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Semiantichains and Unichain Coverings in Direct Products of Partial Orders

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Abstract. We conjecture a generalization of Dilworth's theorem to direct products of partial orders. In particular, we conjecture that the largest "semiantichain" and the smallest "unichain covering" have the same size. We consider a special class of semiantichains and unichain coverings and determine when equality holds for them. This conjecture implies the existence of k-saturated partitions. A stronger conjecture, for which we also prove a special case, implies the Greene-Kleitman result on simultaneous k and $(k+1)$-saturated partitions.

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1. Duality between semiantichains and unichain coverings.

In this paper we study the relationship between semiantichains and unichain coverings in direct products of partial orders. Semiantichains are more general objects than antichains, and unichains are a restricted class of chains. The study of antichains (collections of pairwise unrelated elements) in partially ordered sets admits two approaches. The earlier arises from Sperner's theorem [32], which. characterizes the maximum-sized antichains of a Boolean algebra. In general, Sperner theory obtains explicit values for the maximum size of antichains in partially ordered sets having special properties, and explicit descriptions of their composition. When the poset is ranked and the maximum-sized antichain consists of the rank with most elements, the poset has the *Sperner property.* Generalizations of Sperner's theorem have mostly consisted of showing that various posets have the Sperner property or stronger versions of the Sperner property. Greene and Kleitman [13] have given an excellent survey of results of this type.

Dilworth's theorem [4] bounds the size of the largest antichain by another invariant of the partial order. In particular, covering the partial order by chains is a "dual" minimization problem. No chain hits two elements of an antichain, so a covering always requires more items than any antichain has. Dilworth's theorem asserts that in fact the optimum sizes are always equal. The result does not give the extremal value or extremal collections, but it applies to all partially ordered sets. Generalizations of Dilworth's theorem have flowed less freely. A number of alternate proofs have been given, e.g. [3], [10], but the only broad extension we have is Greene and Kleitman's result [12] on k-families and k-saturated partitions.

The study of k-families began with Erdös. A *k-family* in a partially ordered set is a collection of elements which contains no chain of size $k+1$. An antichain is a 1-family. Erdös [6] generalized Sperner's theorem by showing that the largest k-family in a Boolean algebra consists (uniquely) of the k largest ranks. A (ranked) partial order satisfying this for all k is said to have the "strong Sperner property". Again, further Sperner-type results on k-families can be found in [13]. Clearly any chain contains at most k elements of a k-family, so any partition C of a partial order into chains $\{C_i\}$ gives an upper bound of $m_k(C)=\sum_i \min\ \{k,\ |C_i|\}$ on the size of the largest k-family. If the largest k-family has this size, the partition is called ***k*-saturated.** Greene and Kleitman proved there always exists a k-saturated partition, which for $k=1$ reduces to Dilworth's theorem. They showed further that for any k there exists a partition which is simultaneously k and $k+1$-saturated. They applied lattice methods generalizing Dilworth's less well-known result [5] about the lattice behavior of antichains. Saks [30] gave a shorter proof of the existence of k-saturated partitions of P by examining the direct product of P with a k-element chain.

We consider a generalization of the Dilworth-type idea of saturated partitions to the direct product of any two partial orders. Sperner theory has also discussed direct products. A *semiantichain* in a direct product is a collection of elements no two of which are related if they are identical in either component. The class of semiantichains includes the class of antichains. If the largest semiantichain still consists of a single rank, then the direct product has the two-part *Sperner property*. Results of this nature have been proved by Katona [21], [23], Kleitman [24], and Griggs [15], [17], with extensions to k-families by Katona [22], Schonheim [31], and recently by Proctor, Saks, and Sturtevant [27]. Examples where maximum-sized semiantichains are not antichains were examined by West and Kleitman [33] and G. W. Peck [26].

To generalize Dilworth's theorem to semiantichains we need a dual covering problem. Semiantichains are more general objects than antichains, so we need more restricted objects than chains. We define a *unichain* (one-dimensional chain) in a direct product to be a chain in which one component remains fixed. Alternatively it is the product of an element from one order with a chain from the other. Two elements on a unichain are called *unicomparable*. Clearly no semiantichain can contain two elements of a unichain, so the largest semiantichain is bounded by the smallest covering by unichains. After [33], West and Saks conjectured that equality always holds. We have not proved equality for general direct products, but we prove a special case here. Also, we make a stronger conjecture analogous to Green and Kleitman's simultaneous k and $k+1$-saturation. If one of the partial orders is a chain of $k+1$ elements, the conjecture reduces to their result.

Note that maximizing semiantichains and minimizing unichain coverings are dual integer programs. One such formulation has as constraint matrix the incidence matrix between elements and unichains. Showing that the underlying linear program has an integral optimal solution would prove the conjecture, by guaranteeing that the integer program has no "duality gap".

These dual programs form an example of the frequent duality between "packing" problems and "covering" problems (see [1], [2], [7], [8], [11], [19], [25], [29]. Dilworth's theorem is another example; Dantzig and Hoffman [3] deduced it from duality principles. Hoffman and Schwartz [20] also used integer programming ideas to prove a slight generalization of Greene and Kleitman's k-saturation result by transforming the problem into a transportation problem. These methods work partly because any subset of a partial order is still a partial order. However, a subset of a direct product need not be a direct product. Indeed, subsets of direct product orders frequently have duality gaps between their largest semiantichains and smallest unichain coverings. (The smallest example is a particular 7-element subset of the product of a 2-element chain with a 3-element chain.)

