MULTIDIMENSIONAL ADDITIVE SPLINE APPROXIMATION

by

Jerome H. Friedman
Eric Grosse
and
Werner Stuetzle

COMPUTER SCIENCE DEPARTMENT
Stanford University

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Jerome H Friedman

Stanford Linear Accelerator Center

Eric Grosse

Computer Science Department

Werner Stuetzle

Stanford Linear Accelerator Center

and

Department of Statistics

Abstract. We describe an adaptive procedure that approximates a function of many variables by a sum of (univariate) spline functions s_m of selected linear combinations $a_m \cdot x$ of the coordinates

$$\phi(x) = \sum_{1 \le m \le M} s_m(a_m \cdot x).$$

The procedure i's nonlinear in that not only the spline coefficients but also the linear combinations are optimized for the particular problem. The sample need not lie on a regular grid, and the approximation is affine invariant, smooth, and lends itself to graphical interpretation. Function values, derivatives, and integrals are cheap to evaluate.

1. Introduction

Multidimensional surface approximation is recognized as an important problem for which several methodologies -have been developed. The aim is to construct an approximation $\phi(x)$ to a p-dimensional surface $y = f(x)$ on the basis of (possibly noisy) observations $\{(y_i, x_i)\}_{1 < i < n}$. Most existing methods, such as tensor product spliues, kernels, and thin plate splines (for a survey, see Schumaker [1976]), are linear in that

$$\phi(x) = \sum_{1 \leq i \leq n} w_i y_i,$$

where the weights $\{w_i\}$ depend only on x and $\{x_i\}_{1 \leq i \leq n}$, but not on $\{y_i\}_{1 \leq i \leq n}$. These methods have the advantage that they are straightforward to compute and their theory is tractable. In practice, however, they are limited because they cannot take advantage of special properties of the surface. Due to the inherent sparsity of high-dimensional sampling, procedures successful in high dimensions must be adaptive and thus nonlinear.

In this paper we describe an adaptive procedure that approximates $f(x)$ by a sum of (univariate) spline functions s_m 0f selected linear combinations a, . x of the coordinates

$$\phi(x) = \sum_{1 \leq m \leq M} s_m(a_m \cdot x). \tag{1}$$

The procedure is nonlinear in that not only the spline coefficients but also the linear combinations are optimized for the particular problem.

2. The algorithm

The spline function s_m along a, $\cdot$ x is represented as a sum of j_m B-splines [de Boor, 1979] of order q

$$s_m(a_m \cdot x) = \sum_{1 \leq j \leq j_m} \beta_{mj} B_{mj}(a_m \cdot x). \tag{2}$$

The approximation $\phi(\mathbf{x})$ (given by equations (1) and (2)) is specified by the directions $\{\mathbf{a}_m\}_{1 \leq m \leq M}$, the knot sequences along a, .x for $1 \leq m \leq M$, and the B-spline coefficients $\{\beta_{mj}\}_{1 \leq m \leq M, 1 \leq i \leq m}$. For particular $\{$ a,$\}$, the knots are placed heuristically and then the $\{\beta_{mj}\}$ are determined by (linear) least squares. The residual sum of squares from this fit is taken to be the inverse figure of merit for $\{\mathbf{a}_m\}_{1 \leq m \leq M}$.

Following Friedman, Jacobson, and Stuetzle [1980], the approximation is constructed in a stepwise manner: given $\{\mathbf{a}_m\}_{1 \leq m \leq M-1}$, find $\mathbf{a}_M$ to optimize the figure of merit of $\{\mathbf{a}_m\}_{1 \leq m \leq M}$. Terminate when the figure of merit is not significantly improved.

3. Xmplcmentation

The most difficult part of the algorithm is finding each direction $\mathbf{a}_m$. We perform a numerical search using a Rosenbrock method [Rosenbrock, 1966] modified for the unit sphere, starting at the best coordinate direction. On any given search, there is no guarantee that the global optimum will be found. If the local optimum is insignificant, the search is restarted at random directions. This guards against premature termination. If the local optimum is significant but not identical to the global optimum, no great harm is done because a new search **is** performed in the next iteration on an object function for which the previous optima have -been deflated. Each iteration of the optimizer requires 3.5 p function evaluations, on the average, where p is the dimension of x. Two iterations are nearly always sufficient.

'For high dimensionality, the computation is dominated by the evaluations of the object function. Since it is not crucial to find the precise optimum, considerable savings can be achieved by substituting a similar, but much less expensive figure of merit during the search for a new direction. For this figure of merit not only the previously found directions but also the corresponding spline coefficients arc held fixed. The new direction can thus be found by considering the residuals from the model of the previous iteration. For **a** given direction, the

residuals are modelled by a smooth based on local linear fits [Cleveland, 1979], [Friedman, Jacobson, and Stuetzle, 1980]. The characteristic bandwidth (that is, the average fraction of observations used in each local fit) is taken to be inversely proportional to the number of knots. The residual sum of squares from the smooth is the figure of merit used for the smooth. Solving the least squares problem for the original figure of merit requires

$$O\left(n\left(\sum_{1 \le m \le M} j_m\right)^2\right)$$

operations, while the new figure of merit can be evaluated in roughly n operations using updating formulas for the moving average. The least squares problem has to be solved only once for each iteration to determine the new model after a_m has been found.

To solve the least squares problem, we form the normal equations and use a pseudo-inverse, since the design matrix might not have full rank. The singularity which arises form the inclusion of a constant term for each direction is remedied by simply dropping one column per direction from the design matrix. Higher order singularities caused, for example, by the linear terms for three co-planar directions, are not explicitly taken care of, but are handled by the pseudo-inverse.

Our knot placement procedure is motivated by the sequential nature of the. algorithm. At each iteration, the knot positions are required for the least squares fit, after the new direction has been found. Our model at this point is the spline fit of the previous iteration, plus the moving average smooth along the newly found direction. The knot placement is based on the residuals $\{r_i\}$ from this model. Multidimensional structure in these residuals due to incompleteness of the model manifests itself as high local variability in the scatterplots of r_i against $a_m.x_i$. In order to preserve the ability of fitting this structure in further iterations, it is important to avoid accounting for it by spurious fits along existing directions. For this reason we place fewer knots in regions of higher local variability. Since the residuals change, the knots are replaced along all directions at each iteration.

