order at most N which interpolates $v(x)$ at the points x_ν for $0 < \nu \le 2N-1$ given by

(5) $$E_N v(x) = \sum_{\xi=-N}^{N}{}' a(\xi) e^{2\pi i \xi x}$$

which also satisfies

$$a(-N) = a(N) .$$

The Σ' notation indicates that the first and last terms are multiplied by 1/2. The coefficients are given by

(6) $$a(\xi) = h \sum_{\nu=0}^{2N-1} v(x_\nu) e^{-2\pi i \xi x_\nu} .$$

Provided that the Fourier series for $v(x)$ converges at the points x_ν for $0 \le \nu \le 2N-1$, we have

(7) $$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}(\xi + j(2N)) \qquad |\xi| \le N$$

Define for $1 < K < N$

(8) $$E_{N,K} v(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i \xi x}$$

where $a(\xi)$ is given by (6), and let $E_{N,N} v = E_N v$. If $v(x)$ is real-valued, so is $E_{N,K} v$ for $K < N$. If $w(x)$ is a trigonometric polynomial of order at most N and $\hat{w}(N) = \hat{w}(-N)$, then $E_N w = w$.

3. Accuracy Estimation

Define

$$\delta(v,m,N,K) = \|D^m v - D^m(I_{N,K}v)\|_\infty$$

$$\epsilon(v,m,N,K) = \|D^m v - D^m(E_{N,K}v)\|_\infty$$

The $m = 0$ case of the following lemma appears in Theorem **5.16** of Chapter 10 in Zygmund [7].

Lemma 1. Let $m \geq 0$ be an integer, and suppose that $u = D^m v \in A$. Then

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi|>K} |\hat{u}(\xi)|$$

Proof. Let

(9) $$v_L(x) = \sum_{\xi=-K}^{K} \hat{v}(\xi)e^{2\pi i\xi x} \qquad v_R(x) = \sum_{|\xi|>K} \hat{v}(\xi)e^{2\pi i\xi x}$$

(10) $$w_L = I_{N,K}v_L \qquad w_R = I_{N,K}v_R$$

Then $v = v_L + v_R$ and $I_{N,K}v = w_L + w_R$. Since $w_L = v_L$,

(11) $$v - I_{N,K}v = v_R - w_R$$

so

(12) $$\delta(v,m,N,K) \leq \|D^m v_R\|_\infty + \|D^m w_R\|_\infty \,.$$

By (3),

$$w_R(x) = \sum_{\xi=-K}^{K} \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j\ 2N+1\ )e^{2\pi i\xi x}$$

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-K}^{K} |2\pi\xi|^m \sum_{j=-\infty}^{\infty} |\hat{v}_R(\xi + j(2N+1))|$$

$$\leq \sum_{\xi=-K}^{K} \sum_{j=-\infty}^{\infty} |2\pi(\xi + j(2N+1\ )|^m |\hat{v}_R(\xi + j(2N+1))|$$

$$\leq \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^m |\hat{v}_R(\xi)|$$

so

(13) $$\|D^m w_R\|_\infty \leq \sum_{|\xi| > K} |\hat{u}(\xi)|$$

Also

(14) $$\|D^m v_R\|_\infty \leq \sum_{|\xi| > K} |\hat{u}(\xi)|$$

Combining (12), (13), and (14) gives the lemma.

Lemma 2. Let $m \geq 0$ be an integer, and suppose that $u = D^m v \in A$. Then

$$\epsilon(v,m,N,K) \leq 2 \sum_{|\xi| > K} |\hat{u}(\xi)| \qquad \text{for} \qquad K < N$$

$$\epsilon(v,m,N,N) \leq 2 \sum_{|\xi| \geq N} |\hat{u}(\xi)|$$

Proof. For $K < N$, the proof is the same as in Lemma 1.

Using (9) with K = N - 1 and replacing (10) by

(15) $$w_L = E_N v_L \qquad\qquad w_R = E_N v_R$$

we obtain

(16) $$\epsilon(v,m,N,N) \leq \|D^m v_R\|_\infty + \|D^m w_R\|_\infty$$

By (7),

$$w_R(x) = \sum_{\xi=-N}^{N}{}' \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N))e^{2\pi i\xi x}$$

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-N}^{N}{}' \sum_{j=-\infty}^{\infty} |2\pi(\xi + j(2N))|^m |\hat{v}_R(\xi + j(2N))|$$

$$= \sum_{\xi=-\infty}^{\infty} |2\pi\xi|^m |\hat{v}_R(\xi)|$$

and the lemma follows as'in the proof of Lemma 1.

