AN ANALYSIS OF A MEMORY ALLOCATION SCHEME FOR IMPLEMENTING STACKS

by

Andrew C. Yao

STAN-CS-79-7Ø8
January 1979

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

An Analysis of a Memory Allocation Scheme for Implementing Stacks */

Andrew C. Yao

Computer Science Department
Stanford University
Stanford, California 94305

Abstract.

Consider the implementation of two stacks by letting them grow towards each other in a table of size m . Suppose a random sequence of insertions and deletions are executed, with each instruction having a fixed probability p ($0 < p < 1/2$) to be a deletion. Let $A_p(m)$ denote the expected value of $\max\{x,y\}$, where x and y are the stack heights when the table first becomes full. We shall -prove that, as $m \to \infty$, $A_p(m) = m/2 + \sqrt{m/(2\pi(1-2p))} + O((\log m)/\sqrt{m})$. This gives a solution to an open problem in Knuth [The Art of Computer Programming Vol. 1, Exercise 2.2.2-13].

*/ This research was supported in part by National Science Foundation under grant MCS77-05313. Part of this paper was prepared while the author was visiting Bell Laboratories, Murray Hill, N.J.

1. Introduction.

The purpose of this paper is to give a solution to an open problem of Knuth [2, Exercise 2.2.2-13], regarding the effectiveness of implementing two stacks by letting them grow towards each other.

Consider a contiguous block of m locations, which we use to implement two stacks. One stack grows from the leftend of the block and the other from the rightend; we denote the heights of the stacks by x and y (see Figure 1). One measure*/ of the effectiveness of the memory utilization for this scheme is the expected value of $\max\{x,y\}$ when the two stacks first meet, i.e., when $x+y = m$. For example, suppose the value of $\max\{x,y\}$ is $2m/3$. If we had used one block for each stack, then we should have reserved at least $4m/3$ locations instead of the present m locations. The following model was proposed in [2], with p $(0 \le p < 1)$ as a parameter. Consider a sequence of stack operations to be carried out, until the two stacks meet. Each instruction is either on the left stack or on the right stack with equal probability; and for each choice, there is a -probability p for it to be a <u>deletion</u> and probability 1-p to be an <u>insertion</u>. A deletion on an empty stack will not have any effect. Let $A_p(m)$ denote the expected value of $\max\{x,y\}$ when the two stacks first meet. It was shown in Knuth [2, Exercise 2.2.2-12] that $A_0(m) = m/2 + \sqrt{m/(2\pi)} + O(m^{-1/2})$. It was also stated [2, Exercise 2.2.2-13] that $\lim_{p\to 1} A_p(m) = 3m/4$ for fixed m . Thus, in this model, there is little gain in memory utilization for large m when only insertions are

---

*/ This measure is somewhat conservative. An alternative measure might be the expected value of $\max\{x,y\}$ at any time before the two stacks meet.

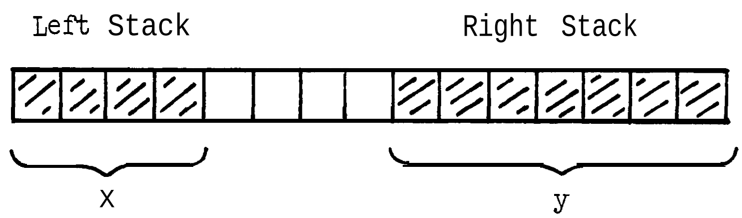

Figure 1. Two stacks growing towards each other.

present; whereas substantial gain results when deletions are overwhelmingly dominant. The question asked was the behavior of $A_p(m)$ for fixed p and large m.

In this paper we prove the following result.

Theorem 1. Let $p \in (0,1/2)$ be a fixed number. Then*/

$$A_p(m) = \frac{m}{2} + \sqrt{\frac{m}{2\pi(1-2p)}} + O\left(\frac{\log m}{\sqrt{m}}\right) .$$

Thus, for such p, there is no substantial gain in memory utilization asymptotically. Note that the formula is also true for $p = 0$, as mentioned earlier.

