THE ANALYSIS OF SEQUENTIAL EXPERIMENTS WITH FEEDBACK TO SUBJECTS

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ABSTRACT

A problem arising in taste testing, medical, and parapsychology experiments can be modeled as follows. A deck of n cards contains c_i cards labeled i, $1 \leq i \leq r$. A subject guesses at the cards sequentially. After each guess the subject is told the card just guessed (or at least if the guess was correct or not). We determine the optimal and worst case strategies for subjects and the distribution of the number of correct guesses under these strategies. We show how to use skill scoring to evaluate such experiments in a way which (asymptotically) does not depend on the subject's strategy.

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1. Introduction

For a variety of testing situations the following experiment is performed: A subject tries to guess the outcome of a sequence of draws without replacement from a finite population. After each guess, the subject is given feedback information. This might be the name of the object just guessed at--complete feedback--or only the information that the guess just made was correct or not--partial feedback. We are interested in the subject's optimal strategy and in methods for scoring subjects which do not depend on the strategy used by the subject.

The following example illustrates our main results.

1.1. Taste testing and partially randomized clinical trials

Consider Fisher's famous Lady tasting tea (Fisher (1949) p. 11). Eight cups of tea were prepared--four of one type and four of a second type. The cups of tea were presented to the lady in a random order, and she was to guess the type for each cup. With no ability and no feedback, the lady is expected to have four of her eight guesses correct. We propose the following variation: to help calibrate her guesses, the lady is told after each guess if it was correct or not. If the lady has no tasting ability but is trying to maximize the number of correct guesses, her optimal strategy, knowing

that a of type one and b of type two remained, is to guess the type corresponding to max(a,b). The expected number of correct guesses under the optimal strategy is $373/70 \doteq 5.3$.

Mathematically, this problem is the same as a problem discussed by Blackwell and Hodges (1956) and Efron (1971) in connection with clinical trials. In comparing two treatments on 2n patients, suppose it is decided that n patients are to get each treatment, the allocation being otherwise random. Assume that the patients arrive sequentially and must either be ruled ineligible or assigned one of the two treatments. A physician observing the outcome of each trial would know which treatment was most probable on each trial. This information could be used to bias the experiment if the physician ruled less healthy patients ineligible on trials when a favored treatment was more probable. A natural measure of the selection bias is the number of correct guesses the experimenter can make by guessing optimally. Blackwell and Hodges showed that with 2n subjects the optimal guessing strategy leads to

$$n + \frac{1}{2}(2^{2n}/\binom{2n}{n} - 1) = n + \frac{1}{2}\sqrt{\pi n} - \frac{1}{2} + 0(\frac{1}{n})$$ correct expected guesses.

The same problem arises in card-guessing experiments. The usual ESP experiment uses a 25-card deck with the 5 symbols 0, +, ∫∫∫, □, * repeated five times each. The deck is shuffled; a sender looks at the cards in sequence from the top down, and a subject guesses at each card after the sender looks at it. We discuss three types of feedback:

Case 1--No feedback. If no feedback is provided, then any guessing strategy has five correct guesses as its expected value. The distribution of the number of correct guesses depends on the guessing strategy. Several writers have shown that the variance is largest when the guessing strategy is some permutation of the 25 symbols. This is further discussed at the beginning of Section 3.

Case 2--Complete feedback. If the subject is shown the card guessed each time, then the optimal strategy is to guess the most probable remaining type at each stage. The expected number under the optimal strategy is 8.65, a result first derived by Read (1962). In Section 2 we give closed form expressions for the expected number of correct guesses for the optimal and worst case strategies for a deck of arbitrary composition.

Case 3--Yes or no feedback. The situation becomes complex with partial feedback--telling the subject if each guess was correct or not. No simple description of the optimal strategy is known. An example in Section 3 shows that the "greedy algorithm" which guesses the most probable symbol at each stage is not optimal. The optimal strategy and the expected number of correct guesses under the optimal strategy can be determined by solving a recurrence relation numerically. For a standard ESP deck the expectation is 6.63 correct guesses. In Theorems 5 and 6 we show that the greedy algorithm is optimal for partial feedback experiments with no repeated values (that is, for a deck labeled $(1,2,\ldots,n)$). For an empirical attempt to solve

these problems, see Thouless (1977). A thorough discussion of statistical problems in ESP research may be found in Burdick and Kelly (1978), and Diaconis (1978).

How should feedback experiments be evaluated? Consider a numerical example made explicit in Table 1. A deck of 20 cards, 10 labeled "red" and 10 labeled "black," was well mixed. A sender looked at the cards in sequence from the top down, and a subject guessed at each card after the sender looked at it. After each trial the guesser was told whether the guess was correct or not. There were 14 correct guesses. If this experiment was naively evaluated by neglecting the availability of feedback information (a widely used approach, see Tart (1977), Chapters 1,2 for references), each trial would be regarded as an independent binomial variable with success probability 1/2. Binomial tables show that P(14 or more correct out of 20) $\doteq$ .058. The choice sequence that the guesser actually made is fairly close to the optimal strategy. There were 7 times that the number of red cards remaining was equal to the number of black cards remaining. At these trials, red and black have the same probability of being correct and either choice is optimal. The guesses made agree with the optimal strategy on 9 of the 13 remaining trials. Perhaps the 14 correct guesses should be compared with 12.30, the expected number of correct guesses under the optimal strategy. Neglecting the availability of feedback information can lead to crediting a subject using an optimal (or near optimal) strategy with having "talent." On the other hand, demanding that a subject

TABLE 1

EXAMPLE OF SKILL SCORING IN AN EXPERIMENT WITH 10 RED AND 10 BLACK CARDS AND FEEDBACK TO THE GUESSING SUBJECT

Trial No.	Guess	Feedback	Optimal	p_i	Card
1	B	Yes	Tie	1/2	B
2	B	No	R	9/19	R
3	B	No	Tie	1/2	R
4	B	Yes	B	9/17	B
5	R	No	Tie	1/2	B
6	B	Yes	R	7/15	B
7	R	Yes	R	8/14	R
8	B	Yes	R	6/13	B
9	R	Yes	R	7/12	R
10	R	Yes	R	6/11	R
11	R	No	Tie	1/2	B
12	R	Yes	R	5/9	R
13	B	No	Tie	1/2	R
14	R	Yes	B	3/7	R
15	B	Yes	B	4/6	B
16	B	Yes	B	3/5	B
17	B	No	Tie	1/2	R
18	B	Yes	B	2/3	B
19	R	Yes	Tie	1/2	R
20	B	Yes	B	1	B
		14 Correct		11.049	

Column 1 is trial number, Column 2 is subject's guess, Column 3 is feedback information, Column 4 is optimal guess (tie means either color is optimal), Column 5 is probability that subject's guess is correct, and Column 6 is card actually present.

significantly exceed the expected number under the optimal strategy can lead to failure to detect a "talented" subject who doesn't use the feedback information. In Section 4 we describe a method of evaluation called skill scoring. The skill score compares the number of correct guesses to a base line score calculated from the conditional expectation of the i^{th} guess given the feedback information. The statistic is particularly simple in the present example. If at the time of the i^{th} guess there are r_i red cards and b_i black cards remaining in the deck, then the probability of the next card being (say) red is $\frac{r_i}{n - i + 1}$. The numbers p_i--the probability of the i^{th} guess being correct--are given in the fifth column of Table 1. If Z_i is one or zero as the i^{th} guess is correct or not, then the skill score statistic S is defined as $S = \Sigma_{i=1}^{2n} \{Z_i - p_i\}$. For this example S = 14-11.049 = 2.95.

In Theorem 7 we show that for any guessing strategy $S/\sqrt{2n/4}$ has a limiting standard normal distribution. In the example of Table 1, $S/\sqrt{5} \doteq 1.32$. Further discussion of this example is in Section 4.

Clearly experiments which combine feedback with sampling with replacement are easier to analyze. Our motivation for considering sampling without replacement is twofold. First, reanalysis of a previously performed feedback experiment done without replacement may be desirable. Second, experiments are often designed without replacement to insure balance between treatments for moderate samples. Efron (1971) gives a nice discussion of these issues and references to standard literature.

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2. Complete Feedback Experiments

In this section we consider experiments with a deck of n cards containing c_i cards labeled i, $1 \leq i \leq r$, so $n = \sum_{i=1}^{r} c_i$. We write $\bar{c} = (c_1, c_2, \ldots, c_r)$ for the composition vector. A subject tries to guess what card is at each position and after each guess is shown the card at this position. The optimal strategy for a subject trying to maximize the total number of correct guesses is to guess the most probable symbol at each stage. (This is easily proved by backward induction.) Let $H = H(\bar{c})$ be the number of correct guesses when the optimal strategy is used. We can derive the distribution of H when r= 2 by using variants of an argument in Blackwell and Hodges (1957). We give the limiting distribution of H here, the exact distribution is derived in the course of the proof.

Theorem 1.

If c_1 and c_2 tend to infinity in such a way that $c_1/(c_1 + c_2) \to p$, $0 < p < 1$, $p \neq 1/2$, then

$$E(H) = \max(c_1, c_2) + \frac{1}{2}\left(\frac{1}{|p - q|} - 1\right) + o(1) . \tag{2.1}$$

$$P(H - \max(c_1, c_2) = k) \to Y(1-Y)^k \tag{2.2}$$

for $k = 0,1,2,\ldots$ where $\gamma = \frac{2|p - q|}{1 + |p - q|}$.