Theorem 1. Let $m > 0$ be an integer and $v \in H^s$ with $s > \frac{1}{2} + m$. Then for each K,

(17) $$\sup_{N \geq K} \delta(v,m,N,K) < CR_K(v)K^{1/2+m-s}$$

where

$$c = \frac{2\,(2\pi)^{m-s}}{\sqrt{s - \frac{1}{2} - m}}$$

and

$$R_K(v) = \left(\sum_{|\xi| > K} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 \right)^{1/2} .$$

Also

(18) $$\sup_{N > K} \epsilon(v,m,N,K) \leq CR_K(v)K^{1/2+m-s}$$

and

(19) $$\epsilon(v,m,K,K) \leq CR_{K-1}(v)(K-1)^{1/2+m-s}$$

Note that since $v \in H^s$, $R_K(v) \to 0$ as $K \to \infty$.

Proof. By Lemma 1, for $N \geq K$ we have

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi| > K} |2\pi\xi|^m |\hat{v}(\xi)|$$

$$\leq 2\left(\sum_{|\xi| > K} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 \right)^{1/2} \left(\sum_{|\xi| > K} |2\pi\xi|^{2(m-s)} \right)^{1/2}$$

$$\leq 2 R_K(v)(2\pi)^{m-s}\left(2 \frac{K^{1+2(m-s)}}{2(s-m) - 1}\right)^{1/2}$$

and (17) follows. (18) and (19) follow similarly from Lemma 2.

Theorem 2. Let $k \geq m \geq 0$ be integers, and suppose $D^k v \in A$. Then for each K,

(20) $$\sup_{N \geq K} \delta(v,m,N,K) \leq Cr_K(v)K^{m-k}$$

where

$$C = 2(2\pi)^{m-k}$$

and

$$r_K(v) = \sum_{|\xi| > K} |2\pi\xi|^k |\hat{v}(\xi)| \quad .$$

Also

(21) $$\sup_{N > K} \epsilon(v,m,N,K) < Cr_K(v)K^{m-k}$$

and

(22) $$\epsilon(v,m,K,K) \leq Cr_{K-1}(v)K^{m-k}$$

Note that since $D^k v \in A$, $r_K(v) \to 0$ as $K \to \infty$.

Proof. By Lemma 1, for $N \geq K$ we have

$$\delta(v,m,N,K) \leq 2 \sum_{|\xi| > K} |2\pi\xi|^m |\hat{v}(\xi)|$$

$$\leq 2(2\pi K)^{m-k} \sum_{|\xi| > K} |2\pi\xi|^k |\hat{v}(\xi)|$$

and (20) follows. (21) and (22) follow similarly from Lemma 2.

Theorem 3. Let $m \geq 0$ be an integer and $v \in C^{k,\alpha}$ with $k + \alpha > \frac{1}{2} + m$. Then for each K,

(23) $$\sup_{N \geq K} \delta(v,m,N,K) < C[D^k v]_\alpha K^{1/2+m-k-\alpha}$$

where

$$C = \frac{2^{\alpha+1/2}\pi^{m-k}}{1-2^{1/2+m-k-\alpha}}$$

Also

(24) $$\sup_{N \geq K} \epsilon(v,m,N,K) < C[D^k v]_\alpha K^{1/2+m-k-\alpha}$$

Proof. The method of proof is similar to that of Bernstein's theorem that $C^{0,\alpha} \subset A$ for $\alpha > \frac{1}{2}$. See Katznelson [3]. Let $u = D^m v$ and $f = D^k v$. If $t = \frac{1}{3} 2^{-\nu}$ and $2^\nu \leq |\xi| \leq 2^{\nu+1}$, then $|e^{2\pi i \xi t} - 1| > \sqrt{3}$, so since

$$f(x+t) - f(x) = \sum_{\xi=-\infty}^{\infty} (e^{2\pi i \xi t} - 1)\hat{f}(\xi)e^{2\pi i \xi x}$$