The knots along a direction a_m are placed as follows: the smooth described above is applied to $\{(r_i, a_m \cdot x_i)\}_{1 \le i \le n}$ and the local variability v_i at each point is taken to be the average squared residual from its local linear fit. The Winsorized local variabilities are defined by

$$w_i = \begin{cases} 2\bar{v} & \text{if } v_i > 2\bar{v} \\ \frac{1}{2}\bar{v} & \text{if } v_i < \frac{1}{2}\bar{v} \\ v_i & \text{otherwise} \end{cases}$$

(where $\bar{v} = \frac{1}{n}\sum_{1 \le i < n} v_i$), and then are scaled so that $\sum_{1 < i < n} \frac{1}{w_i} = 1$. The knots $\{t_l\}$ are placed to divide the line into intervals with equal content of $\frac{1}{w_i}$:

$$\text{for each I,} \qquad \frac{1}{j_m - q + 1} = \sum_{a_m \cdot x_i \in [t_l, t_{l+1}]} \frac{1}{w_i}.$$

4. Examples

In this section we present and discuss the results of applying the Multidimensional Spline Approximation method (MASA) to four examples. (A FORTRAN program implementing MASA is available from the authors.) The first three examples were suggested elsewhere for testing surface approximation procedures. The function in the fourth example was studied in connection with a problem in mathematical genetics.

The first example is taken from Friedman [1979]. In this example uniformly distributed random points $\{x_i | 1 \le i < 200)$ were generated in the six-dimensional hypercube $[0, 1]^6$. Associated with each point x_i was a surface value

$$y_i = 10 \sin(\pi x_i(1)x_i(2)) + 20[x_i(3) - 0.5]^2 + 10x_i(4) + 5x_i(5) + 0x_i(6) + \epsilon_i,$$

where the $\{\epsilon_i\}$ were independent identically distributed standard normal. The inverse figures of merit for the approximation with $M = 1, \ldots, 4$ terms were $6.71, 4.29, 1.87, 0.97$. In three restarts, the figure of merit did not decrease below

0.86, so $M = 4$ was chosen. The four linear combinations and the corresponding spline functions arc shown in figures 1.1-1.4. (For completeness, the program parameters are also listed; see the program comments for a detailed explanation.) The spline along the first linear combination (figure 1.1) is seen to model the linear part of the surface. The second term in the approximation (figure 1.2) models the additive quadratic dependence on z(3). The final two terms (figures 1.3, 1.4) model the interaction between $x(1)$ and $x(2)$. The L_2 norm of the error $\|f - \phi\|_2$ was 0.57.

Although the full advantages of MASA compared to other procedures are realized in higher dimensional or noisy settings, we applied it to two bivariate examples used by Franke [1979] to compare a number of interpolatory surface approximation schemes.. For both examples 100 uniformly distributed random points in the unit square $[0,1]^2$ were generated. The function in Franke's first example is

$$f(x,y) = 0.75 \exp[-x\frac{(9\ 2)^2 + (9y\ 2)^2}{4}]$$
$$+ 0.75 \exp[-\frac{\cdots}{49}(9x+1)^2 - \frac{9y+1}{10}]$$
$$+ 0.5 \exp[-\frac{(9x-7)^2 + (9y-3)^2}{4}]$$
$$+ 0.2 \exp[-(9x-4)^2 - (9y-7)^2].$$

Considerations similar to those in the previous example led to an approximation with three terms, The linear combinations and corresponding spline functions are shown in figures 2.1–2.3.

The function in Franke's second example is

$$f(x, y) = \frac{1}{9}[\tanh(9y - 9x) + 1].$$

For this case the approximation used only one term, shown in figure 3.1.

Since different randorn points were used in Franke's and our tests, precise comparisons are not possible. On the first example, MASA gave roughly an order

of magnitude larger errors than the best methods in Franke's trials (global basis function methods) while on the second example, MASA gave an order of magnitude smaller errors than the best methods. These results are not surprising since the peak-shaped basis functions of the global basis methods are especially suited for representing the peaks of the first example, whereas the ridge-shaped basis functions of MASA are especially suited to the second example. Unfortunately, peak-shaped basis functions are not appropriate for moderate or higher dimensionality. The difficulty is that in order to achieve a smooth fit, the width of the basis peaks needs to be comparable to the distance between data points. For n uniformly distributed random points in a p-dimensional hypercube $[0,1]^p$, the typical nearest neighbor distance is $(\frac{1}{n})^{\frac{1}{p}}$. In particular for $n = 1000$ and $p = 10$, this distance is 0.5, and for $p = 20$ is 0.7. Thus variation of the surface over distances small compared to such large interpoint distances cannot be well approximated with these global basis functions methods.

Our final example is a 19-dimensional function encountered by Carmelli and Cavalli[1979]. An important question is the structure of this function near its minimum. We sampled the function at 200 points uniformly distributed in a small hypercube centered at the minimum found by numerical optimization and applied MASA. The inverse figure of merit for the best constant fit was 13.3. The inverse figure of merit for $M = 1$ was 0.78. In 30 restarts, the figure of merit did not decrease below 0.42. Figure 4.1 gives the linear combination and corresponding spline function. This picture shows considerable structure that was not revealed in the original study.

5. Discussion

MASA can be expected to work well to the extent that the surface can be approximated by a function of the form (1). Of course in the limit $M \to \infty$ all smooth surfaces can be represented by (1), but even for moderate M functions of this form constitute a rich class.

As seen in the previous section, an advantage of using essentially one-dimensional basis functions is the possibility of graphical interpretation. The

entire model can be represented by graphing $s_m(a_m \cdot x)$ against a, . x and by specifying $\{a_m\}_{1 \leq m \leq M}$ (perhaps graphically for $p = 2$ or 3). Additionally, adequacy of the knot placement can be judged using the M plots of the residuals from the final model against $a_m \cdot x$. Proper termination of the algorithm can be assured by monitoring at each iteration the plot of the residuals from the model of the previous iteration along the newly found direction.