We leave open the question of the asymptotic behavior of $A_p(m)$ when $p \geq 1/2$.

*/ Here and throughout this paper, p is assumed to be fixed and the constants in the O-notations may depend on p. Logarithms are the natural logarithms (i.e., with base e).

2. Random Walks.

It is convenient to cast the above model in random walk terminologies (see Feller [1] for backgrounds on random walks). Let I_L , I_R denote an insertion instruction for the left and the right stack, respectively, and D_L , D_R a respective deletion instruction. We can regard the execution of a sequence of such instructions as a "particle" performing a "walk" on the integer lattice points in the plane, with the coordinates (x,y) being the current heights of the stacks. For example, an instruction I_L causes the particle to move from its current position (x,y) to $(x+1,y)$. An instruction D_L will cause the particle to move from (x, y) to $(x-1,y)$, unless $x = 0$ (i.e., an empty left stack), in which case the position does not change. We shall call the line $x = 0$ a reflecting barrier, the line $y = 0$ being also a reflecting barrier. The line $x+y = m$ will be referred to as the absorbing barrier.

By a $(p,m;a,b)$ -random walk, we mean a random walk on the plane that starts at an integer point (a,b) , moves according to the transition rules given below, and stops when any point on the absorbing barrier is reached (the point reached is called the absorption point). We assume hereafter that $0 < p < 1/2$, $m > 0$, $a \geq 0$, $b \geq 0$, and $a+b \leq m$.

The Transition Rules (cf. Figure 2): Suppose (x,y) is the present position. 5e next position (x',y') is given below.

$$\text{(a) If } x \neq 0,\ y \neq 0 \text{, then } (x',y') = \begin{cases} (x+1,\ y) & (1-p)/2 \\ (x,y+1) & (1-p)/2 \\ (x-1,y) & p/2 \\ (x,y-1) & p/2 \end{cases} \quad \text{with probability},$$

$$\text{(b) If } x = 0,\ y \neq 0 \text{, then } (x',y') = \begin{cases} (1,y) & (1-p)/2 \\ (0,y+1) & (1-p)/2 \\ (0,y) & p/2 \\ (0,y-1) & p/2 \end{cases}$$

$$\text{(c) If } x \neq 0,\ y = 0 \text{, then } (x',y') = \begin{cases} (x+1,0) & (1-p)/2 \\ (x,\ 1) & (1-p)/2 \\ (x-1,0) & p/2 \\ (x,0) & p/2 \end{cases},$$

$$\text{(d) If } x = 0,\ y = 0 \text{, then } (x',y') = \begin{cases} (1,0) & (1-p)/2 \\ (0,1) & (1-p)/2 \\ (0,\ 0) & p \end{cases}.$$

Let $(X_{a,b},Y_{a,b})$ denote the pair of random variables that have as their values the coordinates (x,y) of the absorption point if the walk ends on the absorbing barrier, and have values $(0,0)$ if the walk never reaches the absorbing barrier. The value $(0,0)$ in this latter assignment is not important, as we shall see later (see the remark at the end of this section) that it occurs only with probability 0 . Let $Z_{a,b} = \max\{X_{a,b},Y_{a,b}\}$. The quantity of interest, $A_p(m)$, is clearly equal to $\overline{Z_{0,0}}$.

