Stanford Department of Computer Science
Report No. STAN-CS-80-807

June 1980

PATH-REGULAR GRAPHS

by

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Research sponsored by

National Science Foundation
and
Office of Naval Research

COMPUTER SCIENCE DEPARTMENT
Stanford University

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Abstract: A graph is vertex-[edge-]path-regular if a list of shortest paths, allowing multiple copies of paths, exists where every pair of vertices are the endvertices of the same number of paths and each vertex [edge] occurs in the same number of paths of the list. The dependencies and independencies between the various path-regularity, regularity of degree, and symmetry properties are investigated. We show that every connected vertex-[edge-]symmetric graph is vertex-[edge-]path-regular, but not conversely. We show that the product of any two vertex-path-regular graphs is vertex-path-regular but not conversely, and the iterated product $G \times G \times \cdots \times G$ is edge-path-regular if and only if G is edge-path-regular. An interpretation of path-regular graphs is given regarding the efficient design of concurrent communication networks.

Keywords: concurrent network flow, product graphs, regularity, shortest paths, symmetry.

*/ This research was supported in part by National Science Foundation MCS-77-23738 and by Office of Naval Research contract N00014-76-C-0330. Reproduction in whole or in part is permitted for any purpose of the United States government.

**/ This research was supported in part by a Chaim Weizmann Postdoctoral Fellowship.

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I. Introduction and Summary.

A combinatorial regularity property of a graph is expressed by a numerical requirement on the consistency of structure within the graph. The standard property that a graph is regular of degree k , requiring simply that each vertex be adjacent to exactly k other vertices, has received the most thorough investigation in the literature. The more stringent conditions of "strongly regular" [C78] and "distance-regular" [B74] have also received considerable treatment. In this paper we characterize and investigate the regularity of connectivity that can exist between all pairs of vertices concurrently. This regularity is realized by the identification of equitable numbers of shortest paths between all pairs of vertices that at the same time make equitable use of either each vertex, each edge, or both. To develop the concept of path-regularity we explicitly specify our terminology for describing shortest paths in a graph. Other graph theoretic terms not defined here may be found in Harary [H69].

For $n \geq 0$, the sequence $v_0, v_1, \ldots, v_n$ of distinct vertices of the graph G , where $v_i v_{i+1}$ is an edge of G for all $0 \leq i \leq n-1$, shall denote a path of length n . Paths are assumed unordered, so $v_0, v_1, \ldots, v_n$ and $v_n, v_{n-1}, \ldots, v_0$ denote the same path. Vertices v_0 and v_n are endvertices of the path $v_0, v_1, \ldots, v_n$, with all other vertices of the path then being interior vertices. The path $v_0, v_1, \ldots, v_n$ is a shortest path of G whenever any other path with endvertices v_0 and v_n has length at least n , with $d(v_0, v_n) = n$ then denoting the distance between v_0 and v_n . All paths of length at least one have a distinct pair of endvertices. The path v_0 of length zero has the single endvertex v_0, and is also said to have the nondistinct pair v_0, v_0 of endvertices. Hence a connected graph may be taken to contain shortest paths between every (unordered distinct or nondistinct) pair of vertices.

For the complete bipartite graph $K_{1,5}$ of Figure 1(a), every pair of vertices are the endvertices of a unique shortest path. A list of all the resulting shortest paths would then have each edge, but not each vertex, occur in the same number (5) of paths of the list, providing a concept of regularity for the shortest paths versus the edges of $K_{1,5}$. The list of all shortest paths for the cycle C_5 of Figure 1(b) would then have each vertex occur in the same number (6) of shortest paths and each edge occur in the same number (3) of shortest paths, yielding a stronger concept of regularity encompassing the shortest paths, vertices, and edges of C_5. If we allow multiple copies of shortest paths of the graph K_3 K_2 of Figure 1(c) in composing a path list, then it is possible to exhibit a list of shortest paths of $K_3 \times K_2$ where each pair of vertices $v, v' \in V(K_3 \times K_2)$ are the endvertices of the same number of shortest paths of the list and where each vertex, but not each edge, is in the same number of shortest paths of the list. This then provides a concept of regularity for shortest paths versus vertices of $K_3 \times K_2$.

Formally, let a *list* (equivalently multiset) denote a finite collection of elements where multiple copies of each element may occur in the list. A nontrivial graph G is termed *vertex-path-regular* [respectively, *edge-path-regular*] with parameters (k, m_v) [respectively (k, m_e)] if an *associated list* $\mathcal{L}$ of shortest paths of G exists where every pair of vertices are the endvertices of exactly $k > 1$ paths of $\mathcal{L}$

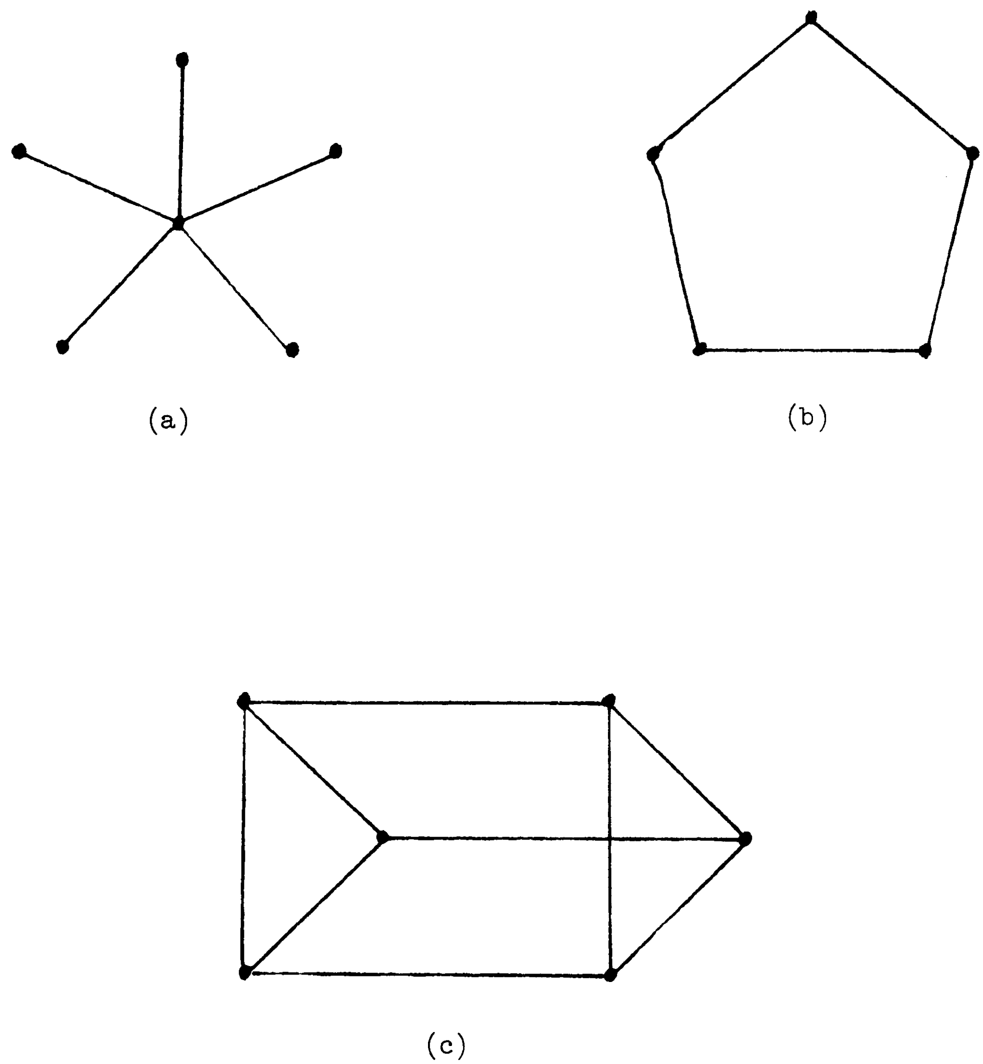

Figure 1. Examples of (a) edge-path-regular,
(b) strongly path-regular, and
(c) vertex-path-regular graphs.

and each vertex [respectively, edge] occurs in exactly m_v [respectively, m_e] paths of $\mathcal{L}$. A graph is strongly path-regular with parameters (k,m_v,m_e) and associated list $\mathcal{L}$ of shortest paths if it is both vertex-path-regular with parameters (k, m_v) and edge-path-regular with parameters (k,m_e) for the same associated list $\mathcal{L}$. For completeness the trivial graph is taken to be vertex-, edge-, and strongly path-regular with parameters (k,k) , (k,k) and (k,k,k) , respectively, for every $k \geq 1$. A graph is said to be vertex-, edge-, or strongly path-regular whenever there exist some parameters for which the graph has the specified path-regularity property, and a graph is said to be path-regular if it is at least either vertex- or edge-path-regular.