The problem of sparse sampling in high dimensions is not encountered, since MASA is fitting one-dimensonal projections of the entire sample. The sample need not lie on a regular grid, and the approximation is affine invariant and smooth. Function values, derivatives, and integrals are cheap to evaluate. In addition, since the approximation is locally quadratic for $q = 3$, optimization algorithms can be expected to converge rapidly.

References

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SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR   1
   0.4260   0.4295   0.0416   0.7118   0.3541   -0.0146

PLOT STATISTICS =   0.   0.   0.   /   0.   9.   0.   /   0.   0.   0.

```
 PROJ                                                                        PROJ
 ON X                                                                        ON Y
 AXIS      300000000030000~010000000001000000000010000000000000000000003     AXIS
 22.4121 +------------+--------+--------+--------+--------+--------+------
 21.9552 I                                                         ****I      0
 21.4982 I                                                     **** I         0
 21.0412 I                                                  ***     I         0
 20.5843 I                                               ***        I         0
 20.1273 I                                             ***          I         0
 19.6704 I                                          ***             I         0
 19.2134 I                                        **                I         0
 18.7565 I                                     ***                  I         0
 18.2995 I                                   **                     I         0
 17.8426 +                                 **                       +         0
 17.3856 I                               ***                        I         0
 16.9287 I                             **                           I         0
 16.4717 I                           **                             I         0
 16.0148 I                         **                               I         0
 15.5578 I                       ***                                I         0
 15.1009 I                     **                                   I         0
 14.6439 I                   ***                                    I         0
 14.1869 I                  **                                      I         0
 13.7300 I                ***                                       I         0
 13.2730 +              **                                          +         0
 12.8161 I            ***                                           I         0
 12.3591 I           **                                            I         0
 11.9022 I         ***                                             I         0
 11.4452 I        **                                               I         0
 10.9883 I       ***                                               I         0
 10.5313 I     **                                                  I         0
 10.0744 I     **                                                  I         0
  9.6174 I   ***                                                   I         0
  9.1604 I  **                                                     I         0
  8.7035 +  ***                                                    +         0
  8.2465 I **                                                      I         0
  7.7896 I***                                                      I         0
  7.3326 I***                                                      I         0
  6.8757 I***                                                      I         0
  6.4187 I ***                                                     I         0
  5.9518 I***                                                      I         0
  5.5048 I*                                                        I         0
  5.0479 I                                                         I         0
  4.5909 I                                                         I         0
  4.1339 +3-----------+--------+--------+--------+--------+--------3+        9
          +-----------+--------+--------+--------+--------+--------+------
 LEFT     000000000000000000000000000000000000001111111111111111111111111111
 BIN      ..........................................................
 EDGE     333334444455555566667777788889999990000111112222333334444455556
          135792468035'791468025791368024791357024691357024681357924~80
          02468~~2468824680257913579135791357913579135791357913579
          88888888899999999900000000011111111112222222223333333334444444
```

?

```
SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR    2
 -0.0211    0.0897    0.9956   -0.0136    0.0        0.0

PLOT STATISTICS =    0.    0.    0.    /    0.    9.    0.    /    0.    0.    0.
     PROJ                                                                      PROJ
     ON X                                                                      ON Y
     AXIS     30000000000000000000001000000000000000010000000000000100000000003  AXIS
    0.0006  +————————————————————  ·    ——————————+————————+————————+————————  +
   -0.1612  I*                                                                I    0
   -0.3230  I*                                                                I    0
   -0.4848  I**                                                               I    0
   -0.6466  I *                                                               I    0
   -0.8084  I **                                                            *I    0
   -0.9702  I  *                                                            *I    0
   -1.1320  I  **                                                           *I    0
   -1.2938  I   *                                                          **I    0
   -1.4556  I   *                                                          * I    0
   -1.6174  +    **                                                      **  +    0
   -1.7792  I    *                                                       *   I    0
   -1.9411  I    **                                                     **    I    0
   -2.1029  I     *                                                     *     I    0
   -2.2647  I     **                                                   **     I    0
   -2.4265  I      **                                                  *      I    0
   -2.5883  I      *                                                  **      I    0
   -2.7501  I      **                                                 *       I    0
   -2.9119  I       *                                                **       I    0
   -3.0737  I       **                                              **        I    0
   -3.2355  +        **                                            *          +    0
   -3.3973  I        *                                            **          I    0
   -3.5591  I        **                                           *           I    0
   -3.7209  I         **                                         **           I    0
   -3.8828  I         *                                         **            I    0
   -4.0446  I         **                                        *             I    0
   -4.2064  I          **                                      **             I    0
   -4.3682  I           **                                     **             I    0
   -4.5300  I           **                                     *              I    0
   -4.6918  I            **                                   **              I    0
   -4.8536  +            **                                 **                +    0
   -5.0154  I             **                               **                 I    0
   -5.1772  I             **                              **                  I    0
   -5.3390  I             ***                            **                   I    0
   -5.5008  I              ***                          **                    I    0
   -5.6626  I               ***                        ***                    I    0
   -5.8244  I                *****                   ***                      I    0
   -5.9863  I                   *******                                       I    0
   -6.1481  I                                                                 I    0
   -6.3099  I                                                                 I    0
   -6.4717  +3                +                +                +           3+    9
         +——  ·  . . . ——————+————————+— .  ·  ——+————————  ———————  ———————
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    EKE-     . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
             00000001111111222222333333344444445555555666666777777788888889999999~
             1024579124689135680135780245791246~9134680135780245790246791
             0741963186308520742974196318530852074296419631~53~7529742964
             3148259360471482593603714825926037148159260370481159269370481

 ?
```