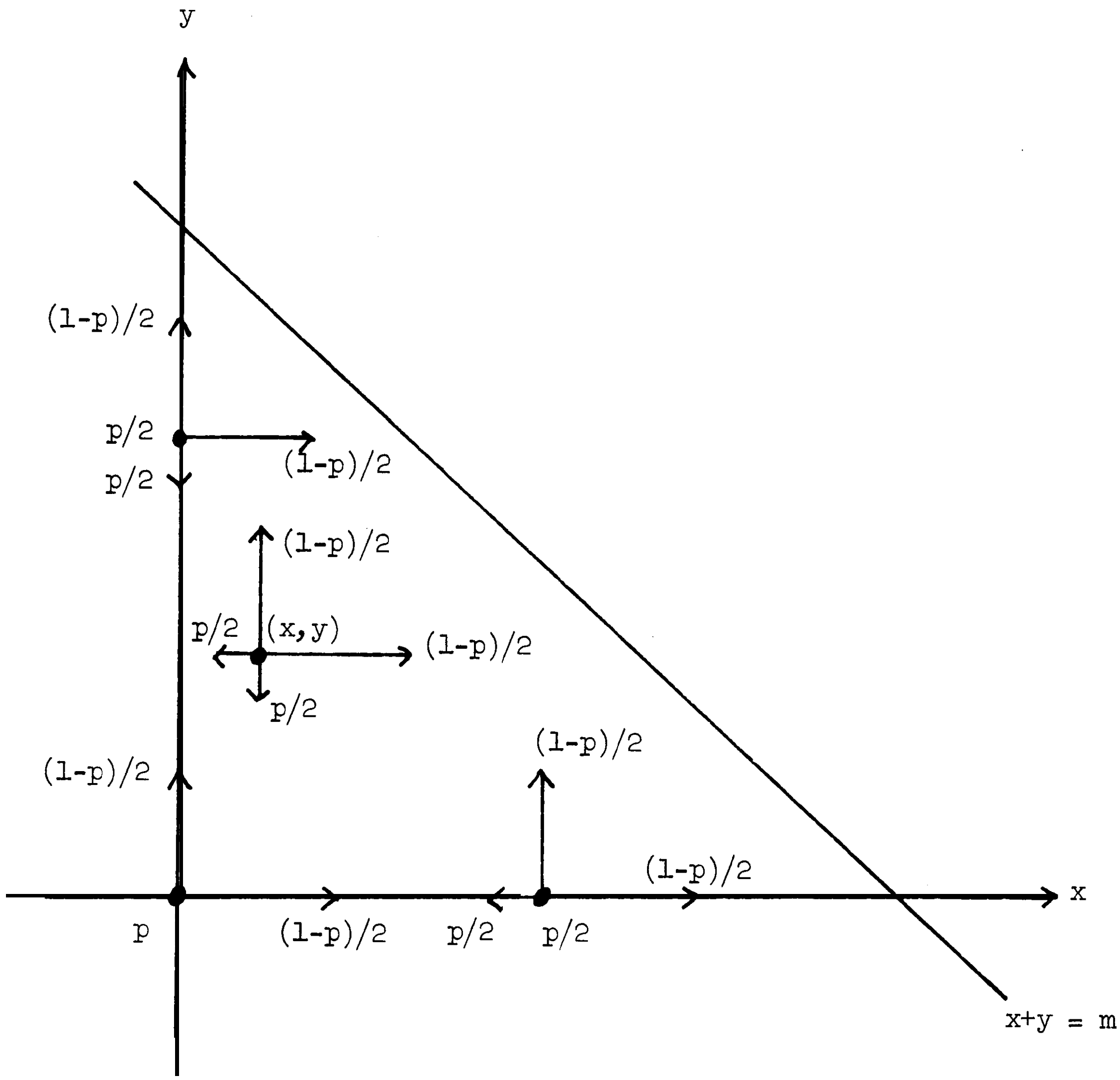

Figure 2. The transition rules for the (p,m;a,b) -random walk.

We begin by considering a related random walk that is easier to analyze. In a $(p,m;a,b)'$ -random walk, a particle starts at the point (a,b) , moves according to the following transition rule

$$(x,y) \to \begin{cases} (x+1,y) & \text{with probability } (1-p)/2 \\ (x,y+1) & \text{with probability } (1-p)/2 \\ (x-1,y) & \text{with probability } p/2 \\ (x,y-1) & \text{with probability } p/2 \,, \end{cases}$$

and stops when it hits the absorbing barrier $x+y = m$. We use $X'_{a,b}$, $Y'_{a,b}$, $Z'_{a,b}$ for the random variables defined in the same way as $X_{a,b}$, $Y_{a,b}$, $Z_{a,b}$. Again, we shall see later that the particle will eventually hit the absorbing barrier with probability 1 .

The value of $\overline{Z'_{a,b}}$ can be evaluated rather precisely. In particular, we have the following result when (a,b) is close to the origin.