If $c_1 = c_2 = k$ (so $p = \frac{1}{2}$), then, as k tends to infinity,

$$E(H) = k + \frac{1}{2}\left(\frac{2^{2k}}{\binom{2k}{k}} - 1\right) = k + \frac{1}{2}\sqrt{\pi k} - \frac{1}{2} + O\left(\frac{1}{k}\right) , \tag{2.3}$$

$$(2.4)\qquad P\left(\frac{H-k}{\sqrt{k/4}} \le x\right) \to \begin{cases} 0 & \text{if } x \le 0 \\ 2\Phi(x) - 1 & \text{if } x \ge 0 \end{cases},$$

where $\Phi(x)$ is the standard normal cumulative distribution.

Results (2.3) and (2.4) are essentially given by Blackwell and Hodges (1957). The results show that there is a big difference between balanced decks where $c_1 = c_2$ and unbalanced decks. In the unbalanced situation the optimal strategy does not do much better than the strategy which always guesses the type corresponding to $\max(c_1, c_2)$. An intuitive explanation is that when $c_1 >> c_2$ the optimal guess will almost always guess type 1.

When $r > 2$, we have not actively pursued the problem of finding the distribution of H, but we have determined the mean of H. If $h(\bar{c}) = E(H(\bar{c}))$, then elementary considerations show that when $\sum_{i=1}^{r} c_i > 0$, h satisfies the recursion

$$(2.5)\qquad h(\bar{c}) = \sum_i \frac{c_i}{c_1 + \ldots + c_r} h(\bar{c} - \bar{\delta}_i) + \frac{\max(\bar{c})}{c_1 + \ldots + c_r}, \quad h(\bar{0}) = 0,$$

where $\bar{\delta}_i$ has a one in the i^{th} position and zeros elsewhere, and $\bar{0}$ is the vector of all zeros.

We will show that $h(\bar{c})$ has the following closed form expression:

Theorem 2. The solution of the recursion (2.5) is

$$(2.6)\qquad h(\bar{c}) = \max(\bar{c}) + \sum_{\bar{0} \ne \bar{i} \le \bar{c}} \left\{ \binom{c_1}{i_1} \cdots \binom{c_r}{i_r} \Big/ \binom{c_1 + \ldots + c_r}{i_1 + \ldots + i_r} \right\} \frac{\max^*(\bar{i})}{i_1 + \ldots + i_r},$$

where

$$\max^*(\bar{i}) = \begin{cases} 0 \text{ if there is a unique } j \text{ such that } i_j = \max(\bar{i}) \\ \max(\bar{i}) \text{ otherwise} \end{cases}$$

The sum in (2.6) is over the nonnegative orthant of the integer lattice in r dimensions,

The recursion (2.5) was used by Read (1962) to numerically determine certain values of h. We recomputed the following values of $h(\bar{c})$ confirming Read's calculations: $h(3,3,3) = 4.78690^+$, $h(5,5,5,5,5) = 8.64675^+$. A direct probabilistic interpretation of the right side of (2.6) is given after the proof.

For a deck containing r different types with each type repeated k times, $\bar{c} = k\bar{i}$, where $\bar{1}$ is a vector of r ones. For large k, weak convergence techniques can be used to bound the right side of (2.6):

Theorem 3. As k tends to infinity,

$$h(k\bar{i}) = k + \frac{\pi}{2} M_r \sqrt{k} + o_r(\sqrt{k}) ,$$

where M_r is the expected value of the maximum of r independent -standard normal variates, The notation o_r means the implied constant depends on r.

The numbers M_r are tabled in Teichroew (1956) and Harter (1961). For example,

r	2	3	4	5
M_r	.564	.863	1.029	1.163

.

Of course, Theorem 3 agrees with (2.3) when r = 2. When k = r = 5, the approximation given by Theorem 3 is about 9.08 as compared with 8.65 from exact evaluation.

In the complete feedback problem it is possible for a subject to try to minimize the expected number of correct guesses by guessing the least probable symbol on each trial. We call this worst case guessing. This can lead to strategies with a strange appearance. For example, with n cards labeled $\{1,2,\ldots,n\}$ the worst strategy guesses any card, (say 1) on the first trial and thereafter guesses a card known not to be in the deck. This leads to l/n as the expected number of correct guesses. Analysis of worst case guessing is valuable in determining how widely the distribution of correct guesses can vary as a function of strategy. The arguments are similar to best case guessing and will not be given in detail. Here are some results:

Theorem 4. Let $d(\bar{c})$ denote the expected number of correct guesses when the worst case strategy is used with complete feedback.

$$d(\bar{c}) = \min(\bar{c}) - \sum_{0 \neq \bar{i} \leq \bar{c}} \left\{ \binom{c_1}{i_1} \cdots \binom{c_r}{i_r} \Big/ \binom{c_1 + \ldots + c_r}{i_1 + \ldots + i_r} \right\} \frac{\min^*(\bar{i})}{i_1 + \ldots + i_r} .$$

where $\min^*(\bar{i}) = \min(\bar{i})(\mathrm{mul}(\bar{i}) - 1)$, $\mathrm{mul}(\bar{i})$ is the number of j such that $i_j = \min(\bar{i})$.

As k tends to infinity,

$$d(k\bar{1}) = k - \frac{\pi}{2} M_r \sqrt{k} + o_r(\sqrt{k})$$

where M_r was defined in Theorem 3.

Some numerical values for d are $d(3,3,3) = 1.48690^{+}$, $d(5,5,5,5,5) = 2.29606^{+}$. When $r = 2$, $\min(c_1,c_2) - d = h - \max(c_1,c_2)$ so (2.1) and (2.3) can be used for similar computations involving d.

Theorems 3 and 4 show that with a bounded number r of distinct types the deviation of either best or worst case guessing from guessing with no feedback is of order $\sqrt{k}$ compared to a lead term of k. This is crucial to results in Section 4 involving the skill scoring statistic.

Proofs for Section 2

Proof of Theorem 1. To determine the distribution of H we follow Blackwell and Hodges (1957) in considering an associated random walk. Without loss of generality suppose $c_1 \geq c_2$. Following the notation of Chapter 3 of Feller (1968), consider a random path composed of lines of slope ± 1. The walk moves up if a card of type 1 is turned up, and down if a card of type 2 turns up. The walk begins at (0,0) and ends at $(c_1 + c_2, c_1 - c_2)$. The optimal strategy is to guess type 1 if the path is below the line $y = c_1 - c_2$, guess type 2 if the path is above this line, and guess arbitrarily at points where the path touches the line. This is because when the path touches $Y = c_1 - c_2$, the number of cards of type 1 remaining equals the number of cards of type 2 remaining. Let T be the number of times the random path touches the line $y = c_1 - c_2$. It is not hard to show by

induction that for any path the number of correct guesses that the optimal strategy makes at time $c_1 + c_2$ equals $c_1 + Z$ where Z is a binomial random variable with parameters 1/2 and T. Thus all randomness in the outcome of a run through the deck using the optimal strategy can be attributed to the outcome of guesses when the remaining numbers of each type were the same.

T takes values $0,1,2,\ldots,c_2$ and a straightforward variant of the proof of Theorem 4 in Section 7 of Feller (1968) shows that

$$(2.7) \qquad p(T=t) = 2^t \frac{c_1 - c_2 + t}{c_1 + c_2 - t} \binom{c_1 + c_2 - t}{c_2 - t} \Big/ \binom{c_1 + c_2}{c_1}, \quad t = 0,1,\ldots,c_2 .$$

Notice that when $c_1 = c_2$, T cannot take on the value 0 and (2.7) is equivalent to equation (2.3) of Blackwell and Hodges (1957). They argue that $T/\sqrt{k}$ tends in distribution to the absolute value of a standard normal, and this implies (2.4). Passing to the limit in (2.7) when c_1 and c_2 tend to infinity with $c_1/(c_1 + c_2) \to p$ $0 < p < 1$, $p \neq \frac{1}{2}$ yields that T has a limiting geometric distribution with $p(T=t) = \tilde{\gamma}(1-\tilde{\gamma})^t$ $t = 0,1,2,\ldots,\tilde{\gamma} = |p - q|$. The limiting distribution of H is obtained from the limiting distribution of T by using the fact that, if H given T=t is binomial with parameters $\frac{1}{2}$ and t, then H unconditionally has the distribution specified by (2.2). The equation for the mean of H can be derived as a special case of (2.6). Thus, when $r = 2$, $\max^*(i_1,i_2) = 0$ unless $i_1 = i_2$. Then (2.6) becomes

$$(2.8) \qquad E(H) = \max(c_1, c_2) + \frac{1}{2} \sum_{i>1} \binom{c_1}{i} \binom{c_2}{i} \Big/ \binom{c_1 + c_2}{2i} .$$