figure 1.2

```
SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR    3
   0.6587    0.7512    0.0251   -0.0340    0.0        0.0083


PLOT STATISTICS =    0.    0.    0.    /    0.    9.    0.    /    0.    0.    0.    PROJ
  PROJ                                                                              ON Y
  AXIS          300~00000000000001000000000010000000000001000000000000000000003    AXIS
   2.264 5    +-------  ------- ---+---------+---------+---------+---------+-------
   1.9643    I                                         *******                    I     0
   1.6641    I                                    ***          ***                I     0
   1.3640    I                                  ***               **              I     0
   1.0638    I                               ***                    *             I     0
   0.7637    I                             ***                       **           I     0
   0.4635    I                           **                            **         I     0
   0.1634    I                         ***                               *        I     0
  -0.1368    I*                       ***                                 **      I     0
  -0.4370    I***                   **                                     *      I     0
  -0.7371    +   ***             ****                                      **     +     0
  -1.0373    I      ***      ****                                           *     I     0
  -1.3374    I        **************                                        **          0
  -1.6376    I                                                              *           0
  -1.9378    I                                                             **     I     0
  -2.2379    I                                                              *      I     0
  -2.5381    I                                                             *             0
  -2.8382    I                                                           **      I      0
  -3.1384    I                                                           *       I      0
  -3.4386    I                                                           *              0
  -3.7387    +                                                         **        +     0
  -4.0389    I                                                         *         I     0
  -4.3390    I                                                         *         I     0
  -4.6392    I                                                       **          I     0
  -4.9394    I                                                       *           I     0
  -5.2395    I                                                      *            I     0
  -5.5397    I                                                    **             I     0
  -5.8398    I                                                    *              I     0
  -6.1400    I                                                    *              I     0
  -6.4402    I                                                  **               I     0
  -6.7403    +                                                  *                +     0
  -7.0405    I                                                  *                I     0
  -7.3406    I                                                  *                I     0
  -7.6408    I                                                **I                      0
  -7.9410    I                                                 *I                      0
  -8.2411    I                                                 *I                      0
  -8.5413    I                                                 *I                      0
  -8.8414    I                                                 *I                      0
  -9.1416    I                                                  I                      0
  -9.4418    I                                                  I                      0
  -9.7419    +3--------+---------+---------+---------+---------+---------+-------3+     9
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             3457891234678012456890234678012356790134578912356790124 5689~
             1470360369269258258147147037036936925825814814704704 70369369259
?
```

figure 1.3

```
SPLINE FUNCTION AND . . KNOTS ALONG DIRECTION NR . . .    4 0.0072
 -0.4506   0.8818      0.1241  -0.0169   0.0605

PLOT STATISTICS =   0.   0.   0.  /  0.   9.   0.  /  0.   0.   0.     PROJ
     PROJ                                                             ON Y
     AXIS   30000000000000000010000000000000010000000001000000000000000000000003  AXIS
    5.1486 +---------+---------+---------+---------+---------+---------+
    5.0070 I                        **********                         I      0
    4.8654 I                      **        ***                        I      0
    4.7238 I                    ***           ***                      I      0
    4.5822 I                  **                ***                     I      0
    4.4406 I                 **                   ***                   I      0
    4.2991 I                **                      ***                 I      0
    4.1575 I                *                         **                I      0
    4.0159 I               **                          ***             I      0
    3.8743 I             **                              **             I      0
    3.7327 +            **                                **            +      0
    3.5911 I            *                                  ***          I      0
    3.4496 I           **                                   **          I      0
    3.3080 I           *                                    **          I      0
    3.1664 I          **                                     **         I      0
    3.0248 I          *                                       **        I      0
    2.8832 I         **                                        **       I      0
    2.7416 I        **                                          **      I      0
    2.6001 I        *                                           **      I      0
    2.4585 I        *                                            **     I      0
    2.3169 +       **                                             **    +      0
    2.1753 I       *                                               **   I      0
    2.0337 I      **                                                *   I      0
    1.8921 I      *                                                 **  I      0
   '1.7505 I     **                                                 ** I      0
    1.6090 I     *                                                   **I      0
    1.4674 I    **                                                    *I      0
    1.3258 I    *                                                      I      0
    1.1842 I    *                                                      I      0
    1.0426 I   **                                                     I      0
    0.9010 +   *                                                      +      0
    0.7595 I **                                                       I      0
    0.6179 I *                                                        I      0
    0.4763 I *                                                        I      0
    0.3347 I**                                                        I      0
    0.1931 I*                                                         I      0
    0.0515 I*                                                         I      0
   -0.0900 I*                                                         I      0
   -0.2316 I                                                          I      0
   -0.3732 I                                                          I      0
   -0.5148 +3          +         +         +                         3:      9
          +---------+---------+---------+---------+---------+---------+

 LEFT    _____
 BIN      000000000000000000000000000000000000000000000000000000000000000000
 EDGE    ..........................................................
          2222211111~000~0000011111222223333344444555556666677777788888
          9753197531975310246802468024680~2468024680246802468024680246802468
          44444433333333336666777777777777777788888888888888889999999999999
          33211099877654478890012234456678894012234456678990112334556 7
 ?
```

figure 1.4

MULTIDIMENSIONAL ADDITIVE SPLINE APPROXIMATION (4/19/80)
PARAMETERS FOR THIS RUN
NOBS   200
NPRED   6
MODE   2
MAXTRY   4
MAXPRO   7
PPCONV
MAXIT
KORDER   3
MAXKNO   9
BANFAC   2.00000
IPRINT   2
NPRINT   1
PLOTRM   .0
AVERAGE SQUARED RESIDUAL AROUND THE MEAN   26.7495
?

```
SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR   1
  0.6398    0.7685

PLOT STATISTICS =   0.   0.   0.   /   0.  11.   0.   /   0.   0.   0.
  PROJ                                                                    PROJ
  ON X                                                                    ON Y
  AXIS        3000000~0~0~00000880100000000010~100000100000000001000~0003  AXIS
   0.4576 +———————+———————+——————————+—————————————+————————+————————+
   0.4447 I****                                                        I    0
   0.4318 I  **                                                        I    0
   0.4188 I   ***                                                      I    0
   0.4059 I     **                                                     I    0
   0.3929 I    **                                                      I    0
   0.3800 I     **                                                     I    0
   0.3670 I      **                                                    I    0
   0.3541 I       **                                                   I    0
   0.3412 I        **                                                  I    0
   0.3282 +         *                                                  +    0
   0.3153 I         **                                                 I    0
   0.3023 I          **                                                I    0
   0.2894 I          *                                                 I    0
   0.2765 I           **                                               I    0
   0.2635 I           **                                               I    0
   0.2506 I           *                                                I    0
   0.2376 I            **                                              I    0
   0.2247 I            *                                               I    0
   0.2117 I            **                                              I    0
   0.1988 +            *                                               +    0
   0.1859 I             **                                             I    0
   0.1729 I             *                                              I    0
   0.1600 I             **                                             I    0
   0.1470 I             *                                              I    0
   0.1341 I             **                                             I    0
   0.1211 I              **                                            I    0
   0.1082 I              *                                             I    0
   0: 0953 I              **                                           I    0
   0.0823 I               **                                          T    0
   0.0694 +                ***   ****                                 7    0
   0.0564 I                  ****  **                                 I    0
   0.0435 I                        **                                 I    0
   0.0306 I                        *              ******Jr**        *I    0
   0.0176 I                        **        **          *********I    0
   0.0047 I                        *      ***                        I    0
  -0.0083 I                       **   **                            I    0
  -0.0212 I                       ****                               I    0
  -0.0342 I                                                          I    0
  -0.0471 I                                                          I    0
  -0.0600 +3        +         +        + +     +    --      +     3:   11
         +———————+———————+——————————+—————————————+————————+————————
  LEFT    0000000000000000000000000000000000000000000000000001111111111111
  BIN
  EDGE:   1111122222233333334444455555666667777778~88899999900000011111122
          1346802468023579135791346802468024579135791346802468024579131
          0098765432109876654321098765432110987654321098777654321098765
          9~01223445667889001223445567789901123345567789901123345567778
?
```