Lemma 1. If $a+b = O(\log m)$, then

$$\overline{Z'_{a,b}} = \frac{m}{2} \pm 2\sqrt{\frac{m}{2\pi(1-2p)}} + O\left(\frac{\log m}{\sqrt{m}}\right) .$$

Proof. See Section 3. □

We also have the following result.

Lemma 2. If $a,b \geq \frac{10}{\log((1-p)/p)} \log m$, then

$$\overline{Z_{a,b}} = \overline{Z'_{a,b}} + O(m^{-9}) .$$

Proof. See Section 3. □

Let

$$\epsilon_p = \min\{(1-2p)/8 , p/8\} ,$$

$$\lambda_p = \max\left\{ \left\lceil \frac{10}{\epsilon_p^2} \right\rceil , \frac{4}{1-2p} \frac{10}{\log((1-p)/p)} \right\} ,$$

and $\lambda'_p = \frac{1-2p}{4} \lambda_p$.

Clearly, $\lambda'_p \geq 10/\log((1-p /p)$. Define $R = [\lceil \lambda'_p \log m \rceil , \lceil \lambda_p \log m \rceil + 1]^2$. Lemmas 1 and 2 combine to give the following formula:

$$\overline{Z_{a,b}} = \frac{m}{2} + \sqrt{\frac{m}{2\pi(1-2p)}} + O\left(\frac{\log m}{\sqrt{m}}\right) \quad \text{for} \quad (a,b) \in R . \qquad (1)$$

We shall now use (1) to evaluate $\overline{Z_{0,0}}$.

Let $t = \lceil \lambda_p \log m \rceil + 1$ and S be the set of all sequences of length t in $\{I_L, I_R, D_L, D_R\}$. For each $s = s_1 s_2 \ . \ . \ . \ s_t \in S$, let $r(s) = \prod_{1 \leq i \leq t} r_0(s_i)$, where $r_0(s_i) = (1-p)/2$ if $s_i \in \{I_L, I_R\}$ and $r_0(s_i) = p/2$ if $s_i \in \{D_L, D_R\}$. For each $s \in S$, let $(f_1(s), f_2(s))$ be the position of the particle in a $(p,m;0,0)$ -random walk after the sequence s has been executed. Clearly, for each k ,

$$\Pr(Z_{0,0} = k) = \sum_{s \in S} r(s) \Pr(Z_{f_1(s), f_2(s)} = k) .$$

As a result, we have

$$\overline{Z_{0,0}} = \sum_{s \in S} r(s) \overline{Z_{f_1(s), f_2(s)}} . \qquad (2)$$

Now, let M_p be any integer such that, if $m \geq M_p$, then $t < m$.

Lemma 3. Suppose $m \geq M_p$. Let $S_0 = \{s \mid s \in S ; (f_1(s), f_2(s)) \notin R\}$. Then

$$\sum_{s \in S_0} r(s) \leq 8m^{-10} .$$

Proof. We need the following fact (see Rényi [3, p. 200]). If the toss of a certain coin has a probability v $(0 < v < 1)$ to result in a "Head", then after tossing the coin N times, we have, for any

$$0 < \delta < \left(2 \max\left\{\frac{1-v}{v}, \frac{v}{1-v}\right\}\right)^{-1} ,$$

$$\Pr(|\#\text{ of "Heads"} - vN| > \delta N) \leq 2e^{-N\delta^2/(4v(1-v))} \tag{3}$$

For each $s \in S$, let $\#I_L(s)$, $\#I_R(s)$, $\#D_L(s)$, $\#D_R(s)$ denote the number of appearances of I_L, I_R, D_L, D_R in s, respectively. It follows from (3) and the fact $4v(1-v) < 1$ that, for a random $s \in S$ (weighted by $r(s)$, of course),

$$\Pr\left(\#I_L(s) - \frac{1-p}{2} t| > \epsilon_p t\right) \leq 2 \exp(-\epsilon_p^2 t) ,$$

$$\Pr\left(\#I_R(s) - \frac{1-p}{2} t| > \epsilon_p t\right) \leq 2 \exp(-\epsilon_p^2 t) ,$$

$$\Pr\left(|\#D_L(s) - \frac{p}{2} t| > \epsilon_p t\right) \leq 2 \exp(-\epsilon_p^2 t) ,$$

$$\Pr\left(|\#D_L(s) - \frac{p}{2} t| > \epsilon_p t\right) \leq 2 \exp(-\epsilon_p^2 t) . \tag{4}$$