When $c_1 = c_2 = k$, we have

$$E(H) = k + \frac{1}{2} \sum_{i \geq 1} \binom{k}{i}^2 \Big/ \binom{2k}{2i} = k + \frac{1}{2}\left(\frac{2^{2k}}{\binom{2k}{k}} - 1 \right)$$

so (2.3) follows. Taking the limit in (2.8) as $\frac{c_1}{c_1 + c_2} \to p$ yields

$$E(H) = \max(c_1, c_2) + \frac{1}{2} \sum_{i \geq 1} \binom{2i}{i} (pq)^i + o(1)$$

$$= \max(c_1, c_2) + \frac{1}{2}\left(\frac{1}{\sqrt{1 - 4pq}} - 1 \right) + o(1)$$

$$= \max(c_1, c_2) + \frac{1}{2}\left(\frac{1}{|p - q|} - 1 \right) + o(1).$$

Proof of Theorem 2. Let $f(\bar{c}) = h(\bar{c}) - \max(\bar{c})$. The recursion (2.5) translates into

$$f(\bar{c}) = \sum_i \frac{c_i}{c_1 + \ldots + c_r} \cdot \{ f(\bar{c} - \bar{\delta}_i) + \max(\bar{c} - \bar{\delta}_i) \} + \frac{\max(\bar{c})}{c_1 + \ldots + c_r} - \max(\bar{c}) .$$

or

$$(c_1 + \ldots + c_r) f(\bar{c}) = \sum_i c_i f(\bar{c} - \bar{\delta}_i)$$

$$+ \left[\sum_i c_i \max(\bar{c} - \bar{\delta}_i) + \max(\bar{c}) - (c_1 + \ldots + c_r) \max(\bar{c}) \right] 1 .$$

The expression in square brackets is easily seen to equal $\max^*(\bar{c})$ as defined in Theorem 2. Now, writing

$$g(\bar{c}) = \frac{(c_1 + \ldots + c_r)!}{c_1! \ldots c_r!} f(\bar{c}) \quad ,$$

the recursion becomes

$$(2.9) \qquad g(\bar{c}) = \sum_i g(\bar{c} - \delta_i) + \frac{(c_1 + \ldots + c_r)!}{c_1! \ldots c_r!} \frac{\max^*(\bar{c})}{c_1 + \ldots + c_r}$$

It is clear from (2.9) that $g(\bar{c})$ can be expressed as a sum over the nonnegative orthant $\bar{0} \neq \bar{i} \leq \bar{c}$ of the function

$$\lambda(\bar{i}) = \frac{(i_1 + \ldots + i_r)!}{i_1! \ldots i_r!} \frac{\max^*(\bar{i})}{i_1 + \ldots + i_r} .$$

At each lattice point $\bar{i}$ the function $\lambda(\bar{i})$ must be multiplied by the number of paths from $\bar{c}$ to $\bar{i}$. This number is

$$\frac{((c_1 - i_1) + \ldots + (c_r - i_r))!}{(c_1 - i_1)! \ldots (c_r - i_r)!} .$$

Thus,

$$g(\bar{c}) = \sum_{0 \neq \bar{i} \leq \bar{i}} \frac{((c_1 - i_1) + \ldots + (c_r - i_r))!}{(c_1 - i_1)! \ldots (c_r - i_r)!} \lambda(\bar{i}) .$$

Transforming g back to f and f back to h completes the proof of Theorem 2.

By considering a multidimensional random walk, taking a step in the direction of the i^{th} coordinate when a card of type i is exposed, we can give a direct probabilistic interpretation to the $\max^*$ of Theorem 2 and $\min^*$ of Theorem 4. Just as when $r = 2$, the only randomness in the number of correct guesses under the optimal strategy comes from lattice points $\bar{i}$ where $\max^*(\bar{i}) > 0$. The number of correct guesses from lattice points where $\max^*(\bar{i}) = 0$ being $\max(\bar{c})$. The probability of a correct guess for a lattice point where $\max^*(\bar{i}) > 0$ is $\frac{\max^*(\bar{i})}{i_1 + \ldots + i_r}$, and the sum in (2.6) is just a sum of these

probabilities multiplied by the probability that the path passes through $\bar{i}$.

Proof of Theorem 3. We are considering a deck of n=rk cards containing k cards marked i, $1 \leq i \leq r$. For $j = 1,2,\ldots,n$, let $\bar{V}_j$ be an r-dimensional random vector which counts how many of each type have been called before time j. Thus, $\bar{V}_1 = \bar{0}$ and $\bar{V}_j(i)$ is the number of cards marked i which have appeared before time j. At the j^{th} trial the optimal strategy is to choose any value ℓ such that $\bar{V}_j(\ell) = \min_i \bar{V}_j(i)$. The probability of a correct guess is then

$$\frac{k - \min_i V_j(i)}{n - i + 1} \qquad j = 1,2,\ldots,n \ . \tag{2.10}$$

To work with (2.10) we use weak convergence techniques from Chapter 4 of Billingsley (1968). The first step is to transform the random vectors $\bar{V}_1,\ldots,\bar{V}_n$ into a random function which will be shown to converge to an appropriate Brownian bridge. Let

$$\bar{X}_j = \sqrt{\frac{r}{k(r-1)}}\left\{\bar{V}_j - \frac{j}{r}\bar{1}\right\} \ .$$

The components of $\bar{X}_j$ have $E(\bar{X}_j(i)) = 0$, $\mathrm{Var}(\bar{X}_j(i)) = 1$. Form a vector valued continuous function ${}_n\bar{X}_t : [0,1] \to \mathbb{R}^r$ by connecting the components $X_j(i)$ by straight lines as in Billingsley ((1968), pp.8-15). Thus, ${}_n\bar{X}_{j/n} = \bar{X}_j$. It follows from Rosen's (1967) results for dependent vector valued random variables that the r-dimensional analog of Theorem 24.1 of Billingsley (1968) holds. That is, ${}_n\bar{X}_t \xrightarrow{D} \bar{W}^o_t$ where $\bar{W}^o_t$ is an r-dimensional mean 0 Gaussian process with the following covariance:

$$\text{for } s < t\,, \quad E\{\bar{W}^o_s(i)\bar{W}^o_t(j)\} = \begin{cases} \dfrac{-s(1-t)}{s-1} & \text{when } i \neq j \\ s(1-t) & \text{when } i = j\,. \end{cases}$$

Thus, each component process $\bar{W}^o_t(i)$ is a Brownian bridge and, for fixed t, $\operatorname{cov} \bar{W}^o_t = t(1-t)\,\Phi$ where

$$\Phi = \begin{pmatrix} 1 & \cdots & -\frac{1}{r-1} \\ -\frac{1}{r-1} & \ddots & 1 \end{pmatrix}\,.$$

This implies that $\sum_i W_t(i) = 0$. Returning to (2.10) and summing yields

$$\sum_{j=1}^{n} \frac{k-6}{n-j+1} - \sqrt{\frac{k(r-1)}{r}} \sum_{j=1}^{n} \frac{\min_i \bar{X}_j(i)}{n-j+1}\,. \tag{2.11}$$

The first sum in (2.11) is easily seen to equal $k + 0_r\left(\frac{\log k}{\sqrt{k}}\right)$ (the notation 0_r means that the implied constant depends on r). We will argue that we may take expectations in (2.11) and pass to the limit as k tends to infinity. Then,

$$E\left\{\sum_{j=1}^{n} \frac{\min_i \bar{X}_j(i)}{n-j+1}\right\} \to \int_0^1 \frac{E(\min \bar{W}^o_t)}{1-t}\,dt\,. \tag{2.12}$$

Assuming the validity of (2.12) for the moment, we have shown that the expected number of correct guesses is

$$k - \tilde{M}_r \sqrt{k} + o_r(\sqrt{k})$$

where

$$\tilde{M}_r = \sqrt{\frac{r-1}{r}} \int_0^1 \frac{E(\min \bar{W}^o_t)}{1-t}\,dt\,.$$

We now show that $\tilde{M}_r = -\frac{\pi}{2} M_r$ where M_r was defined in Theorem 3. To prove this note that one way of constructing $\bar{W}^o_t$ from r independent 1-dimensional Brownian bridges $W^{(1)}_t, \ldots, W^{(r)}_t$ is as follows.

Let $\hat{W}_t = \frac{1}{r}\sum_{j=1}^{r} W_t^{(j)}$ and let $\overline{W}_t^o(i) = \sqrt{\frac{r}{r-1}}\,(W_t^{(i)} - \hat{W}_t)$ for $1 \le i \le r$. It is easy to check that $\overline{W}_t^o$ has the correct covariance, $t(1-t)\wedge$. Now, for fixed t the symmetry of mean 0 Gaussian variables implies that

$$E\{\min_i W_t^{(i)}\} = -E\{\max_i W_t^{(i)}\};\ E\{\min_i \overline{W}_t^o(i)\} = -E\{\max_i \overline{W}_t^o(i)\}\ .$$

Moreover,

$$2E\{\max_i \overline{W}_t^o(i)\} = E\{\text{Range}\{\overline{W}_t^o(i)\}\} = \sqrt{\frac{r}{r-1}}\ E\ \text{Range}\ W_t^{(i)}\}$$

$$= 2\sqrt{\frac{r}{r-1}}\ E\{\max W_t^{(i)}\}\ .$$

For fixed t the variables $W_t^{(i)}$ are independent Gaussian variables with mean 0 and variance $t(1-t)$. It follows that $\tilde{M}_r = -M_r \int_0^1 \sqrt{\frac{t}{1-t}}\,dt = -\frac{\pi}{2} M_r$ as claimed.