figure 2.1

SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR   2
 -0.1465   0.9892

PLOT   STATISTICS =   0,   0.   0.   / 0. 11. 0. / 0. 0. 0.
     PROJ   PROJ
     ON X   ON Y
    AXIS   30~00~0000000001000000~01000000010000001000000000010000003   AXIS
     0.4584 f----------------t------------~-----------t-----------
     0.4454 I   ******   I   0
     0.4325 I   **   **   I   0
     0.4195 I   **   **   I   0
     0.4066 I   *   *   I   0
     0.3936 I   **   **   I   0
     0.3807 3   **   *   I   0
     0.3677 I   *   **   I   0
     0.3548 I   **   *   I   0
     0.3418 I   *   **   4   0
     0.3288 +   **   *   +   0
     0.3159 I   *   *   I   0
     0.3029 I   **   **   I   0
     0.2900 I   *   *   I   0
     0.2770 I   **   *   I   0
     0.2641 I   *   *   I   0
     0.2511 I   *   **   I   0
     0.2382 I . **   *   0
     0.2252 I   *   *   0
     0.2123 I **   **   I   0
     0.1993 +   *   *   +   0
     0.1863 I   *   *   I   0
     0.1734 I **   **   I   0
     0.1604 I *   *   I   0
     0.1475 I *   **   I   0
     0.1345 I **   *   I   0
     0.1216 I *   **   I   0
     0.1086 I *   **   I   0
     0.0957 I *   ****   I   0
     0.0827 I **   ****   I   0
     0.0698 + *   ***   +   0
     0.0568 I *   **   *I   0
     0.0439 I**   **   ***I   0
     0.0309 I*   **   ** I   0
     0.0179 I*   **   *** I   0
     0.0050 I*   **   *** I   0
    -0.0080 I*   ***   ***   I   0
    -0.0209 I   *****   I   0
    -0.0339 I   I   0
    -0.0468 I   0
    -0.0598 +3 ----------+-------+-------+-----$-------t--------+---t-----3:   11
          t   ---+-------+-------+---
     LEFT   ---
   B I N   000000000000000000000000000000000000000000000000000000000000000
     EDGE   ....*..................................
              000000000000111112222222333333444445555555666666777777888889999
              754201246891356802357902467913468013 57~~2457912468913568~235
              792470752075308530853085308530853085308531863186318 63~86
              272724949495050505161616272727283838384949495050505161616272

                                                                           ?

                              figure 2.2

```
SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR    3
  0.9997    0.0259

PLOT STATISTICS =   0.    0.    0.   /   0.   11.    0.   /   0.    0.    0.
  PROJ                                                                      PROJ
  ON X                                                                      ON Y
  AXIS      30000000000100000000100000000100~0000000000100010000000000003  AXIS
   0.2831   +- - - - --—---- - - -  t - - - - - + - - w - f - - - - - - --—+---------
   0.2723              ******
   0.2616   I         **      **                                        I      0
   0.2509   I        **        **                                       I      0
   0.2402   I        *          *                                       I      0
   0.2295   I       **          **                                      I      0
   0.2187           *            *                                      I      0
   0.2080   I      **            **                                     I      0
   0.1973          *              *                                     I      0
   0.1866   I     **              *                                     I      0
   0.1759   +                     **                                    +      0
   0.1651   I    **               *                                     I      0
   0.1544   I    **               *                                     I      0
   0.1437   I     *               **                                    I      0
   0.1330   I    **                *                                    I      0
   0.1223   I    *                 **                                   I      0
   0.1115   I    *                  *                                   I      0
   0.1008   I   **                  *                                   I      0
   0.0901   I  *                    **                                  I      0
   0.0794   I  *                     *                                  I      0
   0.0687   + **                     **            ******               +      0
   0.0580   I *                       *        ***** ***                I      0
   0.0472   I *                       **    *****      ***              I      0
   0.0365   I**                      **  ****            **             I      0
   0.0258   I*                     ******                 **            I      0
   0.0151   I*                                             *            I      0
   0.0044   I*                                             **           I      0
  -0.0064   I*                                              **          I      0
  -0.0171   I                                                *          I      0
  -0.0278   I                                                **         I      0
  -0.0385   +                                                 **        +      0
  -0.0492   I                                                  *        I      0
  -0.0600   I                                                  **       I      0
  -0.0707   I                                                   *       I      0
  -0.0814   I                                                   *       I      0
  -0.0921   I                                                  **I             0
  -0.1028   I                                                  *I             0
  -0.1136   I                                                  *I             0
  -0.1243   I                                                   I             0
  -0.1350   I                                                   I             0
  -0.1457   +3------------+--------+--------+-----------+--+-------3+     11
            +-------+--------+--------+--------+--------+--------+--------
  LEFT      00000000000000000000000000000000000000000000000000000000000000
  BIN       ••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
  EDGE      0~0000011111112222222333333344444445555555666666677777788888889999999
            1346791246791246791246791246791246791246791246791246791246791124679
            306396396296295295285285185185184184174174074073073073063063
            306284062840639517395173952840628406284173951739517306284062
```