As $m \geq M_p$, the particle will not be absorbed in t steps. Since $f_j(s) \leq t$ for $j \in \{1,2\}$, it follows that $s \in S_0$ <u>only if</u> $f_j(s) \leq \lceil \lambda_p' \log m \rceil$ for some $j \in \{1,2\}$. Observe that $f_1(s) \geq \#I_L(s) - \#D_L(s)$ and $f_2(s) > \#I_R(s) - \#D_R(s)$. It is straightforward to verify that $f_j(s) \le \lceil \lambda_p' \log m \rceil$ for some $j \in \{1,2\}$ only if at least one of the conditions $|\#i(s) - r_0(i)t| > \epsilon_p t$, where $i \in \{I_L, I_R, D_L, D_R\}$, is satisfied. It follows then from (4) that,

$$\sum_{s \in S_0} r(s) \le 4 \cdot 2e^{-\epsilon_p^2 t} < 8m^{-10} . \quad \square$$

From (1), (2) and Lemma 3, we obtain that for $m \ge M_p$,

$$\overline{Z_{0,0}} = \sum_{s \notin S_0} r(s) \overline{Z_{f_1(s),f_2(s)}} + \sum_{s \in S_0} r(s) \overline{Z_{f_1(s),f_2(s)}}$$

$$= \left(\frac{m}{2} + \sqrt{\frac{m}{2\pi(1-2p)}} + \left(\frac{\log m}{\sqrt{m}} \right) \right) (1 - O(m^{-10})) + O(m^{-10}) \cdot O(m)$$

$$= \frac{m}{2} + \sqrt{\frac{m}{2\pi(1-2p)}} + O\left(\frac{\log m}{\sqrt{m}} \right) .$$

This proves Theorem 1. $\square$

<u>Remark.</u> Let N be any large integer such that $\frac{1-2p}{8} N > m$. Similar to the proof of Lemma 3, one can show that, with probability $1 - O\left(e^{-\epsilon_p^2 N} \right)$, the particle must have been absorbed in the first N steps in a $(p,m;a,b)$ -random walk (or a $(p,m;a,b)'$ -random walk). Let $N \to \infty$. This shows that the particle will be absorbed with probability 1 .

3. Proofs of Lemma 1 and Lemma 2.

We need some basic facts about 1-dimensional random walks (see Feller [1]). Consider a random walk in 1-dimension that starts at 0, and at each step, moves to the left with probability p $(0 < p < 1/2)$ and to the right with probability 1-p. Let $u_{m,n}(p)$ be the probability that position m $(m > 0)$ is reached for the first time at exactly the n-th step. It is known (see Feller [1, Chap. 14, formula (4.14)]) that

$$u_{m,n}(p) = \frac{m}{n}\binom{n}{(n+m)/2}(1-p)^{\frac{n+m}{2}}\, p^{\frac{n-m}{2}} \quad \text{if } n \geq m \text{ and } n, m \text{ are of the same parity.} \tag{5}$$

All other $u_{m,n}(p) = 0$. Clearly,

$$\sum_n u_{m,n}(p) = 1 . \tag{6}$$

Fact 1. Let $n_0 = m/(1-2p)$ and $c_p = 4p(1-p)/(1-2p)^2$. Then

$$\sum_n u_{m,n}(p)n = n_0$$

$$\sum_n u_{m,n}(p)(n-n_0)^2 = c_p n_0 .$$

Proof. The generating function $U_m(z) = \sum_{n \geq 0} u_{m,n} z^n$ is equal to $(G(z))^m$, where