We now show that the limit step in (2.12) is valid. We will argue in the function space $D[0,1]$. Note first that ${}_n\overline{X}_t \xrightarrow{D} \overline{W}_t^o$ implies $\min_i {}_n\overline{X}_t(i) \to \min_i \overline{W}_t(i)$ in $D[0,1]$. Next consider the continuous functional $T_\epsilon : D[0,1] \to \mathbb{R}$ defined by $T_\epsilon(f) = \int_\epsilon^{1-\epsilon} \frac{f(t)}{1-t}\,dt$. Since $\min_i {}_n\overline{X}_t(i)$ is piecewise constant and equals $\min_i X_j(i)$ on the interval $\frac{j}{n} \le t < \frac{j+1}{n}$, we have that

$$T_\epsilon(\min_i {}_n\overline{X}_t(i)) = \sum_{\epsilon n \le j \le (1-\epsilon)n} \min({}_n\overline{X}_{j/n}(i)) \int_{\frac{i}{n}}^{\frac{i+1}{n}} \frac{1}{1-t}\,dt$$

$$= \sum_{\epsilon n \le j \le (1-\epsilon)n} \min(X_j(i))(-\log\,(1 - \frac{1}{n-i}))$$

$$= \sum_{\epsilon n \le j \le (1-\epsilon)n} \frac{\min X_j(i)}{n-j+1} + 0_\epsilon\left(\sum \frac{\min X_j(i)}{(n-j)^2}\right)\ .$$

To apply Markov's inequality we need to bound $E(|\min X_j(i)|)$.

$$(2.13)\quad E|\min X_j(i)| \le r\,E(|X_j(1)|) \le r(E(X_j^2(1))^{1/2} = r\sqrt{\frac{r}{k(r-1)}}\left(i\,\frac{n-i}{n-1}\right)^{1/2}.$$

Thus, for any $\gamma > 0$,

$$P\{\left|\sum_j \frac{\min_n \bar{X}_j(i)}{(n-j)^2}\right| > \gamma\} \le \frac{1}{\gamma}\, r\sqrt{\frac{r}{k(r-1)}} \sum_j \frac{j^{1/2}}{(n-j)^{3/2}\,(n-1)^{1/2}} < \frac{c}{\gamma\sqrt{k}}$$

where the positive constant c is independent of k and γ. Thus, we have shown that the error converges to 0 in probability and the continuous mapping theorem yields

$$(2.14)\qquad \sum_{\epsilon n \le j \le (1-\epsilon)n} \frac{\min X_j(i)}{n-j+1} \xrightarrow{D} \int_{\epsilon}^{1-\epsilon} \frac{\min \bar{W}_t(i)}{1-t}\,dt\ .$$

To take expectations in (2.14) we must show that the left side is uniformly integrable. Write $M_j = \min_i X_j(i)$ and consider

$$(2.15)\qquad E\left\{\left(\sum \frac{M_i}{n-i+1}\right)^2\right\} \le \sum_{i,j} \frac{E|M_i||M_j|}{(n-i+1)(n-j+1)}\ .$$

When $i \neq j$, $E(|M_i||M_j|) \le \{E(M_i^2)E(M_j^2)\}^{1/2}$ and

$$(2.16)\qquad E(M_i^2) \le r\,E(X_i^2(1)) = \frac{r}{k(r-1)}\,\frac{i}{r}\,(1-\frac{1}{r})\,\frac{n-i}{n-1}\ .$$

Using these bounds in (2.15) shows that

$$E\left\{\left(\sum \frac{M_i}{n-i+1}\right)^2\right\} \le \frac{1}{k(n-1)} \sum_{i,j} \sqrt{\frac{i}{n-i+1}\,\frac{j}{n-j+1}} < \infty \quad \text{as } n \to \infty\ .$$

This implies uniform integrability and thus shows that

$$(2.17)\qquad \sum_{\epsilon n \le i \le (1-\epsilon)n} \frac{E(M_i)}{n-i+1} \to \int_{\epsilon}^{1-E} \frac{E(\min \bar{W}_t^o)}{1-t}\, dt \ .$$

To prove (2.12) note that

$$\int_0^1 \frac{E(\min \bar{W}_t(i))}{1-t}\, dt$$

is a convergent integral so the right side of (2.17) approximates this arbitrarily well for ϵ sufficiently small. Further

$$\Big| \sum_{i \le \epsilon n} \frac{E(M_i)}{n-i+1} \Big| \le \sum_{i \le \epsilon n} \frac{E(|M_i|)}{n-i+1} \le c \sum_{i \le \epsilon n} \frac{1}{n} \sqrt{\frac{i}{n-1}}$$

for some positive c. The last sum is a Reimann sum for

$$\int_0^{\epsilon} \sqrt{\frac{x}{1-x}}$$

and so can be made arbitrarily small for small ϵ. The same argument works for

$$\sum_{(1-\epsilon)n \le i} \frac{E(M_i)}{n-i+1} \ .$$

This completes the proof of (2.12) and thus of Theorem 3.

3. Yes-No Feedback

In this section we discuss problems concerning a deck of n cards with c_i cards of type i, $1 \leq i \leq r$. We again write $\bar{c}$ for the composition vector $\bar{c} = (c_1,\ldots,c_r)$. On each trial the subject is told if the previous guess was correct or not. We refer to this situation as yes-no feedback. The problem is complicated when $\max(\bar{c}) > 1$, so we first state results for a deck of n cards labeled $1,2,\ldots,n$. We begin with no feedback and complete feedback guessing and compare these to yes-no feedback.

No feedback. If no feedback is provided, then any guessing strategy has one correct guess as its expected value. Several writers have shown that the variance of the number of correct guesses is largest when the guessing sequence is a permutation of $\{1,2,\ldots,n\}$ (see J. A. Greenwood (1938) and the references cited there).

Complete feedback. If the subject is shown the card just guessed each time, then the optimal strategy is to guess a card known to remain in the deck. The number of correct guesses has the same distribution as a sum of n independent random variables $X_i, 1 \leq i \leq n$ where $P(X_i = i) = \frac{1}{i} = 1 - P(X_i = 0)$. For large n the number of correct guesses is approximately normally distributed with mean log n and standard deviation $\sqrt{\log n}$.

If the subject is only given yes-no feedback, then the optimal and worst case strategies are described by the following pair of theorems.

Theorem 5. For a deck containing n cards labeled $\{1,2,\ldots,n\}$ a guessing strategy which maximizes the expected number of correct guesses when yes-no feedback is available is the strategy which guesses type 1 until the guess is correct, then guesses type 2 until the guess is correct (or the end of the deck is reached) and so on. If G denotes the number of correct guesses under this strategy, then

$$P(G \geq k) = \frac{1}{k!} \qquad k = 1,2,\ldots,n \quad . \tag{3.1}$$

$$E(G) = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!} = e - 1 + O(\frac{1}{n!}) \quad . \tag{3.2}$$

Theorem 6. For a deck containing n cards labeled $\{1,2,\ldots,n\}$ a guessing strategy which minimizes the expected number of correct guesses when yes-no feedback is available is the strategy which guesses type i on the i^{th} trial until a guess is correct and then repeats the correct guess for the remaining trials. If g denotes the number of correct guesses under this strategy, then g takes values zero and one with probabiltiy:

$$P(g = 0) = \frac{1}{e} + O(\frac{1}{n!}) = 1 - P(g = 1) \quad . \tag{3.3}$$

$$E(g) = 1 - \frac{1}{e} + O(\frac{1}{n!}) \quad . \tag{3.4}$$

Theorems 5 and 6 deal with the only type of deck where we can provide a simple description of the optimal strategy. In each case the optimal stratey is the "greedy" strategy which guesses the most probable (for Theorem 6 the least probable) type. We do not know if the greedy strategy is optimal for decks of 2n cards with composition

vector $(2,2,\ldots,2)$. We will show that the greedy strategy is not optimal for the 9-card deck with composition vector $(3,3,3)$. We first need some notation.

Let $\bar{p} = (p_1,p_2,\ldots,p_r)$ be a vector with integer components $p_i \geq 0$. Define

(3.5) $N(\bar{c};\bar{p}) \equiv$ the number of permutations of $c1 + \ldots + c_r$ symbols which do not have symbol 1 in the first $p1$ positions, nor symbol 2 in positions $p1 + 1,\ldots,p_2$, etc.

Thus, $N(\bar{c};\bar{0}) = (c1 + \ldots + c_r)!$. The numbers $N(\bar{c};\bar{p})$ allow computation of the most probable type at any stage of an experiment with yes-no feedback. They are closely related to rook polynomials described in Chapters 7 and 8 of Riordan (1958) and are discussed further in Diaconis, Graham, and Mallows (1979).

Algorithm to compute probabilities with yes-no feedback. Suppose an experiment started with composition vector $\bar{c}_0$ and that after the j^{th} guess there have been $Y_j(i)$ yes answers on type i and $p_j(i)$ no answers on type i, $1 \leq i \leq r$. The deck now has composition vector $\bar{c} = \bar{c}_0 - \bar{Y}_j$. We will call $\bar{c}$ the reduced composition vector. Writing $\bar{\delta}_i$ for the vector $(0 \ldots 1 \ldots 0)$ with a 1 at position i and 0 elsewhere, the conditional probabilities of a correct (or incorrect) guess on type i on the j + 1st trial given $\bar{Y}_j$ and $\bar{p}_3$ are:

$$(3.6) \qquad P(\text{yes on type } i \mid \bar{Y}_j, \bar{p}_j) = \frac{c_i\, N(\bar{c} - \bar{\delta}_i;\bar{p}_i)}{N(\bar{c};\bar{p}_j)} \quad .$$

$$(3.7) \qquad P(\text{no on type } i \mid \bar{Y}_j; \bar{p}_j) = \frac{N(\bar{c}; \bar{p}_. + \bar{\delta}_i)}{N(\bar{c}; \bar{p}_j)}$$

For $1 \leq i \leq r$.