Parseval's relation implies that

$$\sum_{2^\nu \leq |\xi| \leq 2^{\nu+1}} |\hat{f}(\xi)|^2 \leq \frac{1}{3} \sum_{2^\nu < |\xi| \leq 2^{\nu+1}} |e^{2\pi i \xi t} - 1|^2 |\hat{f}(\xi)|^2$$

$$\leq \frac{1}{3} \|f(x+t) - f(x)\|_2^2$$

$$\leq \frac{1}{3} \|f(x+t) - f(x)\|_\infty^2$$

$$\leq \frac{1}{3} t^{2\alpha} [f]_\alpha^2$$

$$\leq \frac{1}{3} 2^{-2\nu\alpha} [f]_\alpha^2$$

By the Schwarz inequality,

$$\sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)| \le (2^{\nu+1} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)|^2)^{1/2}$$

$$= (2^{\nu+1} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} \frac{|\hat{f}(\xi)|^2}{|2\pi\xi|^{22(k-m)}})^{1/2}$$

$$\le (2\pi)^{m-k}\, 2^{\nu(1/2+m-k)}(2 \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{f}(\xi)|^2)^{1/2}$$

$$\le (2\pi)^{m-k}\, 2^{\nu(1/2+m-k-\alpha)} [f]_\alpha$$

Given K, let j satisfy $2j < K < 2^{j+1}$. Then by Lemma 1, for $N \ge K$ we have

$$\delta(v,m,N,K) \le 2 \sum_{|\xi| \ge K} |\hat{u}(\xi)|$$

$$\le 2 \sum_{\nu=j}^{\infty} \sum_{2^\nu \le |\xi| < 2^{\nu+1}} |\hat{u}(\xi)|$$

$$\le 2(2\pi)^{m-k} [f]_\alpha \sum_{\nu=j}^{\infty} 2^{\nu(1/2+m-k-\alpha)}$$

$$\le 2(2\pi)^{m-k} [f]_\alpha \frac{(2^j)^{1/2+m-k-\alpha}}{1 - 2^{1/2+m-k-\alpha}}$$

and (23) follows since $\frac{K}{2} \ge 2^j$ and $\frac{1}{2} + m - k - \alpha < 0$. **(24)** follows similarly from Lemma 2.

4. Sharpness of Estimates

Theorem 1 shows that if $v \in H^s$ and $s > \frac{1}{2} + m$, then $\delta(v,m,N,K)$ and $\epsilon(v,m,N,K)$ are $o(K^{1/2+m-s})$, independent of $N > K$. Theorem 2 shows that if $D^k v \in A$ and $k > m$, then $\delta(v,m,N,K)$ and $\epsilon(v,m,N,K)$ are $o(K^{m-k})$, independent of $N > K$. We prove in this section that these estimates are sharp: they cannot be improved for these two classes of functions.

Theorem 4. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $m \geq 0$ be an integer, and $s > \frac{1}{2} + m$. Then there exists a $v \in H^s$ such that

$$\limsup_{K \to \infty} \frac{\inf_{N > K} \delta(v,m,N,K)}{\gamma_K K^{1/2+m-s}} = \infty \tag{25}$$

Proof. Let $p_0 = 1$ and define a strictly increasing sequence $\{p_j\}$ of positive integers inductively such that for $j > 1$, if j is odd $p_j = 2p_{j-1}$, and if j is even p_j is a power of 2 such that

$$\gamma_\nu \leq 2^{-j} \qquad \text{for} \qquad \nu \geq p_j/4 \ . \tag{26}$$

Define the sequence $\{b_\nu\}$ for $\nu \geq 1$ by

$$b_\nu = \left(\frac{2^{-j}}{p_{j+1} - p_j}\right)^{1/2} \qquad \text{for} \qquad p_j \leq \nu < p_{j+1} \tag{27}$$

Then $\sum_{\nu=1}^{\infty} b_\nu^2 = \sum_{j=0}^{\infty} \sum_{p_j \leq \nu < p_{j+1}} b_\nu^2 = \sum_{j=0}^{\infty} 2^{-j} < \infty.$

Note that $b_\nu \geq b_{\nu+1}$ for $\nu > 1$ since $p_j \geq 2p_{j-1}$ for $j>0$. Let

(28) $$v(x) = \sum_{\nu=1}^{\infty} (-1)^\nu \frac{1}{(2\pi\nu)^s} b_\nu e^{2\pi i\nu x}$$

Since $\sum_{\xi=-\infty}^{\infty} |2\pi\xi|^{2s} |\hat{v}(\xi)|^2 = \sum_{\nu=1}^{\infty} b_\nu^2 < \infty$, $v \in H^s$. Define v_L, v_R, w_L, and w_R as in (9) and (10). By (11),

(29) $$\delta(v,m,N,K) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty .$$