$$G(z) = \left(1 - \sqrt{1-4p(1-p)z^2}\right)\Big/(2pz) ,$$

as can be directly verified. The first sum is given by

$$\sum_n u_{m,n}(p)n = U_m'(1) = mG'(1) = n_0 .$$

The second sum is then the variance of the sequence $u_{m,n}(p)$, $n = 0,1,2,\ldots$, regarded as a probability distribution. Thus, after some calculations, we find

$$\begin{aligned}\sum_n u_{m,n}(p)(n-n_0)^2 &= U_m''(1) + U_m'(1) - (U_m'(1))^2 \\ &= m(G''(1) + G'(1) - (G'(1))^2) \\ &= c_p n_0 \ . \qquad \square\end{aligned}$$

We also need the following result (see Feller [1, Chap. 14, formula (2.8)]).

<u>Fact 2</u>. The probability that the above random walk ever reaches $-z$ (where $z > 0$) is equal to $(p/(1-p))^z$.

We state one more fact. Let ℓ be any number. For each se $\{\alpha,\beta\}^n$, let $W_n^{(\ell)}(s)$ denote the quantity $|\#\text{ of }\beta - \#\text{ of }\alpha - \ell|$. Let $w_n^{(\ell)}$ be the average value of $W_n^{(\ell)}(s)$, assuming that all 2^n sequences are equally likely.

<u>Fact 3</u>. $w_n^{(\ell)} = \sqrt{\frac{2n}{\pi}} + O\left(\frac{|\ell|+1}{\sqrt{n}}\right)$,

<u>Proof</u>.

$$\begin{aligned}w_n^{(\ell)} &= \sum_{0\le k\le n} \frac{1}{2^n}\binom{n}{k}|(n-k) - k - \ell| \\ &= \frac{1}{2^n}\left[\sum_{k<\frac{n-\ell}{2}}\binom{n}{k}(n-2k-\ell) + \sum_{k\ge\frac{n-\ell}{2}}\binom{n}{k}(2k-(n-\ell))\right]\end{aligned}$$

$$= \frac{1}{2^n}\left[\sum_{k<\frac{n}{2}} \binom{n}{k}(n-2k-\ell) + \sum_{k\geq\frac{n}{2}} \binom{n}{k}(2k-n+\ell)\right] + O\left(\frac{|\ell|}{\sqrt{n}}\right)$$

$$= \frac{1}{2^n}\left(2\sum_{k<\frac{n}{2}} \binom{n}{k}(n-2k) + O\left(\frac{2^n}{\sqrt{n}}\right)\right) + O\left(\frac{|\ell|}{\sqrt{n}}\right) . \qquad (7)$$

We have used the fact $\binom{n}{k} = O\left(\frac{2^n}{\sqrt{n}}\right)$ in the derivation.

Fact 3 follows from (7) and the following formulas, which can be obtained in the standard way (see Knuth [2, Chapter 1]):

$$\sum_{k<\frac{n}{2}} \binom{n}{k}(n-2k) = \lceil n/2\rceil \binom{n}{\lceil n/2\rceil} ,$$

$$\binom{n}{\lceil n/2\rceil} = \sqrt{\frac{2}{\pi n}}\, 2^n \,(1 + O(1/n)) . \qquad \square$$

Proof of Lemma 1. Let $m' = m-(a+b)$ and $\ell = a-b$. A $(p,m;a,b)'$-random walk can be generated in the following way. First generate a sequence $\xi \in \{I,D\}^*$ one symbol at a time, each has a probability p to be a " D " and probability 1-p to be an " I ", until $(\#I-\#D) = m'$ for the first time.*/ Then convert ξ into a sequence $s \in \{I_x,I_y,D_x,D_y\}^*$ probabilistically by attaching with equal probability a suffix x or y , to each symbol in ξ . We now associate with s a walk starting from the point (a,b) to an absorption point on $x+y = m$, by interpreting each I_x , I_y, D_x , D_y as a step moving from position (x,y) to $(x+1,y)$, $(x,y+1)$, $(x-1,y)$, $(x,y-1)$, respectively, It is easy to verify that

*/ We have ignored here the possibility that ξ may be infinite. However, our discussion is valid as the probability is zero for ξ to be infinite (see the remark at the end of Section 2).

this procedure indeed generates a $(p,m;a,b)'$ -random walk. It is also not difficult to see that, for each such s generated, the value of $Z'_{a,b}$ is given by (see Figure 3),

$$Z'_{a,b}(s) = \frac{m}{2} + \frac{h(s)}{2} ,$$

where $h(s) = |(\#\text{ of } I_y + \#\text{ of } D_x) - (\#\text{ of } I_x + \#\text{ of } D_y) - \ell|$.