As implied by (3.6) and (3.7), the function N satisfies the recursion

$$(3.8) \qquad N(\bar{c}; \bar{p} + \bar{\delta}_k) = N(\bar{c}; \bar{p}) - c_k \, N(\bar{c} + \bar{\delta}_k; \bar{p}), \quad 1 \leq k \leq r$$

$$\text{with } N(\bar{c}; \bar{0}) = (c_1 + \ldots + c_r)! \, .$$

This recursion can be solved in closed form to allow computation of N:

$$(3.9) \quad N(\bar{c}; \bar{p}) = \sum_{\bar{i} \leq \bar{c}} (-1)^{i_1 + \ldots + i_r} \binom{p_1}{i_1} \ldots \binom{p_r}{i_r} \frac{((c_1 - i_1) + \ldots + (c_r - i_r))}{(c_1 - i_1)! \ldots (c_r - i_r)!} \, .$$

The proof of (3.9) is given in Diaconis, Graham, and Mallows along with a host of other properties of $N(\bar{c}; \bar{p})$.

Let $E(\bar{c}; \bar{p})$ be the expected number of correct guesses under an optimal strategy starting from the reduced composition vector $\bar{c}$. $E(\bar{c}; \bar{p})$ is well defined since there are only a finite number of strategies and one (or more) of them maximizes the expected number of correct guesses. It is straightforward to show that E satisfies the recurrence:

$$(3.10) \quad E(\bar{c}; \bar{p}) N(\bar{c}; \bar{p}) = \max_k \{ E(\bar{c}; \bar{p} + \bar{\delta}_k) N(\bar{c}; \bar{p} + \bar{\delta}_k)$$

$$+ E(\bar{c} - \bar{\delta}_k; \bar{p}) c_k \, N(\bar{c} - \bar{\delta}_k; \bar{p}) + c_k \, N(\bar{c} - \bar{\delta}_k; \bar{p}) \}$$

where $N(\bar{c}; \bar{p})$ was defined in (3.5). We have not been able to solve

this recurrence for $E(\bar{c};\bar{0})$ in closed form even though N is known through (3.9). The recurrence can be solved numerically. For example, Mary Ann Gatto (Gatto (1978)) generated values for all composition vectors smaller than (5,5,5,5,5). Some results are:

$$E(3,3,3;\bar{0}) = 4.26,\ E(4,4,4,4;\bar{0}) = 5.47,\ E(5,5,5,5,5;\bar{0}) = 6.63\ .$$

The details of computing a number like $E(5,5,5,5,5;\bar{0})$ are not simple. The computation required 15 hours of cpu time on a Honeywell 6070 computer along with clever use of both recursions (3.8) and (3.10).

The optimal strategy at each stage is determined by finding a k which maximizes the right side of (3.10). Formula (3.6) implies that the greedy strategy at each stage is determined by choosing a k maximizing $c_k\ N(\bar{c} - \bar{\delta}_k;\bar{p})$. We now give an example which shows that the greedy strategy is not optimal.

Consider a 9-card deck with 3 each of 3 different types of card. A complete listing of $N(\bar{c};\bar{p})$ and $E(\bar{c};\bar{p})$ for all $(\bar{c};\bar{p})$ that arise with this 9-card deck is given in Diaconis and Graham (1978). In the situation summarized by (231;003) the optimal strategy is to choose type 3 on the next guess. However, type 2 is more probable than type 3 on the next guess. The situation summarized by (231;003) could arise under the optimal strategy from starting position (333;000) as follows: the first guess is type 1, and this is correct. The next three guesses are type 3, and all three guesses are wrong. The next guess on type 3 is correct. At this point the situation is summarized by (232;003) and the optimal guess is type 3. If this is correct, then the situation is summarized by (231;003).

Even though the greedy strategy is not optimal, computations reported in Diaconis, Gatto, and Graham (1979) show that the expected number of correct guesses under the.greedy strategy is extremely close to the expected number under the optimal strategy for decks with composition vector (3,3,3) or (5,5,5,5,5).

If $e(\bar{c};\bar{p})$ is the expected number of correct guesses for the worst possible strategy, then $e(\bar{c};\bar{p})$ satisfies a recurrence obtained from replacing max by min in (3.10). We have not pursued the problem of numerical computation of e.

Even though the optimal strategy seems to be extremely complex, we believe that the following simple persistence conjecture holds: In any problem with partial feedback, if symbol 1 is the optimal guess on trial i and a guess of 1 is answered by "no," then symbol 1 is optimal on guess i + 1.

Proofs for Section 3.

Proof of Theorem 5. When the given strategy is used, the permutations with k or more correct guesses are those in the set $A_k = \{\pi : \pi^{-1}(1) < \pi^{-1}(2) < \ldots < \pi^{-1}(k)\}$. Thus, $P\{G \geq k\} = P(\pi \in A_k) = \frac{1}{k!}$. This proves (3.1) and implies (3.2).

We now argue that the outlined strategy is optimal. In this problem a strategy S may be regarded as a sequence of n functions $s = (S_1, S_2, \ldots, S_n)$ where $S_i : \{0,1\}^{i-1} \to \{1,2,\ldots,n\}$. The interpretation is that a point in $\{0,1\}^{i-1}$ represents a sequence of i-1 yes or no answers, 0 standing for no and 1 for yes. The expected value of a strategy is $E(S) = \sum_{i=1}^{n} E(\delta_{\pi(i)S_i})$ where δ_{ij} is one or zero as i=j

or not. We will say that strategy S dominates strategy S' if $E(S) \geq E(S')$. Strategies S and S' will be called equivalent if $E(S) = E(S')$.

We first argue that the given strategy calls the most probable symbol at each stage. This is implied by the following monotonicity property of the function N:

(3.11) $p_i > p_j$ if and only if $N(1;p+\delta_i) < N(1;p+\delta_j)$.

This property of N is proved and further discussed in Diaconis, Graham, and Mallows (1979). Inequality (3.11) implies, and is implied by, the following combinatorial fact which was first established by Efron (1963).

(3.12) (Efron's Lemma). Let two decks of n cards be prepared. The first deck labeled $(1,2,\ldots,n)$, the second deck labeled $(a_1,a_2,\ldots,a_n)$ with $a_i \in \{1,2,\ldots,n\}$. Each deck is mixed and the cards turned over simultaneously, one pair at a time. The probability of no matches is largest if and only if there are no repeated symbols among the a_i. That is, if $\{a_i\} = \{1,2,\ldots,n\}$.

We have thus argued that the given strategy calls a most probable symbol at each stage. We want to show that any strategy which achieves the maximum number of correct guesses in this problem has this property. We note that a maximizing strategy exists since there are only finitely many strategies.

To begin with we may restrict attention to strategies which do not guess symbols known not to be left in the deck since such

strategies may be improved uniformly over all permutations by modifying them to guess only symbols which have not been definitely eliminated.

We will argue by backward induction that any strategy can be strictly improved by being modified to choose a most probable symbol at each stage. This is clear at trial n since modifying a strategy S so that it chooses the most probable symbol on the final guess can only increase E(S). Consider a strategy S which chooses the most probable symbol on trials n-k, n-k+1,...,n, for fixed $k \geq 0$. Consider a history $h \in \{0,1\}^{n-k-2}$ for which $S_{n-k-1}(h) = a$ where $b \neq a$ is the most probable guess and strictly more probable than a. By (3.11) we must have $p_b > p_a$, i.e., $p_b \geq p_a + 1$. No matter what the outcome of the guess $S_{n-k-1}(h) = a$ is, no symbol is more probable than b just before trial n-k. Thus, by induction we may assume $S(h,0) = S(h,1) = b$ (i.e., we can modify S to have this property without decreasing E(S)).

Consider the portion of the "strategy tree" of S following h (see Figure 1). Form the strategy $\hat{S}$ from S by defining $\hat{S}_{n-k-1}(h) = b$, $\hat{S}_{n-k}(h,0) = \hat{S}_{n-k}(h,1) = a$ and interchanging the two parts T_{01} and T_{10} of S which follow (h,0,1) and (h,1,0) (see Figure 1).