figure 2.3

MULTIDIMENSIONAL ADDITIVE SPLINE APPROXIMATION (4/19/80)
PARAMETERS FOR THIS RUN
```
NOBS            100
NPRED             2
MODE              2
MAXTRY            2
MAXPRO            1
PPCONV      .150000
MAXIT             2
KORDER            3
MAXKNO           11
BANFAC      1.60000
IPRINT            3
NPRINT            1
PLOTRM          .0
```
AVERAGE SQUARED RESIDUAL AROUND THE MEAN   .109703
3

```
SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR    1
  0.7088  -0.7054

PLOT STATISTICS =    0.    0.    0.   /   0.   30.   0.   /   0.   0,   0.
  PROJ                                                                        PROJ
  ON X                                                                        ON Y
  AXIS      300000010100000000101011010101030~111100010010100001000010003     AXIS
   0.2223   +---------------------------------------------------------------+
   0.2161   I************************                                        I      0
   0.2100   I                       **                                       I      0
   0.2039   I                        **                                      I      0
   0.1978   I                         *                                      I      0
   0.1917   I                         **                                     I      0
   0.1856   I                          *                                     I      0
   0.1795   I                          **                                    I      0
   0.1734   I                           *                                    I      0
   0.1672   I                           *                                    I      0
   0.1611   +                          **                                    +      0
   0.1550   I                           *                                    I      0
   0.1489   I                           *                                    I      0
   0.1428   I                           *                                    I      0
   0.1367   I                           *                                    I      0
   0.1306   I                          **                                    I      0
   0.1245   I                           *                                    I      0
   0.1183   I                           *                                    I      0
   0.1122   I                           *                                    I      0
   0.1061   I                          **                                    I      0