Now

$$|D^m v_R(\tfrac{1}{2})| = \left| \sum_{|\xi|>K} (2\pi i\xi)^m \hat{v}(\xi) e^{\pi i\xi} \right| = \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu$$

so

(30) $$\|D^m v_R\|_\infty \geq \sum_{\nu>K} (2\pi\nu)^{m-s} b_\nu .$$

By (3),

$$w_R(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i\xi x}$$

where for $|\xi| \leq K$,

$$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N+1)) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N+1))$$

Since 2N + 1 is odd, this last series is an alternating series of terms decreasing in absolute value, so

$$|a(\xi)| \leq |\hat{v}(\xi + 2N + 1)| .$$

Hence

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-K}^{K} |2\pi\xi|^m |a(\xi)|$$

$$\leq \sum_{\xi=-K}^{K} |2\pi(\xi + 2N + 1)|^m |\hat{v}(\xi + 2N + 1)|$$

$$= \sum_{\nu=2N+1-K}^{2N+1+K} (2\pi\nu)^{m-s} b_\nu$$

$$\leq \sum_{\nu=K+1}^{3K+1} (2\pi\nu)^{m-s} b_\nu$$

since the b_ν's form a decreasing sequence. Combining this with (29) and (30) yields

$$\delta(v,m,N,K) \geq \sum_{\nu=3K+2}^{\infty} (2\pi\nu)^{m-s} b_\nu .$$

For even $j > 4$, let $K_j = p_j/4$. Then since $p_{j+1} = 2p_j$,

$$\delta(v,m,N,K_j) \geq \sum_{\nu=p_j}^{\infty} (2\pi\nu)^{m-s} b_\nu$$

$$> \sum_{p_j \leq \nu < p_{j+1}} (2\pi\nu)^{m-s} (p_j 2^j)^{-1/2}$$

$$\geq (p_j 2^j)^{-1/2} (2\pi)^{m-s} \int_{p_j}^{2p_j} \frac{dx}{x^{s-m}}$$

Now $\int_{p_j}^{2p_j} \frac{dx}{x^\beta} = c_\beta p_j^{1-\beta}$ where

$$c_\beta = \begin{cases} \frac{2^{1-\beta}-1}{1-\beta} & \text{for} \quad \beta \neq 1 \\ \log 2 & \text{for} \quad \beta = 1 \end{cases}$$

so if $d_\beta = 2^{1-3\beta}\pi^{-\beta}c_\beta$,

$$\delta(v,m,N,K_j) \geq c_{s-m}\, 2^{-j/2}(2\pi)^{m-s}p_j^{1/2+m-s}$$

$$= d_{s-m}2^{-j/2}K_j^{1/2+m-s}$$

Thus (26) implies that

$$\frac{\delta(v,m,N,K_j)}{\gamma_{K_j} K_j^{1/2+m-s}} \geq d_{s-m}2^{j/2}$$

and the theorem follows.

Theorem 5. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $k > m \geq 0$ be integers. Then there exists a v with $D^k v \in A$ such that

(31)
$$\limsup_{K\to\infty} \frac{\inf_{n\geq K} \delta(v,m,N,K)}{\gamma_K\, K^{m-k}} = \infty \ .$$

Proof. Same as the proof of Theorem 4 with the following alterations. Replace s by k throughout the proof. Replace (26) by

(26')
$$\gamma_\nu \leq 2^{-2j} \qquad \text{for} \qquad \nu \geq p_j/4 \ .$$

Define $b_\nu = \dfrac{2^{-j}}{p_{j+1} - p_j}$ for $p_j \leq \nu < p_{j+1}$.

Then $\sum_{\nu=1}^{\infty} b_\nu < \infty$ and $\sum_{\xi=-\infty}^{\infty} |2\pi\xi|^k |\hat{v}(\xi)| < \infty$ so $D^k v \in A$. We have for even $j > 4$

$$\begin{aligned}
\delta(v,m,N,K_j) &\geq \sum_{\nu=p_j}^{\infty} (2\pi\nu)^{m-k} b_\nu \\
&> \sum_{p_j \leq \nu < p_{j+1}} (2\pi\nu)^{m-k} (p_j 2^j)^{-1} \\
&\geq (p_j 2^j)^{-1} (2\pi)^{m-k} \int_{p_j}^{2p_j} \frac{dx}{x^{k-m}} \\
&= c_{k-m} 2^{-j} (2\pi)^{m-k} p_j^{m-k} \\
&= \frac{1}{2} d_{k-m} 2^{-j} K_j^{m-k}
\end{aligned}$$

Thus (26') implies that

$$\frac{\delta(v,m,N,K_j)}{\gamma_{K_j} K_j^{m-k}} \geq \frac{1}{2} d_{k-m} 2^j$$

and the theorem follows.