From our preceeding discussion it is then clear that $K_{1,5}$ of Figure 1(a) is edge-path-regular with parameters $(1,5)$, and C_5 of Figure 1(b) is strongly path-regular with parameters $(1,6,3)$. For $K_3 \times K_2$ of Figure 1(c), consider the list $\mathcal{L}$ containing two copies of every shortest path of length at most one in $K_3 \times K_2$ and one copy of every shortest path of length two. By direct application of the definition this list $\mathcal{L}$ is then sufficient to confirm that $K_3 \times K_2$ is vertex-path-regular with parameters $(2,14)$.

Note that if G is vertex- or edge-path-regular with parameters (k,m_v) or (k,m_e) , respectively, or strongly path-regular with parameters (k,m_v,m_e) , then the associated list must have each vertex of G present as a path of length zero with multiplicity k and each adjacent pair of vertices of G present as a path of length one with multiplicity k . Thus it is sufficient to show that each edge occurs in exactly (m_e-k) paths of length ≥ 2 of the list to confirm the edge-path-regularity property, and/or that each vertex occurs as an interior vertex in exactly $m_v-k|V(G)|$ paths of the list to confirm the vertex-path-regularity property. Consider the wheel W_5 of Figure 2. Giving multiplicity 2 to the shortest paths of length two with interior vertex v_1 and multiplicity 1 to the other shortest paths of length two, we note that each edge then appears in the same number (2) of these paths. Hence W_5 is edge-path-regular with parameters $(4,6)$. Alternatively, giving multiplicity 1 to the same paths containing v_1 and multiplicity 2 to the other shortest paths of length two allows us to confirm that W_5 is vertex-path-regular with parameters $(5,27)$. The graph W_5 is not regular (of degree), so by Theorem 1 of the next section can not be strongly path-regular. Hence the example W_5 demonstrates that a graph can be both vertex-path-regular and edge-path-regular without being strongly path-regular.

Path-regular graphs may be visualized as providing efficient design of communications networks, Let the vertices of an edge-path-regular graph with parameters (k,m_e) represent communication bases in the network and the edges trunk lines each capable of hosting m_e channels of concurrent communication. The edge-path-regularity property then allows for k dedicated communication channels to be provided between every pair of bases concurrently. Furthermore, the channel allocation is efficient both in that all dedicated channels follow shortest paths and that every trunk line is used to full capacity. If the constraint in a communication network is alternatively related to a fixed level of switching capacity at every communication base, then the vertex-path-regular graphs indicate efficient network design. The associated lists

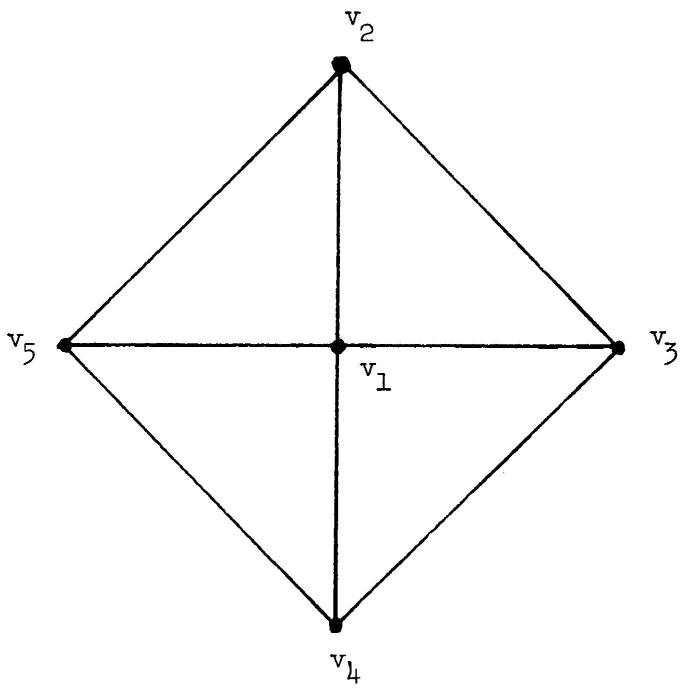

Figure 2. The wheel W_5 which is vertex-path-regular with parameters (5,27) and edge-path-regular with parameters (4,6), yet not strongly path-regular.

of the path-regular graphs, as specified for the examples of Figures 1 and 2, then provide the dedicated channels for such a communication network interpretation. This concurrent communication interpretation provides some motivation and an intuitive appeal to many of our derived results but is not explicitly mentioned in the balance of the paper.

The example graphs of Figures 1 and 2 all possessed considerable symmetry that was instrumental in the demonstration of the respective path-regularity properties of these graphs. As succinctly noted by Biggs in his book Algebraic Graph Theory [B74], "A symmetry property of a graph is related to the existence of automorphisms -- that is, permutations of the vertices which preserve adjacency. A regularity property is defined in purely numerical terms. Consequently, symmetry properties induce regularity properties, but the converse is not necessarily true."

In Section II we investigate the dependencies and independencies between the various regularity, path-regularity, and symmetry properties. Cur main result is in accord with the preceeding observation on regularity and symmetry properties. Specifically, we show that:

(i) a connected vertex-symmetric*/ graph is vertex-path-regular but not conversely,

(ii) a connected edge-symmetric**/ graph is edge-path-regular but not conversely, and

(iii) a connected graph that is both vertex- and edge-symmetric is strongly path-regular but not conversely.

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*/ Also termed vertex-transitive by some authors.

**/ Also termed edge-transitive by some authors, These results insure that many important classes of graphs have a path-regularity property. Cycles, cubes and regular complete k-partite graphs are strongly path-regular, and any complete bipartite graph is edge-path-regular. We also indicate in Section II the considerable extent to which the vertex-, edge-, and strongly path-regular properties are independent of other graph properties and parameter values.

The fact that a graph is vertex- or edge-path-regular does not determine the parameters (k,m_v) or (k,m_e) uniquely, but it uniquely determines their ratio. Hence we define $\sigma(G) = k/m_v$ as the <u>vertex-path-regularity</u> of the vertex-path-regular graph G and $\rho(G) = k/m_e$ as the <u>edge-path-regularity</u> of the edge-path-regular graph G . In Section III we obtain the following formulas for evaluating a(G) and $\rho(G)$:

For any vertex-path-regular graph G with n vertices and ℓ edges,

$$\sigma(G) = \begin{cases} \dfrac{n}{\sum\limits_{v,v' \in V(G)} [d(v,v')+1]} & \text{for } G \text{ of any diameter,} \\[2ex] \dfrac{n}{3n^{2} - n - 2\ell} & \text{for } G \text{ of diameter } 2, \end{cases}$$

and for any edge-path-regular graph G with n vertices and $\ell > 1$ edges,

$$\rho(G) = \begin{cases} \dfrac{a}{\sum\limits_{v,v' \in V(G)} d(v,v')} & \text{for } G \text{ of any diameter} \\[2ex] \dfrac{a}{n(n-1)-\ell} & \text{if } G \text{ has diameter } \le 2 \text{ .} \end{cases}$$

A table of values of $a(G)$ and $\rho(G)$ is then provided for the major classes of path-regular graphs. In Section III we also derive some nontrivial necessary conditions for a graph to be vertex- and/or edge-path-regular involving inequalities between the relative size of the cuts and separating sets of the graph and the required values for σ and ρ from the preceeding formulas.