We claim that for each permutation π of the deck there is a unique permutation $\hat{\pi}$ of the deck such that the number of hits that S has for π is the same as the number of hits that $\hat{S}$ has for $\hat{\pi}$. This correspondence is given by switching coordinates n-k-1 and n-k:

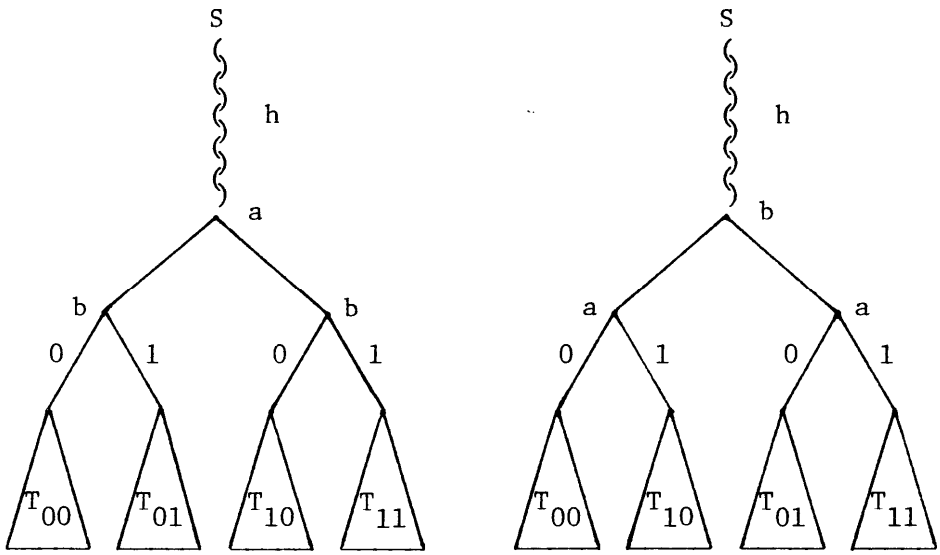

Figure 1

i.e.,

$$G(i) = \begin{cases} \pi(n-k) & \text{for } i = n-k-1 \\ \pi(n-k-1) & \text{for } i = n-k \\ \pi(i) & \text{otherwise.} \end{cases}$$

It is now a simple matter of checking the four cases $\pi(n-k-1)\begin{cases} = a \\ \# a \end{cases}$, $\pi(n-k)\begin{cases} = b \\ \neq b \end{cases}$ to see that $\hat{S}$ has the desired property or $\hat{\pi}$. For example, if $\pi(n-k-1) = a$, $\pi(n-k) \neq b$ (and, of course, π generates the history h), then π generates the history (h,1,0), collects one more hit (at the question $S_{n-k-1}(h) \stackrel{?}{=} a$) and exits into T_{10}. However, in $\hat{S}$, $\hat{\pi}$ gets a no at the question $\hat{S}_{n-k-1}(h) \stackrel{?}{=} b$, a yes at the question $\hat{S}_{n-k}(h,0) \stackrel{?}{=} a$ (collecting one hit) and also exits into T_{10}. Thus,

$$E(S) \geq E(S) \ .$$

However, by induction if we replace $\hat{S}_{n-k}(h,0) = a$ by $\hat{S}'_{n-k}(h,0) = b$, then since b is (still) more probable than a, this gives a strict improvement to $\hat{S}$. This shows that an optimal strategy must also guess the most probable symbol on trial n-k-l. This completes the induction step and the theorem is proved.

Proof of Theorem 6 Under the given strategy the number g of correct guesses is either zero or one. The probability of one correct guess is the probability that two permutations have one or more matching coordinates. This probability is well known (Feller 1968, p. 100) to be

$$P(g = 1) = 1 - P(g = 0) = 1 - \frac{1}{2!} + \frac{1}{3!} + \ldots + (-1)^{n-1}\frac{1}{n!} = 1 - \frac{1}{e} \approx O(\frac{1}{n!}) \ .$$

This proves (3.3) and (3.4).

We now show that the strategy given in Theorem 6 achieves the minimum number of expected correct guesses.

Using the notation established in the proof of Theorem 5, a strategy S is a sequence of functions $S = (S_1, S_2, \ldots, S_n)$, $S_i : \{0,1\}^{i-1} \to \{1,2,\ldots,n\}$. To begin with, it is easily shown that the expected value of any strategy can be decreased by modifying it so that

$$(3.13) \quad s_i(0,\ldots,01) = S_{i-1}(0,\ldots,00) \quad \text{for} \quad i = 2,3,\ldots,n$$

and so that S never achieves more than 1 correct guess.

Restricting attention to strategies which satisfy (3.13) we see that the strategy S is determined by the n numbers $S_1, S_2(\bar{0}), S_3(\bar{0}), \ldots, S_n(\bar{0})$. The expected value of S is the probability of one or more matches of a random permutation π to n symbols labeled $S_1, S_2(\bar{0}), \ldots, S_n(\bar{0})$. Efron's Lemma (3.12) shows that this probability is smallest when $\{S_1, S_2(\bar{0}), \ldots, S_n(\bar{0})\} = \{1, 2, \ldots, n\}$. This proves Theorem 6.

4. Evaluation of Feedback Experiments

The evaluation of feedback experiments is problematic because it is impossible to know what use a subject will make of the feedback information. In this section we introduce an evaluation approach called skill scoring. The idea is to compare the number of correct guesses with a base line rate calculated from the conditional expected number of correct guesses given the available information.

One example of skill scoring in the present setting was given in Table 1. To motivate the abstract definitions we are about to present, we review this example. The problem considered was card guessing with two types (call them type 1 and type 2), k of each type (so n= 2k cards in all) and complete feedback. We can model this by considering the basic probability space to be S_n, the set of permutations on $\{1,2,\ldots,n\}$, with the uniform probability measure. A permutation π is chosen at random and the ith trial is declared "type 1" if $\pi(i)$ is odd and "type 2" if $\pi(i)$ is even. On the ith trial the guessing subject is given feedback.

$$f_i = \begin{cases} 1 & \text{ith guess is correct,} \\ 2 & \text{ith guess is incorrect.} \end{cases}$$

This particular feedback function only depends on the current coordinate. Some possible variations are:

(4.1a) Feedback might depend on previous outcomes. This is realistic in card guessing experiments with unconscious cuing due to subjects being within sight or earshot of one another. If there were

few correct guesses in the early stages, more active feedback might be made available as the experiment progressed.

(4.1b) In addition to telling if the previous guess was correct or not, feedback might indicate if incorrect guesses were "close." Fisher (1924), (1928), and (1929) gives some examples of measures of closeness.

(4.1c) Feedback might only be available on some outcomes. For instance, the subject might be given feedback after red guesses but no feedback after black guesses.

We formulate the general situation in terms of S_n, the set of permutations of $\{1,2,\ldots,n\} = \Omega_n$. To model a pack of cards with c_i cards labeled i we need the idea of an evaluation function.

For example, to model red-black card guessing we might consider

$$\lambda_i(\pi(i)) = \begin{cases} 1 \text{ if } \pi(i) \text{ is odd} \\ 2 \text{ if } \pi(i) \text{ is even} \end{cases}$$

(4.2a) An evaluation function $\bar{\lambda}$ is a sequence of functions $\bar{\lambda} = (A,,\ldots,\lambda_n)$ where $\lambda_i(\pi) = \lambda_i(\pi(i))$ for $\pi \in S_n$. Let the range of λ_i be denoted by R_1, $= \{\lambda_i(\pi(i)) : \pi \in S_n\}$. An evaluation function is of type r if $\lambda_i(\pi) - 1 \equiv \pi(i) \pmod r$. Let Λ_i denote the algebra in S_n generated by $\lambda_1,\lambda_2,\ldots,\lambda_i$.

We will restrict attention to guessing strategies which take values in R_1. For each sequence of guesses and each history up to

time i, we must define a feedback function f_i. For complete feedback guessing, $f_i = \lambda_i(\pi)$. For yes-no feedback $f_i = \delta_{\lambda_i G_i}$, where G_i is the i^{th} guess.

(4.2b) A feedback function $\bar{f}$ is a sequence of functions $\bar{f} = (f_1, \ldots, f_n)$ where $f_i : R_1 \times R_2, \ldots, \times R_i \times S \to \Omega_n$. For each fixed $\bar{r} = (r_1, r_2, \ldots, r_i)$, we may regard f_i as a function $f_i(r_1, \ldots, r_i; \cdot)$ from S_n into Ω_n. This function is to be measurable when S_n is equipped with the algebra Λ_i defined in (4.2a) for any $\bar{r}$. We also define the algebra $\mathfrak{F}(\bar{r}) = \sigma\{f_1(r_1;\cdot), f_2(r_1, r_2;\cdot), \ldots, f_i(r_1, \ldots, r_i;\cdot)\}$.

This frightening terminology has the following interpretation: that $\bar{f}$ is measurable means that f_i only depends on the first i guesses and the values $\lambda_1(\pi), \ldots, \lambda_i(\pi)$. A function from S_n will be measurable with respect to $\mathfrak{F}(r_1, \ldots, r_i)$ if it only depends on the first i components of the permutation through the feedback information given when guesses $r_1, r_2, \ldots, r_i$ are made on trials $1, 2, \ldots, i$.

(4.2c) A feedback function is adapted if $\delta_{r_i \lambda_i}$ is $\mathfrak{F}(r_1, \ldots, r_i)$ measurable for each $r_1, r_2, \ldots, r_i$, $1 \leq i \leq n$. Adaptability means that the feedback includes the information that the last guess was correct or not.

(4.2d) A guessing strategy $\bar{g}$ is a sequence of functions $\bar{g} = (g_1, g_2, \ldots, g_n)$ where g_1 is a constant and $g_i : R_1 \times \ldots \times R_{i-1} \times S_n \to R_i$ satisfies $g_i(r_1, \ldots, r_{i-1}; \bullet)$ is $\mathfrak{F}(r_1, \ldots, r_{i-1})$ measurable. The value of g_i will be denoted G_i.

(4.2e) The collection of functions $\bar{\lambda}, \bar{f}$ will be called an experiment.