The following lemma is geometrically obvious.

<u>Lemma 3</u>. Let $\{\beta_\nu\}$ be a decreasing sequence of positive numbers converging to 0. Then $\sum_{\nu=1}^{\infty} \beta_\nu e^{2\pi i\nu/3}$ converges and

$$\left| \sum_{\nu=1}^{\infty} \beta_\nu e^{2\pi i\nu/3} \right| \leq \beta_1 .$$

Theorem 6. Let $\{\gamma_\nu\}$ be a sequence of positive numbers converging to 0. Let $m \geq 0$ be an integer, and $s > \frac{1}{2} + m$. Then there exists a $v \in H^s$ such that

$$\limsup_{K \to \infty} \frac{\inf_{N > K, 3 \nmid N} \epsilon(v,m,N,K)}{\gamma_K K^{1/2+m-s}} = \infty \tag{32}$$

and

$$\limsup_{N \to \infty} \frac{\epsilon(v,m,N,N)}{\gamma_N N^{1/2+m-s}} = \infty \ . \tag{33}$$

If k is an integer with k > m, then there exists a v with $D^k v \in A$ such that

$$\limsup_{K \to \infty} \frac{\inf_{N > K, 3 \nmid N} \epsilon(v,m,N,K)}{\gamma_K K^{m-k}} = \infty \tag{34}$$

and

$$\limsup_{N \to \infty} \frac{\epsilon(v,m,N,N)}{\gamma_N N^{m-k}} = \infty \ . \tag{35}$$

Proof. The proof of (32) is the same as the proof of Theorem 4 with the following alterations. Replace (28) by

$$v(x) = \sum_{\nu=1}^{\infty} e^{2\pi i\nu/3} \frac{1}{(2\pi\nu)^s} b_\nu e^{2\pi i\nu x} \ .$$

For N > K, we have

$$\epsilon(v,m,N,K) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty$$

where v_R is given by (9) and $w_R = E_{N,K} v_R$. Now

$$|D^m v_R(\tfrac{2}{3})| = |\sum_{|\xi| \gg K} (2\pi i\xi)^m \hat{v}(\xi) e^{4\pi i\xi/3}| = \sum_{\nu > K} (2\pi\nu)^{m-s} b_\nu$$

so

$$\|D^m v_R\|_\infty \geq \sum_{\nu > K} (2\pi\nu)^{m-s} b_\nu \quad .$$

By (7),

$$w_R(x) = \sum_{\xi=-K}^{K} a(\xi) e^{2\pi i\xi x}$$

where for $|\xi| \leq K$,

$$a(\xi) = \sum_{j=-\infty}^{\infty} \hat{v}_R(\xi + j(2N)) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N)) \quad .$$

Suppose $3 \nmid N$. Then $j(2N)$ cycles through the equivalence classes mod 3, so by Lemma 3,

$$|a(\xi)| \leq |\hat{v}(\xi + 2N)| \quad .$$

Hence, as before,

$$\|D^m w_R\|_\infty \leq \sum_{\nu=K+1}^{3K+1} (2\pi\nu)^{m-s} b_\nu$$

and the rest of the proof goes through, establishing (32).

To prove (33) for this v, imitate the proof of Theorem 4 as above with-the following changes. Define v_L and v_R by (9) with $K = N - 1$, and define w_L and w_R by (15). Then

$$\epsilon(v,m,N,N) \geq \|D^m v_R\|_\infty - \|D^m w_R\|_\infty \; .$$

As above,

$$\|D^m v_R\|_\infty \geq \sum_{\nu > N} (2\pi\nu)^{m-s} b_\nu \quad .$$

By (7),

$$w_R(x) = \sum_{\xi=-N}^{N}{}' \, a(\xi) e^{2\pi i \xi x}$$

where

$$a(\xi) = \sum_{j=1}^{\infty} \hat{v}(\xi + j(2N)) \qquad \text{for} \quad |\xi| < N$$