There is an intimate relation between shortest paths in the product graph $G \times H$ and the shortest paths of G and H . This relation is exploited in Section IV to obtain our major results on the products of path-regular graphs:

(i) The product*/ graph $G \times H$ is vertex-path-regular whenever G and H are both vertex-path-regular, but not conversely, and

(ii) the product graph $G \times H$ is edge-path-regular if and only if G and H are both edge-path-regular with $|V(G)|\rho(G) = |V(H)|\rho(H)$, where specifically $G \times G \times \ldots \times G$ is edge-path-regular if and only if G is edge-path-regular.

Finally, in Section V, we propose and discuss several interesting open questions that arose in our investigation of path-regular graphs, of which the most intriguing to us is the following: Is there an edge-path-regular graph G with $p(G) = r$ for every nonzero rational r in the unit interval?

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*/ Also termed the Cartesian product graph, The product graph is defined in Section IV.

II. Regularity, Path-Regularity and Symmetry.

The primary goal of this section is to determine the dependencies and independencies between the various regularity, path-regularity, and symmetry properties. Our first theorem provides some affirmative implications between regularity (of degree) and path-regularity properties. Although the wheel W_5 of Figure 2 illustrates that a graph can be vertex-path-regular and/or edge-path-regular without being regular, Theorem 1 demonstrates that a strongly path-regular graph must be regular. And conversely, although the property that the connected graph G be regular of degree k is not by itself sufficient to induce either the vertex- or edge-path-regularity property for G , the property that the connected graph G be strongly regular is sufficient to make G strongly path-regular. Note that G is strongly regular with parameters (i_1,i_2,i_3) whenever G is regular of degree i_1 , where also any two adjacent vertices have exactly i_2 common neighbors, and any two nonadjacent vertices have exactly i_3 common neighbors.

Theorem 1. A strongly path-regular graph with parameters (k,m_v,m_e) and n vertices is regular of degree $(2m_v - kn - k)/m_e$. On the other hand, any strongly regular graph with parameters $(i_1, i_2, i_3 \geq 1)$ and $n > 2$ vertices is strongly path-regular with parameters (k,m_v,m_e) where $k = i_3$, $m_v = ni_3 + i_1(i_1-i_2-1)/2$, and $m_e = i_3 + 2(i_1-i_2-1)$.

Proof. Let the n vertex graph G be strongly path-regular with parameters (k,m_v,m_e) , where $\mathcal{L}$ is the associated list of paths, Any specific vertex v of G will occur as an endvertex in k(n-1) paths of length at least one in $\mathcal{L}$, and each of these paths will contain exactly one edge incident to v. Also, v will occur as an interior vertex in $m_v - kn$ paths of $\mathcal{L}$, where each of these paths will contain exactly two edges incident to v. Thus the total number of occurences of edges incident to v in all paths of $\mathcal{L}$ is $2m_v - kn - k$. But the total number of occurences of edges incident to v in all paths of $\mathcal{L}$ is also given by $m_e \times \text{degree}(v)$ since each edge of G occurs in m_e paths of $\mathcal{L}$. Therefore $\text{degree}(v) = (2m_v - kn - k)/m_e$ for any v in G.

For the second part of the theorem let the graph $G = (V,E)$ have $n > 2$ vertices and be strongly regular with parameters $(i_1, i_2, i_3 \geq 1)$. Let the list $\mathcal{L}$ contain i_3 copies of the zero length path v for every $v \in V$, i_3 copies of the path v, w for each edge $vw \in E$, and one copy of the path u, v, w for every nonadjacent pair of distinct vertices $u, w \in V$ and every distinct v adjacent to both u and w. The fact that every two nonadjacent vertices of G have i_3 common neighbors implies that $\mathcal{L}$ contains $k = i_3 \geq 1$ shortest paths between every pair of vertices of G. Any edge $vw \in E$ occurs in i_3 paths of length one of $\mathcal{L}$. Noting that there are $i_1 - i_2 - 1$ vertices other than v adjacent to w and not to v and also $i_1 - i_2 - 1$ vertices other than w adjacent to v and not to w, the edge vw also occurs in $2(i_1 - i_2 - 1)$ of the paths of length two of $\mathcal{L}$, so in total in $m_e = i_3 + 2(i_1 - i_2 - 1)$ paths of $\mathcal{L}$. Every vertex $v \in V$ will occur as an endvertex in ni_3 paths of $\mathcal{L}$ and as the mid-vertex of $i_1(i_1 - i_2 - 1)/2$ paths of length two of $\mathcal{L}$, so in total in $ni_3 + i_1(i_1 - i_2 - 1)/2$ paths of $\mathcal{L}$. Hence G is strongly path-regular with the associated list $\mathcal{L}$. $\square$

As another partial converse to the first part of Theorem 1 we now derive the following lemma which will be employed in the subsequent theorem.

Lemma 2. Every graph which is both regular and edge-path-regular is strongly path-regular.

Proof. Let the n vertex graph G be regular of degree j and edge-path-regular with parameters (k,m_e), where $\mathfrak{L}$ is the associated list of paths. For any vertex v of G, there are jm_e occurences of edges incident to v in the paths of $\mathfrak{L}$. A total of k(n-1) of the jm_e such occurences correspond to v being an en&vertex, the remainder corresponding to v being an interior **vertex** of the paths. Each occurence of v as an interior vertex of a path involves exactly two occurences of edges incident to v in that path, so v must occur as an interior vertex in $[jm_e - k(n-1)]/2$ paths of $\mathfrak{L}$. Hence each vertex v of G occurs in $[jm_e + k(n-1)]/2$ paths of $\mathfrak{L}$, so G is strongly path-regular with parameters $(k, [jm_e + k(n-1)]/2, m_e)$. ❑

As previously noted, the symmetries characterized by the automorphisms of a graph induce extensive numerical regularity properties, although the converse implications generally do not hold. In accord with this maxim, the standard vertex and edge symmetry properties of graphs are now shown to induce the corresponding vertex- and edge-path-regularity properties while the converses are shown to fail by counterexamples.

Theorem 3.

(i) Every connected vertex-symmetric graph is vertex-path-regular, but not conversely;

(ii) every connected edge-symmetric graph is edge-path regular, but not conversely;

(iii) every connected graph that is both vertex- and edge-symmetric is strongly path-regular. However, there exist strongly path-regular graphs that are, respectively, not vertex-symmetric and not edge-symmetric.

Proof. Let $G = (V,E)$ be connected and either vertex-symmetric or edge-symmetric or both. Let $k(u,v)$ be the number of distinct shortest paths between u and v in G, and let $k^* = \mathrm{lcm}\{k(u,v) \mid u,v \in V\}$. Let the list $\mathcal{L}$ contain $k^*/k(u,v)$ copies of each distinct shortest path between u and v for all pairs of vertices of V, so then every pair $u,v \in V$ are the endvertices of k^* paths of $\mathcal{L}$.