We define the skill scoring statistic for an experiment by

$$S = \sum_{i=1}^{n} \{\delta_{G_i\lambda_i} - E\{\delta_{G_i\lambda_i} | \mathfrak{F}(G_1,\ldots,G_{i-1})\}\} \quad . \tag{4.3}$$

The main motivation for considering S is that for a wide variety of experiments S can be normed to have an approximate standard normal distribution uniformly in guessing strategies. This is made precise in:

Theorem 7. For an experiment as defined in (4.2e) and any guessing strategy $\bar{g}$, the skill scoring statistic S defined by (4.3) satisfies

$$E(S) = 0 \quad . \tag{4.4}$$

If the evaluation function is of type r as defined by (4.2a) and the feedback function adapted as defined by (4.2c), then as n tends to infinity,

$$P\left(\frac{S}{\sqrt{n\,\frac{1}{r}(1-\frac{1}{r})}} \leq x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \quad . \tag{4.5}$$

Convergence in (4.5) is uniform in guessing strategies $\bar{g}$.

We now discuss some motivation and properties of S. In the absence of "talent," the distribution of $\delta_{G_i\lambda_i}$ given the feedback information is the conditional permutation distribution. S will be large when there are more successful guesses than chance predicts.

To compute S, only the observed guesses $G_1, G_2, \ldots, G_n$ need be known, not the entire guessing strategy.

For definiteness consider the example in Table l--card guessing with complete feedback from a deck containing k red and k black cards. As shown in Theorem 1, a subject using the optimal (or worst case) strategy expects to obtain approximately $k + \frac{1}{2}\sqrt{\pi k}$ (or $k - \frac{1}{2}\sqrt{\pi k}$) correct guesses. The statistic S compensates for this by subtracting a random correction factor with mean value $k + \frac{1}{2}\sqrt{\pi k}$ (or $k - \frac{1}{2}\sqrt{\pi k}$). This allows us to see if the subject scored more than chance when the strategy has been adjusted for. The conditional expected value in (4.3) may be complicated to compute if $\bar{f}$ is complex. For yes-no partial feedback the conditional expectations may be computed using (3.6) and (3.7).

One penalty that must be paid for the close tracking of $\delta_{G_i \lambda_i}$ by its expected value is as follows: If the feedback information at some stage determines the composition of the remainder of the deck, none of the subjects' guesses from that trial on have an effect on S. This can be seen in the last guess in Table 1 when the feedback information determined that the last remaining card was black. Similarly, the possible corrections due to feedback are less pronounced at the beginning of the deck and more pronounced toward the end of the deck.

Theorem 7 holds because the terms in the sum for S are a Martingale difference sequence with well-behaved variance. The Martingale central limit theorem is in force. If there was a practical reason for doing so, the result could be extended to scoring functions of the form

$$(4.6)\qquad \hat{S} = \sum_{i=1}^{n} \{W_i(G_1,\ldots,G_i;\ \lambda_1,\ldots,\lambda_i) - E[W_i(G_1,\ldots,G_i;\ \lambda_1,\ldots,\lambda_i)\,|\,\mathfrak{F}(G_1,\ldots,G_{i-1})]\}$$

where the functions W_i could be chosen to give desired weights to correct or incorrect guesses depending on previous results. We note that the form and motivation for the statistic S are quite similar to the form and motivation for the Mantel-Haenszel statistic as discussed (for example) by Tarone and Ware (1977). It should be possible to show that S is locally most powerful by arguments similar to those used to show that the Mantel-Haenszel statistic is locally most powerful against Lehmann alternatives.

We now illustrate the hypothesis of Theorem 7 through some examples.

(4.7) Example of the need of adaptability assumptions.

The adaptability assumption (4.2c) simply means that the feedback includes the information that the last guess was correct or not. To see that there is no hope of a normal limiting result without this assumption, consider an experiment with no feedback information, for example, $f_i \equiv 1$. To be specific, suppose there are n each of two types, and that the guessing strategy always guesses type 1. Then the number of correct guesses will always be n, and the conditional probability subtracted off at each stage will always equal 1/2 so that $S \equiv 0$. This example presents a fundamental problem for the widely used normal approximation to classical card guessing

experiments without feedback (this is discussed by Greville (1941), (1944)). It underscores the need for common sense even when Theorem 7 is in force since, if a subject always guesses the same type of card, the randomness captured by the limiting normality will be due to the fluctuation of the conditional expectations in S.

The next example shows the need for the assumption of a deck of type r by exhibiting several non-normal limits (depending on the guessing strategy) for a deck labeled $\{1,2,\ldots,n\}$.

(4.8) Example: Partial Feedback guessing for a deck labeled $\{1,2,3,\ldots,n\}$.

In this problem, as discussed in Section 3, a deck of n cards is labeled $\{1,2,\ldots,n\}$. A subject guesses the value of each card sequentially and is told if each guess is correct or not. Here $\lambda_i(\pi(i)) = n(i)$, $f_i(G_1,\ldots,G_i;\pi(1),\ldots,\pi(i)) = \delta_{\pi(i)G_i}$, and S can be represented as

$$S = \sum_{i=1}^{n} \left(a_{i\pi(i)} - \frac{1}{n-i+1} \sum_{j=1}^{n} \delta_{G_i\pi(j)} \right) .$$

To see that the distribution of S depends on the guessing strategy we consider three cases:

Case I. Worst case guessing. If the guessing strategy is the worst case strategy established in Theorem 6, we will show that the limiting distribution of S converges to a beta distribution on $\frac{1}{2}(1+\frac{1}{e^2})$ to1 with an atom at $-\frac{1}{2}(1-\frac{1}{e^2})$. More precisely,

(4.9) $P(S \leq t) \to G(t)$ as n tends to infinity where the distribution function G(t) is defined by

$$G(t) = 0 \quad \text{for} \quad t < -\frac{1}{2}(1-\frac{1}{e^2}) \ ,$$

$$= \frac{1}{e} \quad \text{for} \quad -\frac{1}{2}(1-\frac{1}{e^2}) \leq t \leq \frac{1}{2}(1+\frac{1}{e^2}) \ ,$$

$$= \sqrt{2}\,(t-t,^{1/2} \quad \text{for} \quad \frac{1}{2}(1+\frac{1}{e^2}) \leq t \leq 1 \ ,$$

$$= 1 \quad \text{for} \quad t > 1 \ .$$

Case II. $G_i \equiv 1$. We will show that when G_i always guesses 1, the distribution of S converges to an exponential distribution on $(-\infty,1]$. More precisely,

(4.10) $$P\{1-S \leq x\} \to 1 - e^{-x} \quad \text{for} \quad 0 \leq x < \infty \ .$$

Note that while the expected value of S agrees with the limiting expected value of 0, computation shows that

$\mathrm{Var}(S) = 2 \log n + O\left(\frac{\log^3(n)}{n}\right)$, as n tends to infinity.

Case III. Best case guessing. In Theorem5 the rule for maximizing the expected number of hits was shown to be the rule which guesses the most probable card at each stage. When this rule is used, we will show that, as $n \to \infty$, the statistic S tends to a countable mixture of continuous distributions:

$$P(S \leq t) = \sum_{i=1}^{\infty} p_i F_i(t) \tag{4.11}$$

where

$$p_i = \frac{1}{i!} - \frac{1}{(i-1)!}, \quad F_i(t) = P\{\prod_{j=1}^{i+1} L_i \leq e^{t-i}\}$$

where $L_1, L_2, \ldots, L_{i+1}$ are the lengths of the i + 1 intervals the unit interval is partitioned into by dropping i points at random.

Proofs for Section 4.

Proof of Theorem 7. Consider the basic probability space S_n with the uniform distribution. Let $G_1, \ldots, G_n$ be any sequence of guesses. Let $B0 = \{\phi, S_n\}$, $B_i = \mathfrak{F}(G_1, \ldots, G_{i+1})$ for $i = 1,2,\ldots,n-1$, $B_n = 2^{S_n}$. Thus, $B0 \subset B_1 \subset \ldots \subset B_n$. Let

$$Z_i = \frac{1}{\sqrt{n \frac{1}{r}(1 - \frac{1}{r})}} \{\delta_{G_i \lambda_i} - E\{\delta_{G_i \lambda_i} | B_{i-1}\}\} \quad \text{and} \quad X_i = \sum_{j=1}^{i} Z_j .$$

Because $\bar{f}$ is adapted, X_i is a B_i Martingale with $E(X_i) = 0$. To prove (4.5), we first show that (4.5) holds when $f_i = \lambda_i$ and $\bar{G}_i$ is the result of best case guessing. Further, and without real loss, suppose that $n = rk$. Let M_i denote the minimum of the number of each type seen

before time i, so $M_1 \equiv 0$. The probability of a correct guess on the i^{th} trial is $p_i = \frac{k - M_i}{n - i + 1}$ so Z_i takes values

$$\begin{cases} \dfrac{1}{\sqrt{n \frac{1}{r}(1 - \frac{1}{r})}} (1 - p_i) & \text{with probability } p_i \\ - \dfrac{1}{\sqrt{n \frac{1}{r}(1 - \frac{1}{r})}} p_i & \text{with probability } 1 - p_i \end{cases} .$$