$$a(-N) = a(N) = \sum_{j=0}^{\infty} \hat{v}(N + j(2N))$$

For $N = K_j$ for even $j > 4$, $3 \nmid N$, so by Lemma 3,

$$|a(\xi)| \leq |\hat{v}(\xi + 2N)| \qquad \text{for} \quad |\xi| < N$$

$$|a(-N)| = |a(N)| \leq |\hat{v}(N)| \quad .$$

Hence

$$\|D^m w_R\|_\infty \leq \sum_{\xi=-N}^{N}{}' \, |2\pi\xi|^m |a(\xi)|$$

$$\leq \sum_{\xi=-N+1}^{N-1} |2\pi(\xi + 2N)|^m |\hat{v}(\xi + 2N)| + |2\pi N|^m |\hat{v}(N)|$$

$$= \sum_{\nu=N}^{3N-1} (2\pi\nu)^{m-s} b_\nu$$

So

$$\epsilon(v,m,N,N) \geq \sum_{\nu=3N}^{\infty} (2\pi\nu)^{m-s} b_\nu$$

and (33) follows.

(34) and (35) follow by similar alterations to the proof of Theorem 5.

<u>Remarks.</u> Theorem 4 shows that the $o(K^{1/2+m-s})$ estimate of $\delta(v,m,N,K)$ given by Theorem 1 is sharp by showing that there is no function g(K) going to 0 faster than $K^{1/2+m-s}$ for which $\delta(v,m,N,K) = \mathcal{O}(g(K))$ for all $v \in H^s$. Note that we can obtain a real-valued function in H^s satisfying (25): since the trigonometric interpolants of real-valued functions are real-valued, at least one of the real or imaginary parts of the v constructed must also satisfy (25). Similar statements hold for Theorem 5 and 6. Also, many of the details of the constructions are for convenience, e.g. making the p_j's powers of 2, and placing the singularities at $x = \frac{1}{2}$ in the odd case and at $x = \frac{2}{3}$ in the even case.

5. Corollaries and Summary

Let w_n denote the n-point trigonometric interpolant of v. i.e., if $n = 2N + 1$, $w_n = I_N v$ and if $n = 2N$, $w_n = E_N v$.

Corollary 1. Let $m \geq 0$ be an integer. If $v \in H^s$ with $s > \frac{1}{2} + m$, then

$$\|v - w_n\|_{C^m} = o(n^{1/2+m-s})$$

If $D^k v \in A$ and $k > m$, then

$$\|v - w_n\|_{C^m} = o(n^{m-k})$$

If $v \in C^{k,\alpha}$ and $k + \alpha > \frac{1}{2} + m$, then

$$\|v - w_n\|_{C^m} = \mathcal{O}(n^{1/2+m-k-\alpha}) .$$

The $m = 0$ case gives the improved estimate for C^k functions:

Corollary 2. If $v \in C^k$ and $k > 1$, then

$$\|v - w_n\|_\infty = o(n^{1/2-k}) .$$

These corollaries also hold for the K-th partial sums of w_n if we replace n by K in the estimates.

Although we gain an extra half power of n in the estimate for general C^k functions over the recent $\mathcal{O}(n^{1-k})$ estimate, there are other classes of functions for which Kreiss and Oliger's $\mathcal{O}(n^{1-\beta})$ estimate for functions satisfying $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-\beta})$ yields better results. For example, if $D^k v$ is not necessarily continuous but is of bounded variation, then $\hat{v}(\xi) = \mathcal{O}(|\xi|^{-k-1})$, so $\|v - w_n\|_\infty = \mathcal{O}(n^{-k})$. Or, if $D^{k-1}v$ is absolutely continuous (or equivalently if $D^k v \in L^1$), then $\hat{v}(\xi) = o(|\xi|^{-k})$, and Kreiss and Oliger's proof shows that $\|v - w_n\|_\infty = o(n^{1-k})$ if $k > 1$. See Katznelson [3] and Zygmund [7] for discussions of the growth of Fourier coefficients. We conclude with a table of estimates.

If $D^k v \in$	then $\|v - w_n\|_\infty =$	for
L^1	$o(n^{1-k})$	$k > 2$
L^2	$o(n^{1/2-k})$	$k > 1$
$C^{0,\alpha}$	$\mathcal{O}(n^{1/2-k-a})$	$k + \alpha > \frac{1}{2}$
H^s	$o(n^{112-k-s})$	$k + s > \frac{1}{2}$
B.V.	$\mathcal{O}(n^{-k})$	$k > 1$
A	$o(n^{-k})$	$k > 0$.

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