Assume G is vertex-symmetric. For each $v \in V$, form the sublist $\mathcal{L}_v$ composed of all paths of $\mathcal{L}$ containing the vertex v. For any $v,u \in V$, the assumption that G is vertex-symmetric means there exists an automorphism α mapping v into u. Now any path p of $\mathcal{L}_v$ is mapped by α to a path, $\alpha(p)$, containing the vertex u where $\alpha(p)$ is also a shortest path between its endvertices in G, so $\alpha(p)$ is in $\mathcal{L}_u$. Furthermore, each distinct shortest path between the endvertices of the path p is mapped by α into a distinct shortest path between the endvertices of the path $\alpha(p)$ and vice-versa for the inverse automorphism α^{-1}. Thus p has the same multiplicity in $\mathcal{L}_v$ as $\alpha(p)$

has in $\mathfrak{L}_u$ so $|\mathfrak{L}_v| \leq |\mathfrak{L}_u|$. Since α^{-1} is an automorphism mapping u into v, $|\mathfrak{L}_v| = |\mathfrak{L}_u|$, and G is vertex-path-regular verifying (i).

Now assume G is edge-symmetric and for each edge $e \in E$ form the sublist $\mathfrak{L}_e$ composed of all paths of $\mathfrak{L}$ containing the edge e. For any two edges $e, e' \in E$, the assumption that G is edge-symmetric means there exists an automorphism α which maps edge e into edge e'. By the same argument as preceeding we then obtain that $|\mathfrak{L}_e| = |\mathfrak{L}_{e'}|$ for any edges $e, e' \in E$, so G is edge-path-regular verifying (ii). Noting that the same list $\mathfrak{L}$ was utilized in the proofs of both (i) and (ii) then verifies (iii).

To show none of the converses hold first consider the wheel W_5 of Figure 2. W_5 is neither vertex- nor edge-symmetric, yet it is both vertex- and edge-path-regular, demonstrating that neither the converse of (i) nor (ii) hold.

For counterexamples to the converse of (iii) first note that Folkman [F67; CM78, p. 95] has demonstrated the existence of a regular graph which is edge-symmetric but not vertex-symmetric. By part (ii) of this theorem and Lemma 2 such a graph is then strongly path-regular without being vertex-symmetric, To demonstrate that a strongly path-regular graph need not be edge-symmetric, consider the graph C_5+C_5 composed of two distinct chordless five cycles along with all edges between vertices of these distinct five cycles. The list containing each path of G of length zero or one with multiplicity 7, each path of length two in a chordless five cycle having multiplicity 2, and each path of length two with nonadjacent endvertices in one chordless five cycle and midvertex in the other five cycle having multiplicity 1, is sufficient to confirm that C_5+C_5 is strongly path-regular with parameters (7,77,11) . Although C_5+C_5 is clearly vertex-symmetric, it is not edge-symmetric since some edges are in chordless five cycles and others are not, completing the theorem. □

From Theorem 3 it follows that the class of strongly path-regular graphs is quite broad, including all cycles, complete graphs, regular complete k-partite graphs, and the cubes of every dimension, Also all complete bipartite graphs are edge-path-regular. As might be expected, the condition that a graph be vertex-, edge-, or strongly path-regular is quite independent of most other typical parameter values and/or properties associated with a graph, a partial summary of which is noted in the following.

Corollary 3.1. There exist strongly path-regular graphs of any specified girth, or of any specified diameter, or of any specified edge or vertex connectivity, or of any specified chromatic number.

Proof. The cycle C_n is strongly path-regular of girth n and diameter $\lfloor n/2 \rfloor$, thus realizing any specified girth or diameter, The complete graph K_{n+1} , regular complete bipartite graph $K_{n,n}$, and n dimensional cube are all examples of strongly path-regular graphs of edge and vertex connectivity n . The complete graph K_n and any regular complete n-partite graph have chromatic number n . □

Two properties of a graph will be termed <u>independent properties</u> if there are examples of graphs exhibiting all four possible cases: (a) having both properties, (b) having each specified property without the other, and (c) having neither property. Figure 3 provides examples showing that the property that a regular graph be strongly path-regular is independent of the property that a graph be either (i) Hamiltonian, or (ii) Eulerian, or (iii) planar. Verification that the graphs of Figure 3 satisfy the respective properties is straightforward from standard results in the literature regarding these properties. To confirm that the cited example graphs are not strongly path-regular, consider the following: Every edge of an n vertex graph, other than K_n, that is edge-path-regular with parameters (k,m_e) must have each edge occur in $m_e-k \geq 1$ paths of length at least two in the associated list. Alternatively:

<u>Observation.</u> If G is a graph other than a complete graph where some edge of G does not occur in any shortest path between any nonadjacent endvertices in G, then G is not edge-path-regular, hence also not strongly path-regular.

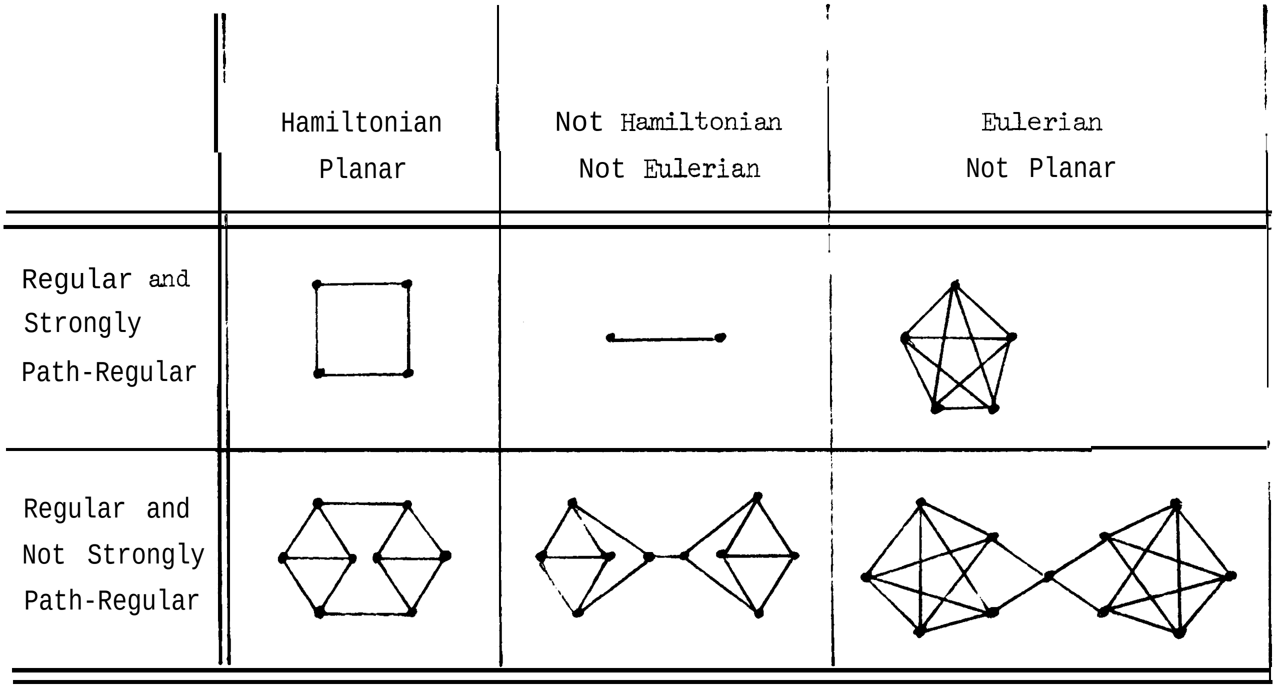


Figure 3. Graphs showing that the property that a connected regular graph be strongly path-regular is indenpendent of the properties that a graph be either (i) Hamiltonian, or (ii) Eulerian, or (iii) planar.