According to the Martingale Central Limit Theorem (Hall (1977)) the limiting normality will be demonstrated if we can show that

$$\frac{1}{n \frac{1}{r}(1 - \frac{1}{r})} \sum_{i=1}^{n} p_i(1 - p_i) \xrightarrow{\text{Prob}} 1 .$$

We show that

(4.12a) $$\frac{1}{n} \sum_{i=1}^{n} p_i \xrightarrow{\text{Prob}} \frac{1}{r} ,$$

(4.12b) $$\frac{1}{n} \Sigma p_i^2 \xrightarrow{\text{Prob}} \frac{1}{r^2} .$$

To demonstrate (4.12a) write

$$Pi = \frac{k - \frac{i}{r} - \tilde{M}_i}{n - i + 1} \quad \text{with} \quad M_i = M_i - \frac{i}{r} .$$

Then

$$\frac{1}{n} \sum_{i=1}^{n} p_i = \frac{1}{r} + O(\frac{\log n}{n}) - \frac{1}{n} \Sigma \frac{\tilde{M}_i}{n - i + 1} .$$

The inequality (2.13) implies that there is a positive constant c_r such that $E(|M_i| \leq c_r\sqrt{\frac{i(n-i)}{n}}$. Using this and Markov's inequality it follows that for any $\epsilon > 0$,

$$P\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{\tilde{M}_i}{n-i+1} > \epsilon\right\} \leq \frac{c_r'}{\epsilon\sqrt{n}}$$

so that (4.12a) is true. The proof of (4.12b) is similar. Hence, we have shown that (4.7) holds when $f_i = \lambda_i$ and $\bar{G}_i$ is best case guessing. A similar proof works if $f_i = \lambda_i$ and $\underline{G}_i$ is worst case guessing. If now f_i is an arbitrary measurable feedback sequence and G_i an arbitrary guessing strategy, let $p_i = E\{\delta_{G_i\lambda_i}|B_i\}$. Recall i defined in (4.2a). Let $\bar{p}_i = E\{\delta_{\bar{G}_i\lambda_i}|\Lambda_{i-1}\}$, $\underline{p}_i = E\{\delta_{\underline{G}_i\lambda_i}|\Lambda_{i-1}\}$. Then $\underline{p}_i \leq p_i \leq \bar{p}_i$; and since (4.12a) and (4.12b) hold for $\underline{p}_i$ and $\bar{p}_i$, they must hold for p_i. This completes the proof of Theorem 7.

Proofs for example (4.8).

Proof of (4.11). For worst case guessing S takes values which depend on T, the time of the first correct guess. Let N(i,n) denote the number of permutations $\pi \in S_n$ which do not have $\pi(j) = j, 1 \leq j \leq i$. Equation (3.9) implies that $N(i,n) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}(n-j)!$ and we see that $P\{T = k\} = \frac{1}{n!}N(k-1, n-1)$ and $P\{i^{th}$ guess is correct$|$past$) = \frac{N(i-1, n-1)}{N(i-1, n)}$. Thus, S takes values

$(1-i)$ with probability $\frac{1}{n}$,

$(1 - \frac{1}{n} - \frac{n-2}{(n-1)^2})$ with probability $\frac{n-2}{n(n-1)}$,

$$1 - \sum_{i=1}^{k} \frac{N(i-1,\ n-l)}{N(i-1,\ n)} \quad \text{with probability } \frac{1}{n!}\, N(i-1,\ n-l)\ ,$$

$$\vdots$$

$$1 - \sum_{i=1}^{n} \frac{N(i-1,\ n-l)}{N(i-1,\ n)} \quad \text{with prdbability } \frac{1}{n!}\, N(n-1,\ n-l)\ ,$$

$$- \sum_{i=1}^{n} \frac{N(i-1,\ n-l)}{N(i-1,\ n)} \quad \text{with probabiltiy } \frac{1}{n!}\, N(n,\ n)\ .$$

We now show that $\frac{1}{n!}\, N(i,n) = (1-\frac{1}{n})^{i} + O(\frac{1}{n})$ uniformly in i. Indeed,

$$\left|\frac{1}{n!}\, N(i,n) - (1-\frac{1}{n})^{i}\right| \le \sum_{j=0}^{i} \binom{i}{j}\left(\frac{(n-j)!}{n!} - \frac{1}{n^{j}}\right) \le \sum_{j=0}^{n} \binom{n}{j}\left(\frac{(n-j)!}{n!} - \frac{1}{n^{j}}\right)$$

$$= e + O(\frac{1}{n!}) - (1-\frac{1}{n})^{n} = O(\frac{1}{n})\ .$$

Thus, for any k, $1 \le k \le n$,

$$\sum_{i=1}^{k} \frac{N(i-1,\ n-l)}{N(i-1,\ n)} = \frac{1}{n} \sum_{i=1}^{k} \left\{ \left(\frac{1-\frac{1}{n-l}}{1-\frac{1}{n}} \right)^{i-1} \right\} + O(\frac{1}{n}) = \frac{1}{2} - \frac{1}{2}(1-\frac{2}{n})^{k} + O(\frac{1}{n}),$$

so that S takes values

$$\frac{1}{2} + \frac{1}{2}(1-\frac{2}{n})^{k} + O(\frac{1}{n}) \quad \text{with probability } \frac{1}{n}(1-\frac{1}{n-1})^{k} + O(\frac{1}{n^{2}})$$

for $1 \le k \le n$ and S takes the value $-\frac{1}{2}(1-\frac{1}{e^{2}}) + O(\frac{1}{n})$ with probability $\frac{1}{e} + O(\frac{1}{n!})$.

Using these estimates shows that $P(S \le t) \to G(t)$ for $t \le \frac{1}{2} + \frac{1}{2}(1-\frac{1}{e^{2}})$. For larger t we have

$$P(S \leq t) = \frac{1}{e} + P\{\frac{1}{2} + \frac{1}{2}(1 - \frac{2}{n})^{T} + O(\frac{1}{n}) \leq t\} + O(\frac{1}{n!})$$

$$= \frac{1}{e} + P\{-\frac{2T}{n} \leq \log(2(t - \frac{1}{2}) + O_t(\frac{1}{n})\} + O(\frac{1}{n!})$$

$$= \frac{1}{e} + (1 - P\{\frac{T}{n} \leq -\frac{1}{2}\log 2(t - \frac{1}{2}) + O_t(\frac{1}{n})\}) + O(\frac{1}{n})$$

$$= \frac{1}{e} + 1 - (\frac{1}{e} + O(\frac{1}{n}) + \frac{1}{n} \sum_{0 \leq j \leq nf(t)} (1 - \frac{1}{n})^{j-1} + O(\frac{1}{n})$$

$$= \sqrt{2}\,(t - \frac{1}{2})^{1/2} + O(\frac{1}{n})$$

where we have written $f(t) = \frac{1}{2}\log 2(t - \frac{1}{2})$. This completes the proof of (4.9).

Proof of (4.10). When the guessing strategy has $G_i \equiv 1$, then S takes values $1 - (H_n - H_T)$, $k = 0,1,2,\ldots,n-1$, where T is uniformly distributed on $\{0,1,2,\ldots,n-1\}$ and $H_k = 1 + \ldots + 1/k$. So,

$$P\{1 - S > t\} = P\{e^{H_T - H_n} < e^{-t}\}$$

$$= P\{e^{H_T - H_n} \leq e^{-t} \,|\, \sqrt{n} \leq T \leq n - \sqrt{n})\}\,(1 + O(\frac{1}{\sqrt{n}})) + O(\frac{1}{\sqrt{n}})$$

$$= P\{e^{\log T - \log n + O(\frac{1}{T})} \leq e^{-t} \,|\, \sqrt{n} \leq T \leq n - \sqrt{n}\} + O(\frac{1}{\sqrt{n}})$$

$$= P\{\frac{T}{n}(1 + O(\frac{1}{T})) \leq e^{-t} \,|\, \sqrt{n} \leq T \leq n - \sqrt{n}\} + O(\frac{1}{\sqrt{n}})$$

$$\to e^{-t} \text{ as n tends to infinity,}$$

Proof of (4.11). With best case guessing it was shown in Theorem 5 that the number of correct guesses, G, takes value i with probability $p_i = \frac{1}{i!} - \frac{1}{(i-1)!}$ $1 \leq i \leq n-1$. When $G = i$, let T_j be the waiting time for the jth correct guess, for $1 \leq j \leq i$. The

random variables $\frac{1}{n}(T_1,\ldots,T_i,\ n - T_1 + \ldots + T_i)$ are easily shown to have as limiting distribution the distribution of the lengths $L_1,L_2,\ldots,L_{i+1}$ of the i + 1 intervals that the unit interval is partitioned into by i random points. de Finetti (Feller 1971, p. 42) has shown that $P\{L_1 \geq x_1,\ldots,L_{i+1} \geq x_{i+1}\} = (1 - x_1 + \ldots + x_{i+1})_+^1$ where + denotes positive part. When G = i, write $\bar{T} = \Sigma_{j=0}^{k} T_i$; then

$$P\{S \leq t \mid G = i\} = P\{i - \sum_{j=1}^{i} \sum_{k=0}^{T_j} \frac{1}{n-j-k} - \sum_{j=0}^{n-\bar{T}} \frac{1}{n-i-k} \leq t\}$$

$$\rightarrow P\{L_1 L_2 \cdot \ldots L_{i+1} \leq e^{t-i}\}$$

by an easy argument. This completes the proof of (4.11).

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