Note that, for each sequence ξ of n symbols, the average value of h(s) for s derived from ξ is in fact equal to $w_n^{(\ell)}$. Thus, we have

$$\overline{Z'_{a,b}} = \frac{m}{2} + \frac{1}{2} \sum_n (\text{Probability that } |\xi| = n) \cdot w_n^{(\ell)} .$$

It is easy to see that the quantity (probability that $|\xi| = n$) is exactly $u_{m',n}(p)$. Hence

$$\overline{Z'_{a,b}} = \frac{m}{2} + \frac{1}{2} \sum_n u_{m',n}(p) w_n^{(\ell)} .$$

Using Fact 3 and the fact $\ell = O(\log m)$, we have

$$\overline{Z'_{a,b}} = \frac{m}{2} + \sqrt{\frac{1}{2\pi}} \sum_n u_{m',n}(p) \left(\sqrt{n} + O\left(\frac{\log m}{\sqrt{n}} \right) \right) . \qquad (8)$$

Write $\sqrt{n}$ and $1/\sqrt{n}$ as

$$\sqrt{n} = \sqrt{n'_0} + \frac{1}{2} (n-n'_0)(n'_0)^{-1/2} + O((n-n'_0)^2 (n'_0)^{-3/2}) ,$$

and

$$\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n'_0}} + O(|n-n'_0| (n'_0)^{-3/2})$$

$$= \frac{1}{\sqrt{n'_0}} + O((n-n'_0)^2 (n'_0)^{-3/2}) \qquad \text{for all } n \geq 1 .$$