Now let us return to the primary theme of this section which is to describe the dependencies and independencies that exist between the various regularity, path-regularity, and symmetry properties. In Figure 4 and the following corollaries we describe the extent to which the vertex-path-regularity and edge-path-regularity properties are distinct and independent of other regularity and symmetry properties,

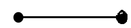
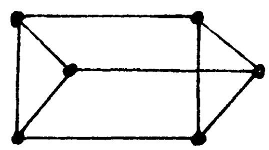
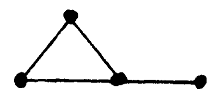
	Vertex-symmetric (⇒ { vertex-path-regular and regular })	Not vertex-path-regular Not regular (⇒ Not vertex-symmetric
Edge-symmetric (⇒ edge-path-regular)		
Not edge-path-regular (⇒ Not edge-symmetric)		

Figure 4. An indication of the independence of regularity, path-regularity, and symmetry properties.

Corollary 3.2. The property that a graph be edge-path-regular is independent of the property that a graph be (i) vertex-symmetric, or (ii) vertex-path-regular, or (iii) regular.

Proof. All possible cases are covered by the examples of Figure 4. Three of the four example graphs are immediately seen to have the indicated properties. The other graph, $K_3 \times K_2$, is the classic example of a graph that is vertex- but not edge-symmetric, and we need only show that it is not edge-path-regular. From the theorems proved in Section IV it follows that $K_i \times K_j$ is vertex symmetric but not edge-path-regular for any $i > j > 2$. We include a separate proof for $K_3 \times K_2$ to keep this section self-contained.

Let the six edges of $K_3 \times K_2$ that are in triangles be type A edges and the other three be type B edges. Note that every shortest path of length two in $K_3 \times K_2$ uses one type A and one type B edge, so any list of shortest paths in which every pair of vertices of $K_3 \times K_2$ are the endvertices of the same number of shortest paths can not have each edge occur in the same number of paths. □

Corollary 3.3. The property that a graph be vertex-path-regular is independent of the property that a graph be (i) edge-symmetric, or (ii) edge-path-regular, or (iii) regular.

Proof. All cases for (i) and (ii) are confirmed by the examples of Figure 4. To show that the property of being vertex-path-regular is independent of the property of being regular, note that K_2 has both properties, $K_{1,2}$ has neither property, and the wheel W_5 of Figure 2 is vertex-path-regular but not regular. Finally the regular graph of Figure 3 (lower right corner) that is Eulerian and not planar and not strongly path-regular is readily seen not to be vertex-path-regular as the separating vertex would have to be an interior vertex of too many paths. □

III. Evaluation of Path-Regularity.

Although knowledge that a graph G is either vertex- or edge-path-regular is not sufficient to determine the parameters (k,m), it is now shown to be sufficient to determine their ratio k/m. The class of vertex- and edge-path-regular graphs of diameter two are of special importance and the ratio k/m takes on a particularly simple formulation in that case.

Theorem 4. Let G be a vertex-path-regular graph with parameters (k, m_v) where G has vertex set $\{v_1, v_2, \ldots, v_n\}$ and ℓ edges. The vertex-path-regularity c(G) is then given by

$$O(G) = \frac{k}{m_v} = \begin{cases} \dfrac{n}{\sum\limits_{i \le j} [d(v_i, v_j)+1]} & \text{for } G \text{ of any diameter,} \quad (1) \\[2ex] \dfrac{2n}{3n^2 - n - 2\ell} & \text{for } G \text{ of diameter } < 2, \quad (2) \end{cases}$$

where $d(v_i, v_j)$ denotes the distance between v_i and v_j.

Proof. Let G be a vertex-path-regular graph with parameters (k, m_v) and associated list $\mathcal{L}$ of shortest paths. The total number of vertices in all paths of $\mathcal{L}$ is given by $k \sum\limits_{i<j} [d(v_i,v_j)+1]$ since each pair of vertices v_i, v_j are the endvertices of k paths of length $d(v_i,v_j)$ where each such path contains $d(v_i,v_j)+1$ vertices. But the total number of vertices in all paths of $\mathcal{L}$ is also given by nm_v since each vertex occurs in m_v paths of $\mathcal{L}$. Thus $k \sum\limits_{i<j} [d(v_i,v_j)+1] = nm_v$, verifying formula (1).

When G has diameter at most two, $\mathcal{L}$ must then contain exactly kn paths of length zero, $k\ell$ paths of length one, with the remaining $k[n(n-1)/2 - \ell]$ paths of length two, yielding formula (2). □

An analogous result is now stated for edge-path-regular graphs, where the proof is immediate by the same arguments utilized in the preceeding theorem.

Theorem 5. Let G be an edge-path-regular graph with parameters (k,m_e) where G has vertex set $\{v_1,v_2,\ldots,v_n\}$ and $\ell \geq 1$ edges. The edge-path-regularity $\rho(G)$ is then given by

$$\rho(G) = \frac{k}{m_e} = \begin{cases} \dfrac{\ell}{\sum_{i\leq j} d(v_i,v_j)} & \text{for G of any diameter,} \quad (3) \\[2ex] \dfrac{\ell}{n(n-1) - \ell} & \text{for G of diameter} \leq 2 . \quad (4) \end{cases}$$

From (1) and (3) we then obtain:

Corollary 5.1. For any strongly path-regular graph with n vertices and $\ell \geq 1$ edges,

$$\frac{n}{\sigma(G)} = \frac{\ell}{\rho(G)} + \frac{1}{2} n(n+1) . \quad (5)$$

Formulas (1)-(4) allow for straightforward computation of $\sigma(G)$ and $\rho(G)$ when G is known to be vertex- and/or edge-path-regular. Complete graphs, cycles, regular complete j-partite graphs, and the cubes of all dimensions are known to be strongly path-regular from the results of Section II, and the values of σ and ρ for these graphs

Classes of Graphs		vertex-path-regularity σ	edge-path-regularity ρ
Complete: K_n		$\frac{1}{n}$	1
Cycle: C_n	n even, $n \geq 4$	$\frac{8}{(n+2)^2}$	$\frac{8}{n^2}$
	n odd, $n \geq 3$	$\frac{8}{(n+2)^2-1}$	$\frac{8}{n^2-1}$
Regular Complete j -partite: $K_{\underbrace{i,i,\ldots,i}_{j}}$		$\frac{2}{2ji+i-1}$	$\frac{ji-i}{ji+i-2}$
j -dimension Cube: Q_j		$\frac{2}{(j+2)2^{j-1}+1}$	$\frac{1}{2^{j-1}}$
Product of Complete Graphs: $K_i \times K_j$		$\frac{2}{3ij-i-j+1}$	Not edge-path-regular for $i > j > 2$
Complete Bi-partite: $K_{i,j}$		Not vertex-path-regular for $i \neq j$	$\frac{ij}{i^2+j^2+ij-i-j}$

Table 1. Values of the vertex-path-regularity and edge-path-regularity for several important classes of path-regular graphs.

are tabulated in Table 1. The product graph $K_i \times K_j$ is vertex symmetric, hence vertex-path-regular. The value of σ for $K_i \times K_j$ along with the value of ρ for the edge-path-regular complete bipartite graph $K_{i,j}$ are also given in Table 1. The relation between σ and ρ given by (5) is seen to hold for the four classes of strongly path-regular graphs in Table 1. The fact that $K_i \times K_j$ is not edge-path-regular for $i > j \geq 2$ follows from Theorem 8 of Section IV. It is also noted in Table 1 that $K_{i,j}$ is not vertex-path-regular for $i \neq j$. For this fact consider that in any list having the same number of shortest paths between all pairs of vertices of $K_{i,j}$ for $i > j$, the number of times a vertex occurs as an interior vertex of a path of the list is greater for vertices of the j membered set than for the i membered set.