   0.1000   +                           *                                    +      0
   0.0939   I                           *                                    I      0
   0.0878   I                           *                                    I      0
   0.0817   I                           *                                    I      0
   0.0756   I                          **                                    I      0
   0.0695   I                           *                                    I      0
   0.0633   I                           *                                    I      0
   0.0572   I                           *                                    I      0
   0.0511   I                          **                                    I      0
   0.0450   I                           *                                    I      0
   0.0389   +                           *                                    +      0
   0.0328   I                          **                                    I      0
   0.0267   I                           *                                    I      0
   0.0206   I                          **                                    I      0
   0.0144   I                           *                                    I      0
   0.0083   I                          ***                                   I      0
   0.0022   I                            ***                                 I      0
  -0.0039   I                              ********************************  I      0
  -0.0100   I                                                                I      0
  -0.0161   I                                                                I      0
  -0.11222  +3     + +       + + ++ + + + 3 3++++    +++    +    +    3+     30
            +---------+---------+---------+---------+---------+---------+------
            --------e-----e-----------
  LEFT
  BIN       00000000000000000000000000000000000000000000000000000000000000000
  EDGE      .................................................................
            6655555544444333332222211111000000000011111222223333344445555 ···
            319753197531974208642086420753102468024691357913579136802468
            998766543321009877654432110988734556788901123445677789~012334
            8~3580358035803580358035702570253085308530853085318631863186
```
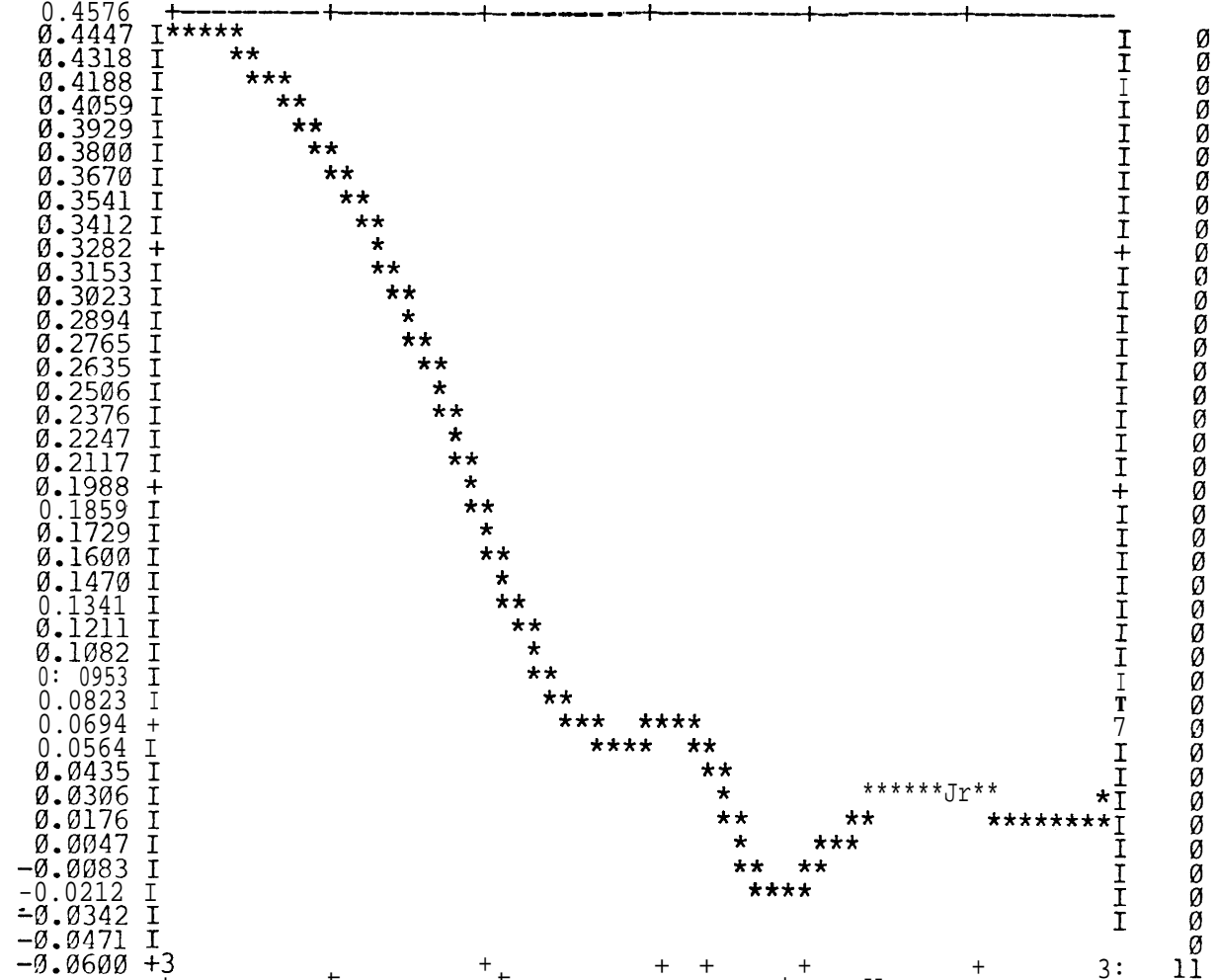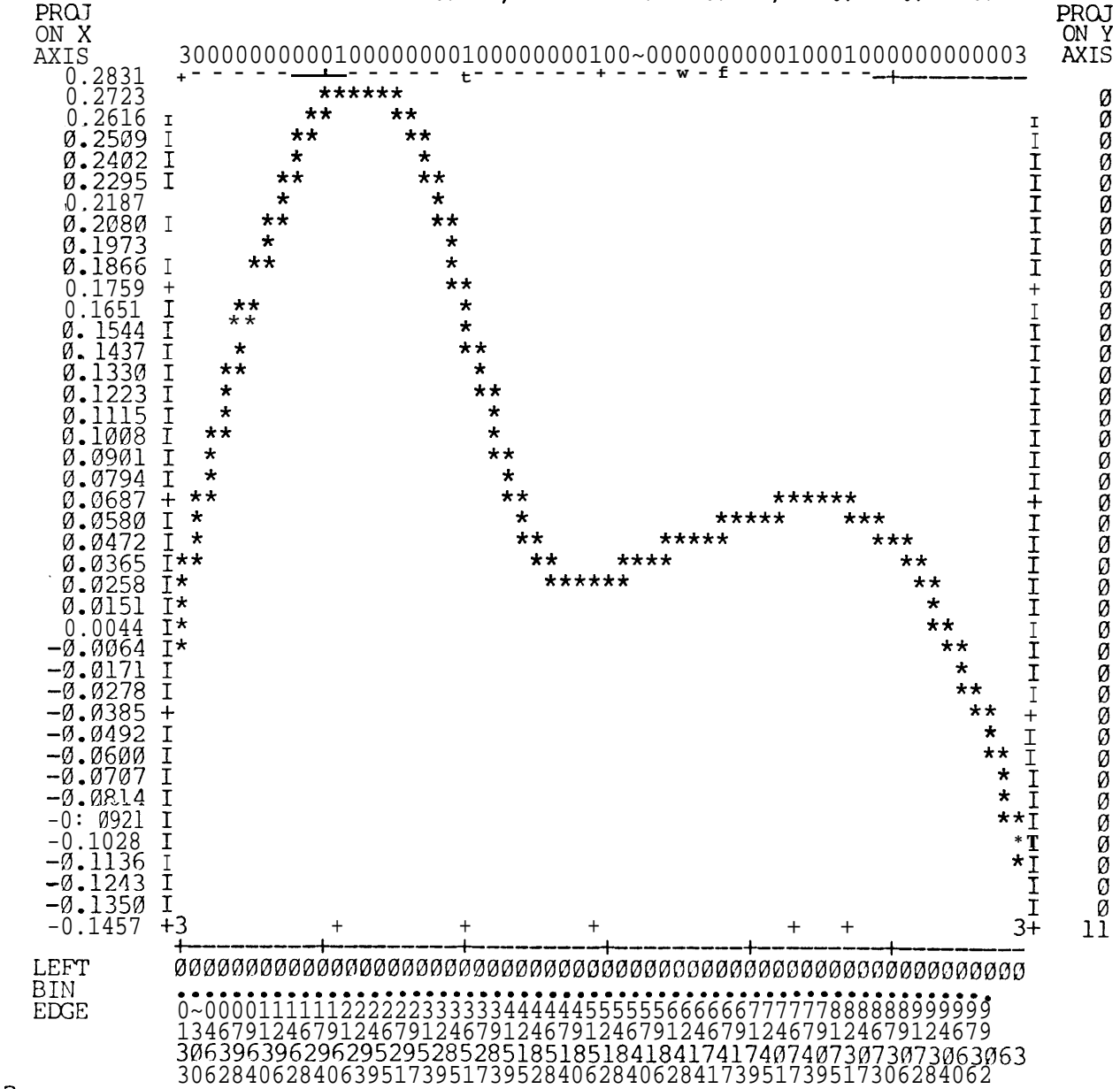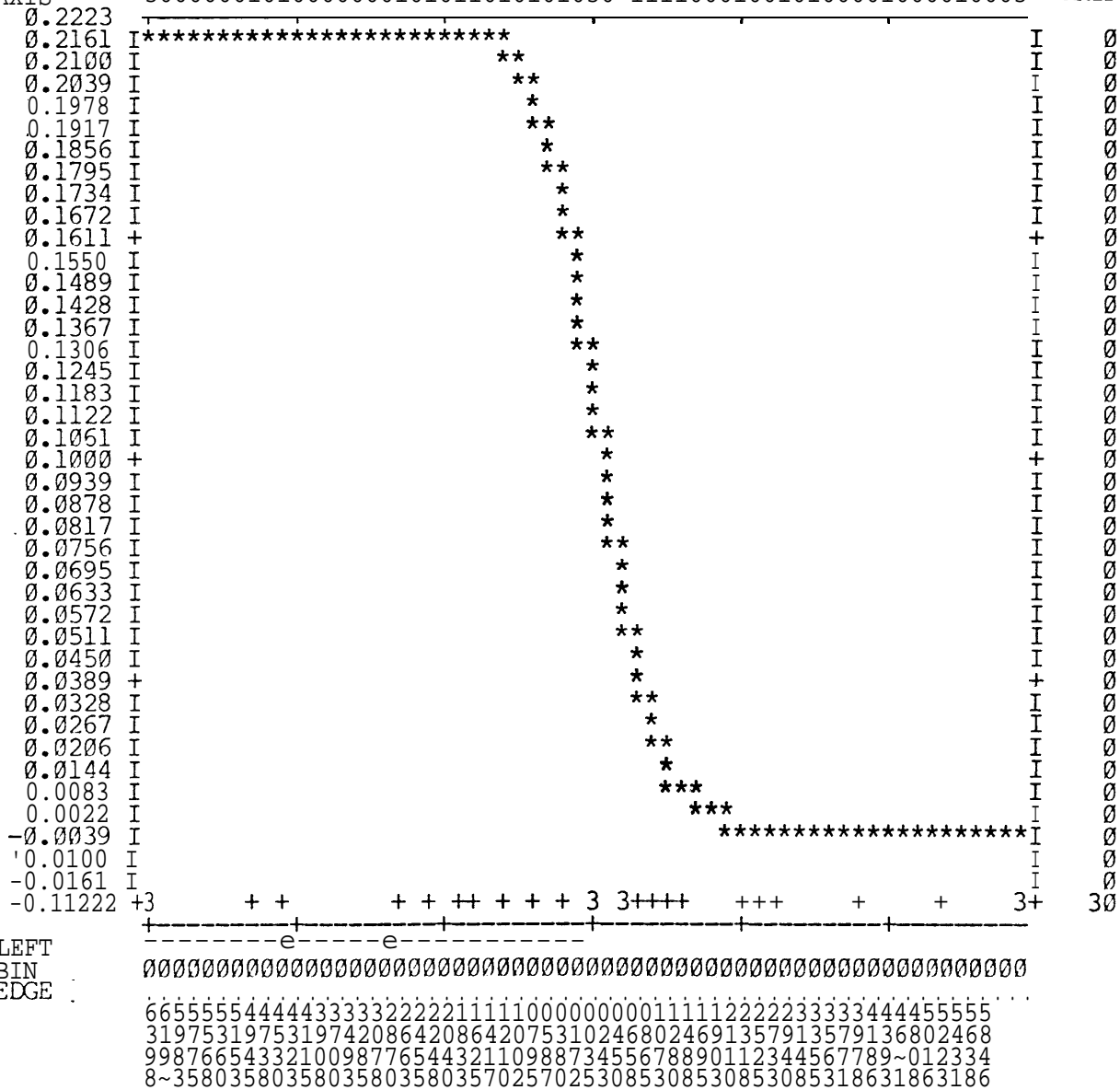
figure 3.1