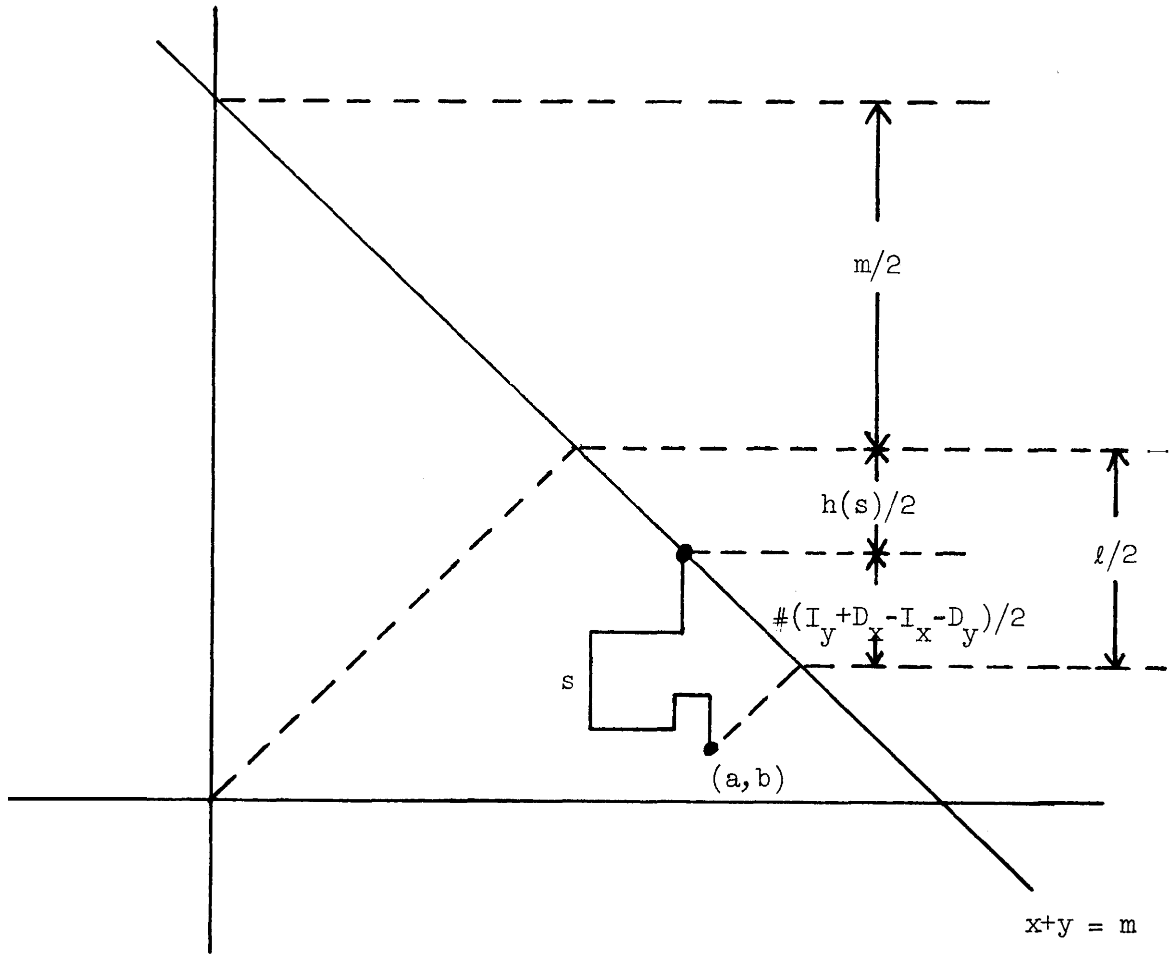

Figure 3. An illustration for $Z'_{a,b}(s = m/2 + h(s)/2$

Substituting these expressions into (8), and making use of Fact 1, we obtain

$$\overline{Z'_{a,b}} = \frac{m}{2} + \sqrt{\frac{1}{2\pi}} \sum_n u_{m',n}(p)\left(\sqrt{n'_0} + \frac{1}{2\sqrt{n'_0}}(n-n'_0) + O(\log m)\cdot\left(\frac{1}{\sqrt{n'_0}} + \frac{(n-n'_0)^2}{n'_0\sqrt{n'_0}}\right)\right)$$

$$= \frac{m}{2} + \sqrt{\frac{n'_0}{2\pi}} + \frac{\log m}{\sqrt{n'_0}}\, O\left(1 + \frac{1}{n'_0}\sum_n u_{m',n}(p)(n-n'_0)^2\right)$$

$$= \frac{m}{2} + \sqrt{\frac{m'}{2\pi(1-2p)}} + \left(\frac{\log m}{\sqrt{m'}}\right)$$

As $m' = m - O(\log m)$, the lemma follows. □

Proof of Lemma 2. Consider a $(p,\infty;a,b)$ -random walk, and let $\Delta(a,b)$ be the probability that the particle will ever touch the reflecting boundaries ($x = 0$ or $y = 0$) . By Fact 2, the probability for it to touch $x = 0$ is $(p/(1-p))^a$ and for it to touch $y = 0$ is $(p/(1-p))^b$. This implies that $\Delta(a,b) \leq (p/(1-p))^a + (p/(1-p))^b \leq 2m^{-10}$.

Since any walk that does not touch the reflecting barriers occurs with the same probability in both the $(p,m;a,b)$ -random walk and the $(p,m;a,b)'$ -random walk, we conclude that

$$|\overline{Z_{a,b}} - \overline{Z'_{a,b}}| \leq m\cdot\Delta(a,b) \leq 2m^{-9} .$$

This completes the proof of Lemma 2. □

References

[1] W. Feller, An Introduction to Probability Theory and Its Applications, Volume I, John Wiley and Sons, New York, 1968, 3rd edition.

[2] D. E. Knuth, The Art of Computer Programming, Volume 1, Addison-Wesley, Reading, Mass., 1975, 2nd edition.

[3] A. Rényi, Foundations of Probability, Holden-Day, San Francisco, 1970.