Utilization of formulas (1)-(4) as in Table 1 requires that we first know that the graphs have the corresponding path-regularity property. A test to determine if a particular graph is vertex- and/or edge-path-regular can be developed utilizing the computational procedure of linear programming. Such a test to determine if a graph is edge-path-regular is outlined in the following.

A Test for Edge-Path-Regularity of G.

Let $P = \{p_1, p_2, \ldots, p_j\}$ be the set of all shortest paths of the graph G. Assign nonnegative weights x_i to the paths of P such that:

(i) the sum of the weights x_i for all paths of P between the endvertices $v, v' \in V(G)$ is unity for every pair $v, v' \in V(G)$,

(ii) the sum of the weights on the paths containing the edge $e \in E(G)$ is less than or equal to z for every $e \in E(G)$, and

(iii) z_{min} is the minimum value of z satisfying the constraints of (i), (ii), where z_{min} can be found efficiently by linear programming techniques. From Theorem 5 we then obtain:

(a) if $\frac{1}{z_{min}} \neq \frac{|E(G)|}{\sum_{v,v' \in V(G)} d(v,v')}$, then G is not edge-path-regular,

(b) if $\frac{1}{z_{min}} = \frac{|E(G)|}{\sum_{v,v' \in V(G)} d(v,v')}$, then G is edge-path-regular, where z_{min} is the value of $1/\rho(G)$ and integral parameters (k, m_e) can be found by rationalizing the fractional values of x_i that are obtained (rational solution values for x_i are guaranteed for such a linear program).

A test for vertex-path-regularity is readily obtained by an analogous linear program utilizing formula (1) for the test criteria.

Although such tests can be reasonably efficient when the number of shortest paths is not prohibitive (e.g. when diameter(G) = 2), they can become computationally intractable. Furthermore, they do not readily identify general classes of graphs that either possess or fail to possess a particular path-regularity property. To complement the results of Section II which determined large classes of graphs that have particular path-regularity properties, it is desirable to identify certain necessary structural properties of path-regular graphs whose absence is then sufficient to insure that certain general classes of graphs do not possess a particular path-regularity property. Some nontrivial necessary conditions for graphs to be vertex- and/or edge-path-regular are obtained by examining the cuts and separating vertex sets of the graphs.

Theorem 6. The graph $G = (V,E)$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $\ell > 1$ edges:

(i) can be edge-path-regular only if for any cut $(A,\bar{A}) \subset E$,

$$\frac{|(A,\bar{A})|}{|A|\ |\bar{A}|} \geq \frac{\ell}{\sum_{i\leq j} d(v_i,v_j)} , \tag{6}$$

where further if G has diameter at most two, only if

$$\frac{(A,\bar{A})}{|A|\ |\bar{A}|} \geq \frac{\ell}{n(n-1) - \ell} ; \tag{7}$$

(ii) can be vertex-path-regular only if for any separating vertex set $S \subset V$, such that no edge joins any point of the non-void set $A \subset V-S$ to any point of the non-void set $B = V-S-A$,

$$\frac{|S|}{|A \cup S|\ |B \cup S|} \geq \frac{n}{\sum_{i<j} [d(v_i,v_j)+1]} , \tag{8}$$

where further if G has diameter at most two, only if

$$\frac{|S|}{|A \cup S|\ |B \cup S|} \geq \frac{2n}{3n^2 - n - 2\ell} . \tag{9}$$

Proof. Let $\mathcal{L}$ be a list of shortest paths of G such that every pair of vertices v_i, v_j are the endvertices of k paths of $\mathcal{L}$, and consider two cases:

(i) Assume further that each edge of G is contained in m_e paths of the list $\mathcal{L}$. Then for any cut $(A,\bar{A}) \subset E$, note that $k|A|\,|\bar{A}|$ paths of $\mathcal{L}$ have one endvertex in A and one in $\bar{A}$ and so must contain an edge of $(A,\bar{A})$, hence $k|A|\,|\bar{A}| \le m_e|(A,\bar{A})|$ or

$$\frac{|(A,\bar{A})|}{|A|\,|\bar{A}|} \geq \frac{k}{m_e} .$$

From (3) and (4) we obtain (6) and (7).

(ii) Assume for this case that each vertex of G is contained in exactly m_v paths of $\mathcal{L}$, and let $S \subset V$ separate $A \subset V\text{-}S$ from $B = V\text{-}S\text{-}A$. Each vertex of S is the endvertex of $k|V|$ paths of $\mathcal{L}$, and any path of $\mathcal{L}$ with one endvertex in A and the other in B contains at least one vertex of S, so $k|S|\,|V| + k|A|\,|B| \le m_v|S|$ and

$$\frac{k}{m_v} \leq \frac{|S|}{|A|\,|B| + |S|\,|V|} = \frac{|S|}{|A \cup S|\,|B \cup S|} .$$

Then from (1) and (2) we obtain (8) and (9). ☐

Theorem 6 will now be utilized to characterize a large class of graphs that are vertex-path-regular but not edge-path-regular.

Corollary 6.1. Let G be edge- and vertex-symmetric with diameter at most two and regular of degree $r \geq 4$. Then $G \times K_2$ is vertex-path-regular but is not edge-path-regular.

Proof. When G is vertex-symmetric and connected $G \times K_2$ will also be vertex-symmetric and connected, hence vertex-path-regular by Theorem 2. Let G have n vertices and $\ell = \frac{nr}{2} > 2n$ edges. The cut $(A, \bar{A})$ separating $G \times K_2$ into two copies of G has $|(A,\bar{A})| = n$ and

$|A| = |A| = n$, so if $G \times K_2$ is edge-path-regular from (6) we obtain

$$\frac{|E(G \times K_2)|}{\sum\limits_{\substack{i \leq j \\ v_i, v_j \in V(G \times K_2)}} d(v_i, v_j)} \leq \frac{1}{n} \quad . \tag{10}$$

$\therefore$ $G \times K_2$ has $2\ell+n$ edges, which are the number of vertex pairs at distance one in $G \times K_2$. Furthermore $G \times K_2$ has $2[\binom{n}{2} - \ell] + 2\ell$ vertex pairs at distance two, and $2[\binom{n}{2} - \ell]$ vertex pairs at distance three, which is the diameter of $G \times K_2$. Thus

$$\sum_{\substack{i \leq j \\ v_i, v_j \in V(G \times K_2)}} d(v_i, v_j) = 2\ell + n + 2(2[\binom{n}{2} - \ell] + 2\ell) + 3(2[\binom{n}{2} - \ell])$$

$$= 5n^2 - 4n - 4\ell \quad .$$

But then from (10) noting $\ell \geq 2n$,

$$\frac{1}{n} \geq \frac{2\ell+n}{5n^2 - 4n - 4\ell}$$

$$\geq \frac{5}{5n-12} \quad ,$$

a contradiction. Hence $G \times K_2$ is not edge-path-regular. $\square$

IV. Products of Path-Regular Graphs.

Given a class of graphs satisfying a specified symmetry or regularity property, it is often possible to determine a broader class of graphs possessing the same symmetry or regularity property by performing certain standard graphical composition operations on member graphs of the class. For the graphs G and H the product, $G \times H$ is the graph with vertex set $V(G) \times V(H)$ where (v,w) is adjacent to (v',w') in $G \times H$ whenever $v = v'$ and w is adjacent to w' in H, or whenever $w = w'$ and v is adjacent to v' in G. Regarding symmetry, it is straightforward to show that the product of any two vertex-symmetric graphs is vertex-symmetric, however, even the product of an edge-symmetric graph with itself need not be edge-symmetric, e.g. $K_{1,2} \times K_{1,2}$ is not edge-symmetric.