MULTIDIMENSIONAL ADDITIVE SPLINE APPROXIMATION (4/19/80)
PARAMETERS FOR THIS RUN
NOBS   100
NPRED   2
MODE   2
MAXTRY   2
MAXPRO   1
PPCONV
MAXIT
KORDER   3
MAXKNO   30
BANFAC
IPRINT
NPRINT   1
PLOTRM   .0
AVERAGE SQUARED RESIDUAL AROUND THE MEAN   .972118E-02
?

SPLINE FUNCTION AND KNOTS ALONG DIRECTION NR 1
```
 -0.0025   0.0        0.7222  -0.2242  -0.3383 -0.5495    0.0351
 -0.0242   0.0        0.0556  -0.0384  -0.0141   0.0      0.0
  0.0     -0.0039    -0.0660  -0.0153  -0.0235
```
PLOT   STATISTICS = 0.   0.   0. / 0. 11.   0. / 0.   0.   0.

```
  PROJ                                                                      PROJ
  ON X                                                                      ON Y
  AXIS    40000000000000000000010000000000000000001000001000000000000000000004 AXIS
 29.8919 +---------+---------+---------+---------+---------+---------+
 29.6401 I*               *****                                            I   0
 29: 3884 I*             ***    **                                         I   0
 29.1366 I*            **       **                                         I   0
 28.8848 I**         **          *                                         I   0
 28.6331 I *        **          **                                         I   0
 28.3813 I **      *            *                                          I   0
 28.1295 I  **   ***           **                                         I   0
 27.8778 I   ** **            *                                           I   0
 27.6260 I   ****            *                                            I   0
 27. 3743 +                 **                                            +   0
 27.1225 I                  *                                             I   0
 26.8707 I                 **                                             I   0
 26.6190 I                 *                                              I   0
 26.3672 I                 *                                              I   0
 26.1154 I                **                                              I   0
 25.8637 I                *                                               I   0
 25.6119 I                *                                               I   0
 25.3601 I               **                                               I   0
 25.1084 I               *                                                I   0
 24.8566 +               *                                                +   0
 24.6048 I               **                                               I   0
 24.3531 I               *                                                I   0
 24.1013 I               *                                                I   0
 23.8496 I               **                                               I   0
 23.5978 I               *                                                I   0
 23.3460 I               *                                                I   0
 23.0943 I               **                                               I   0
 22.8425 I               *                                                I   0
 22.5907 I               **                                             I  0
 22.3390 +               *                                             *+   0
 22.0872 I               *                                            ***I   0
 21.8354 I               **              ********      ***             I   0
 21.5837 I                *         ***             *****              I   0
 21.3319 I               **       **                                   I   0
 21.0802 I              **      **                                     I   0
 20.8284 I             ** ***                                          I   0
 20.5766 I              ***                                            I   0
 20.3249 I                                                             I   0
 20.0731 I                                                             I   0
 19.8214 +4        +         +         +         +                   4+  11
```
```
         +---------+---------+---------+---------+---------+---------+

 LEFT    =========================================================
 BIN     11111111111111111111111111100000000000000000000000000000000000000
 EDGE
         111110~0000000000000000000~0099999999999999999999998888888888888
         2110099887766554433221 1~099887776655443322110099887766554 4
         272838394940505161627273 8384949505 0 6 1617 2 7 2838394950 506161 72
         2570258035803580368136813681469146914692479247925702257025703
```
?

```
MULTIDIMENSIONAL ADDITIVE SPLINE APPROXIMATION (4/19/80)
PARAMETERS FOR THIS RUN
NOBS           200
NPRED           19
MODE             2
MAXTRY           2
MAXPRO           2
PPCONV      150000
MAXIT        .   2
KORDER           4
MAXKNO          11
BANFAC     1.50000
IPRINT           3
NPRINT           i
PLOTRM         .0
AVERAGE SQUARED RESIDUAL AROUND THE MEAN    13.2975
?
```