Regarding path-regularity properties, deeper relations between graphs and their products are obtained beyond those attributable simply to considerations of symmetry. The stronger results are inherent in the relation between shortest paths in G, H, and $G \times H$ as noted in the following. If $v_0, v_1, \ldots, v_p$ is a shortest path from v_0 to v_p in G and $w_0, w_1, \ldots, w_q$ is a shortest path from w_0 to w_q in H, then

$$(v_0, w_0), (v_0, w_1), \ldots, (v_0, w_{q-1}), (v_0, w_q), (v_1, w_q), \ldots, (v_{p-1}, w_q), (v_p, w_q)$$

is a shortest path from (v_0, w_0) to (v_p, w_q) in $G \times H$. Thus certain shortest paths in $G \times H$ may in effect be composed simply by concatenating shortest paths in G and H. Furthermore, every edge of any particular shortest path from (v_0, w_0) to (v_p, w_q) in $G \times H$ is either of type $(v', w)(v'', w)$, denoting an edge $v'v''$ of a path from v_0 to v_q in G, or of type $(v, w')(v, w'')$, denoting an edge $w'w''$ of a path from w_0 to w_q in H, where in fact these paths must be shortest paths in G and H, respectively. Thus a particular shortest path of $G \times H$ may be decomposed by projections into G and H, determining a unique pair of shortest paths in G and H. These relations between shortest paths in G, H, and $G \times H$ provide sufficient foundation to obtain several results on the products of path-regular graphs.

Theorem 7. The product graph $G \times H$ is vertex-path-regular whenever the graphs G and H are both vertex-path-regular.

Proof. Assume G and H are vertex-path-regular with parameters (k_G, m_G) and (k_H, m_H), respectively. Then G and H are also vertex-path-regular for parameters (k', m'_G) and (k', m'_H) where $k' = \mathrm{LCM}(k_G, k_H, 2)$ and $m'_G = m_G k'/k_G$, with m'_H defined similarly. Let $\mathcal{L}_G$ be a list of shortest paths of G where each vertex of G occurs in m'_G paths of the list, and where every pair of vertices of G are the endvertices of k' paths of $\mathcal{L}_G$. Further assume these k' paths are then (arbitrarily) divided into $k'/2$ forward paths and $k'/2$ reverse paths. Define $\mathcal{L}_H$ with designation of forward and reverse paths similarly. The above designations can be viewed as yielding $k/2$ oriented paths between every ordered pair of vertices in G and in H.

Compose a list $\mathcal{L}$ of paths in $G \times H$ as follows. For each distinct pair of vertices $(v_0, w_0), (v_p, w_q) \in V(G \times H)$, pair up each of the $k'/2$ forward paths, from v_0 to v_p of the list $\mathcal{L}_G$, say $v_0, v_1, \ldots, v_p$, with a distinct one of the $k'/2$ forward paths from w_0 to w_q of the list $\mathcal{L}_H$, say $w_0, w_1, \ldots, w_q$, to determine a path

$(v_0, w_0), (v_1, w_0), \ldots, (v_{p-1}, w_0), (v_p, w_0), (v_p, w_1), \ldots, (v_p, w_{q-1}), (v_p, w_q)$

of $\mathcal{L}$, and also pair up each of the $k'/2$ reverse paths $v_p, v'_{p-1}, \ldots, v'_1, v_0$ of the list $\mathcal{L}_G$ with a distinct one of the $k'/2$ reverse paths $w_q, w'_{q-1}, \ldots, w'_1, w_0$ of $\mathcal{L}_H$ to determine

$$(v_p, w_q), (v'_{p-1}, w_q), \ldots, (v'_1, w_q), (v_0, w_q), (v_0, w'_{q-1}), \ldots, (v_0, w'_1), (v_0, w_0)$$

of $\mathcal{L}$. The forward paths can be viewed as oriented from (v_0, w_0) to (v_p, w_q) in $G \times H$, and the reverse paths as oriented from (v_p, w_q) to (v_0, w_0) in $G \times H$. Also include k' copies of the single vertex path (v_0, w_0) in $\mathcal{L}$ for each $(v_0, w_0) \in V(G \times H)$.

Thus for every pair of vertices of $G \times H$, the list $\mathcal{L}$ contains k' shortest paths between those vertices. Each vertex (v,w) of' $G \times H$ is then an endvertex of $k'|V(G \times H)|$ paths of $\mathcal{L}$. The paths of $\mathcal{L}$ in which (v,w) is an interior vertex may be divided into three subclasses. Each of the $m'_G - k'|V(G)|$ paths of $\mathcal{L}_G$ containing v as an interior vertex is utilized in forming $|V(H)|$ paths of $\mathcal{L}$ in which (v,w) is an interior vertex. Similarly, each of the $m'_H - k'|V(H)|$ paths of $\mathcal{L}_H$ containing w as an interior vertex is utilized in forming $|V(G)|$ other paths of $\mathcal{L}$ in which (v,w) is an interior vertex. Finally, there are $(|V(G)|-1)(|V(H)|-1)$ pairs of vertices $(v',w),(v,w')$ of $G \times H$ with $v' \neq v$, $w' \neq w$, where each such pair are the endvertices of $k'/2$ other paths of $\mathcal{L}$ containing (v,w) as an interior vertex. This accounts for all occurences of (v,w) in the paths of $\mathcal{L}$, and confirms that (v,w) occurs in the same number of paths, specifically $m'_G|V(H)| + m'_H|V(G)| - \frac{k'}{2}(|V(G \times H)| + |V(G)| + |V(H)| - 1)$, for any $(v,w) \in V(G \times H)$. Hence $G \times H$ is vertex-path-regular. $\square$

Surprisingly, the converse of Theorem 7 does not hold. Specifically, it is straightforward to show by enumerating appropriate paths that $C_3 \times K_{1,2}$ is vertex-path-regular with parameters $(2,24)$, even though $K_{1,2}$ is not vertex-path-regular.

Explicitly contained in the proof of Theorem 7 is the fact that the parameter k' for $G \times H$ can be as small as the least common multiple of k_G and k_H except for only an additional factor of 2 when k_G and k_H are both odd.

Corollary 7.1. If G and H are vertex-path-regular with parameters (k_G, m_G) and (k_H, m_H), then $G \times H$ is vertex-path-regular for parameters $(k_{G \times H}, m_{G \times H})$ where $k_{G \times H} = \text{LCM}(k_G, k_H, 2)$.

To achieve the edge-path-regular property for a product graph, $G \times H$, of edge-path-regular graphs G and H, we must be able to choose respective parameters (k_G, m_G) and (k_H, m_H) so that (i) the number of occurences of a vertex of G as an endvertex in the associated list $\mathcal{L}_G$, given by $|V(G)|k_G$, is the same as the number of occurences of a vertex of H as an endvertex in the associated list $\mathcal{L}_H$, given by $|V(H)|k_H$, and (ii) the edge multiplicities m_G and m_H are equal. The quantity, $|V(G)|\rho(G)$, is termed the end-degree of the edge-path-regular graph G, and its critical significance is evident in the following theorem.

Theorem 8. The product graph $G \times H$ is edge-path-regular if and only if G and H are both edge-path-regular of the same end-degree, i.e., with $|V(G)|\rho(G) = |V(H)|\rho(H)$. Furthermore, if $G \times H$ is edge-path-regular, then $G \times H$, G, and H all have the same end-degree and

$$\rho(G \times H) = \rho(G) / |V(H)| = \rho(H) / |V(G)| . \qquad (11)$$

Proof. Assume G and H are edge-path-regular with the same end-degree, We may then assume G and H are edge-path-regular for the parameters (k, m_G) and (k, m_H) where $|V(H)| / m_H = |V(G)| / m_G$. Let $\mathcal{L}_G$ be a list of shortest paths of G where each edge of G occurs in m_G paths of the list, and where every pair of vertices of G are the endvertices of k paths of G. Define $\mathcal{L}_H$ similarly. By the same construction utilized in the proof of Theorem 7, compose a list $\mathcal{L}$ of shortest paths in $G \times H$. For every pair of vertices of $G \times H$, the list $\mathcal{L}$ as previously noted contains k shortest paths with those vertices as end-vertices. Furthermore, each of the m_G paths of $\mathcal{L}_G$ containing vv' as an edge is utilized in forming $|V(H)|$ paths of $\mathcal{L}$ containing $(v,w)(v',w)$ as an edge for each $w \in V(H)$, and these are the only occurences of $(v,w)(v',w)$ as an edge in the paths of $\mathcal{L}$. Similarly each of the m_H paths of $\mathcal{L}_H$ containing ww' as an edge is utilized in forming $|V(G)|$ paths of $\mathcal{L}$ containing $(v,w)(v,w')$ as an edge for each $v \in V(G)$, and these are the only paths of $\mathcal{L}$ containing $(v,w)(v,w')$. since $m_G|V(H)| = m_H|V(G)|$, $G \times H$ is edge-path-regular.

Assume $G \times H$ is edge-path-regular, and that $\mathcal{L}$ is a list of shortest paths of $G \times H$ where every pair of vertices of $G \times H$ are the endvertices of k paths of $\mathcal{L}$ and where each edge of $G \times H$ occurs in m paths of $\mathcal{L}$. For any fixed $v_0, v_p \in V(G)$, each of the $k|V(H)|^2$ paths of $\mathcal{L}$ with endvertices (v_0,w) and (v_p,w') for some

$w, w' \in V(H)$ identifies [by considering only the constituant edges $(v', w^*)(v'', w^*)$] a shortest path between v_0 and v_p in G. For each of the $|V(G)|^2$ ordered pairs of vertices of G we then so identify $k|V(H)|^2$ paths of G, and out of all these paths exactly $m|V(H)|$ of them will contain any specified edge $vv' \in E(G)$. Hence G is edge-path-regular with $\rho(G) = |V(H)|\rho(G \times H)$. The corresponding argument for $\rho(H)$ then completes the theorem. ❑

Analogous to Corollary 7.1 and by the same reasoning we obtain:

Corollary 8.1. If G and H are edge-path-regular with parameters (k_G, m_G) and (k_H, m_H) where $|V(G)|k_G/m_G = |V(H)|k_H/m_H$, then $G \times H$ is edge-path-regular for parameters $(k_{G \times H}, m_{G \times H})$ where $k_{G \times H} = \mathrm{LCM}(k_G, k_H, 2)$.

By noting that the process of constructing the paths of $G \times H$ in the list $\mathcal{L}$ from the paths of the associated lists $\mathcal{L}_G$ and $\mathcal{L}_H$ was identical in Theorems 7 and 8, we obtain:

Corollary 8.2. The product graph $G \times H$ is strongly-path-regular whenever G and H are both strongly-path-regular of the same end-degree.

For an iterated product $G \times G \times \ldots \times G$ of a path-regular graph we immediately obtain from Theorems 7, 8 and their corollaries:

Theorem 9. For the graph G let $G^{[1]} = G$, and $G^{[j]} = G \times G^{[j-1]}$ for $j \geq 2$. Then for any $j \geq 1$

(i) $G^{[j]}$ is strongly path-regular with parameters $(k, m_v^{[j]}, m_e^{[j]})$ whenever G is strongly path-regular with parameters $(k, m_v^{[1]}, m_e^{[1]})$ and k is even.

(ii) $G^{[j]}$ is vertex-path-regular with parameters $(k, m_v^{[j]})$ whenever G is vertex-path-regular with parameters $(k, m_v^{[1]})$ and k is even.

(iii) $G^{[j]}$ is edge-path-regular with parameters $(k, m_e |V(G)|^{j-1})$ whenever G is edge-path-regular with parameters (k, m_e) and k is even.

Significant from Theorem 9 is the ability to readily identify a large class of graphs from which the edge-path-regular property does not derive from the edge-symmetry property. For example, the graph $K_{m,n}^{[j]}$ for any $j \geq 2$ and any $m \neq n$ is edge-path-regular by Theorem 9, but is clearly not edge-symmetric.

For cases where G is vertex-symmetric, Theorem 9 demonstrates that $G^{[j]}$ is vertex-path-regular for parameters (k, m_v) where k can be chosen independently of j . This is in sharp contrast to the dependence of k on j that would be implicit from the earlier proof of Theorem 2. More specifically note that there are m! distinct shortest paths between opposite corners of the m-dimensional cube $K_2^{[m]}$. Yet from Theorem 9 it is possible to specify just two shortest paths between any and every pair of vertices of $K_2^{[m]}$ such that the resulting list of shortest paths has the same number of paths containing any specified vertex and the same number of paths containing any specified edge.

V. Open Questions Regarding Path-Regular Graphs.

The previous sections have developed some fundamental properties regarding path-regular graphs. At the same time some interesting questions have arisen that suggest further directions for research concerning the property of path-regularity.

Question 1. Given that p(G) is always rational and in the range $0 < \rho(G) \leq 1$, is there an edge-path-regular graph G with $\rho(G) = i/j$ for any rational $0 < i/j \leq 1$?

Comment. From Table 1 it is seen that $\rho(K_{1,j}) = 1/j$ for $j \geq 1$, and $\rho(K_{\underbrace{2,2,\ldots,2}_{j}}) = (j-1)/j$ for any $j > 2$, yielding the extreme irreducible fraction values for the range $0 < i/j \leq 1$. Many other intermediate rational values are obtained by the classes of edge-path-regular graphs so far identified, and composition rules such as in Theorem 9 provide further classes of achievable rational values for ρ. The more camprehensive problem of characterizing all realizable parameter values (k,m_v), (k,m_e) and (k,m_v,m_e) for vertex-, edge-, and strongly path-regular graphs may also yield interesting results, but appears less tractable.

Question 2. For which directed graphs D is it possible to construct a list $\mathfrak{L}$ containing exactly one directed path between each pair of vertices, such that each vertex and/or each directed edge occurs in the sane number of directed paths of $\mathfrak{L}$?

Comment. From the discussion at the end of Section IV it is clear that the symmetric directed graph whose directed edges correspond to the edges of the m-dimensional cube $Q_m = K_2^{[m]}$ has the property described. Furthermore, the constructions utilized in the proofs of Theorems 7 and 8 should provide for the identification of numerous classes of directed graphs having this desired property. Such graphs could be applicable to the problem of synthesis of communication networks requiring a single dedicated directed channel concurrently between every pair of vertices where the network must utilize the same type of multichannel cable for all arcs. A more comprehensive task would be to develop and investigate the concept of path-regularity for general directed graphs.

Question 3. Is there a good characterization for the classes of vertex-path-regular, edge-path-regular, and strongly path-regular graphs?

Comment. Several results in this paper lead to the conclusion that a simple characterization of path-regular graphs may not be possible. The fact that many nonsymmetric as well as symmetric graphs have particular path-regularity properties probably precludes a constructive approach starting from a limited set of path-regular graphs and using identified composition procedures. The fact that $C_3 \times K_{1,2}$ is vertex-path-regular even though $K_{1,2}$ is not, suggests further difficulties in fashioning a characterization. Attempts at characterization using procedures similar to the linear programming test of Section III probably will yield only variations of the definition of path-regularity rather than genuine alternative characterizations. A more promising approach is to generalize and extend the concept of cut [M80] and separating vertex set as employed in Theorem 6 in view of the Mengerian duality [H69, p.47] that exists in the non concurrent case, i.e., regarding paths between a single fixed pair of vertices and their separating cuts and vertex sets.

As a final observation we note that many simplifications and further specialized results for path-regularity properties can be obtained regarding the specific class of graphs of diameter 2 , and we are pursuing that approach in a subsequent paper [DM 80].

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