Dilworth's theorem can also be proved by transforming it to a bipartite matching problem or a network flow problem (see [9], [10]). The difficulty in applying these latter methods to direct products is that unicomparability, unlike comparability, is not transitive. Much is known about the integrality of optima when the constraint matrix is totally unimodular, balanced, etc., as summarized by Hoffman [18]. Unfortunately, none of the several integer programming formulations we know of for this direct product problem have any of those properties. Finally, Greene and Kleitman use lattice theoretic methods because the set of k-families and maximum k-families form well behaved lattices. We have found no reasonable partial order on semiantichains or maximum semiantichains.

In the case where the largest semiantichain is also an antichain, network flow methods can be used to prove the conjecture. This result will appear in a subsequent paper. In Section 2 we find necessary and sufficient conditions for equality to hold when semiantichains and unichain coverings are required to have a particularly nice property called "decomposability". When this happens, the size of the optimum is determined by the sizes of the largest k-families in the two components. In Section 3 we develop the stronger form of the conjecture and show it holds in this case. We note with boundless ambition that if the first conjecture is true we can begin to ask about the existence of "k-saturated partitions" of direct products into unichains, analogously to k-saturated partitions of posets.

Before embarking on the subject of decomposability, we note that this duality question can be phrased as a problem in graph theory. The "comparability graph" of a partially ordered set is formed by letting (x, y) be an edge in $G(P)$ if x

is related to y in P. An antichain becomes an independent set of vertices; a chain becomes a complete subgraph. Dilworth's theorem states that the independence number $\alpha(G)$ equals the clique covering number $\theta(G)$. When we take direct products, the "unicomparability graph" is just the product graph $G(P) \times G(Q)$.* Now independent sets are semiantichains and cliques are unichains, and again we want to show $\alpha = \theta$. Comparability graphs are perfect graphs, but it is not true in general for products of perfect graphs that $\alpha = \theta$. (Example x |, where the left factor is perfect, but not a comparability graph.) We can ask for what subclasses of perfect graphs does $\alpha(G \times H) = \theta(G \times H)$?

2. Decomposability.

We consider semiantichains and unichain coverings which arise from partitions of the component orders. We will use $d(P, Q)$ to denote the size of the largest semiantichain in $P \times Q$.

Partition P and Q into collections of antichains $\mathcal{A}$ and $\mathcal{B}$. Any matching of antichains in $\mathcal{A}$ with antichains in $\mathcal{B}$ induces a semiantichain when the complete direct product of each matched pair is included. An antichain which can be formed in this way is called *decomposable.*

Given partitions of P and Q into antichains, it is a simple algebraic consequence that the largest decomposable semiantichain we can form from them is obtained by matching the largest from each, then the next largest, and so on. We call this the "greedy product" of two partitions, and its size is

$$g(\mathcal{A}, \mathcal{B}) = \sum_i |A_i||B_i|, \quad \text{where } A_i \geq A_{i+1} \text{ and } B_i \geq B_{i+1}.$$

Now partition P and Q into collections of chains $\mathcal{C}$ and $\mathcal{D}$. This induces a unichain covering of $P \times Q$. For each pair (C_i, D_j), we cover the sub-product $C_i \times D_j$. It is easy to see we do this with fewest unichains if we take $\min\{|C_i|, |D_j|\}$ copies of the longer chain. Again, a unichain covering so formed is called a *decomposable* covering. Its size, a "pairwise minimum" function generalizing m_k,

*Independence number = size of largest set of mutually non-adjacent vertices. Clique covering number = size of smallest collection of complete subgraphs which together touch all vertices. Product graph $G \times H$ has as vertices the Cartesian product of the vertex sets of G and H. (u, v) and (u', v') are joined by an edge if $u = u'$ and (v, v') is an edge of H or $v = v'$ and (u, u') is an edge in G.

is

$$m(C, D) = \sum_{i,j} \min\{|C_i|, |D_j|\}.$$

Using Greene and Kleitman's terminology, we let $d_k(P)$ denote the size of the largest k-family in P and put $\Delta_k(P) = d_k(P) - d_{k-1}(P)$. Let $\Delta^P \cdot \Delta^Q = \sum_k \Delta_k(P)\Delta_k(Q)$. (We note this is a quantity which appears independently in [29], where Saks proved $d_1(P \times Q) \leq \Delta^P \cdot \Delta^Q$.)

To further simplify notation, let a_i and b_i be the size of the i-th largest antichains in $\mathcal{A}$ and $\mathcal{B}$. Since $g(\mathcal{A}, \mathcal{B})$ depends only on the sizes in the partition, we will speak interchangeably of $g(\mathcal{A}, \mathcal{B})$ and $g(\langle a_i \rangle, \langle b_i \rangle)$ even if there is no decomposition corresponding to those numbers.

Theorem 1. For any antichain *partitions* A *and* $\mathcal{B}$ *and* chain partitions C and $\mathcal{D}$ *of partial orders* P *and* Q,

(0) $g(\mathcal{A}, \mathcal{B}) \leq \Delta^P \cdot \Delta^Q \leq m(\mathcal{C}, \mathcal{D})$.

Furthermore, equality holds on the left if and only ***if***

(1) $b_k > b_{k+1} \Rightarrow \sum_{i \leq k} a_i = d_k(P)$.

(2) $a_k > a_{k+1} \Rightarrow \sum_{i \leq k} b_i = d_k(Q)$.

(3) $b_k = b_{k+1}$ *and* $a_k = a_{k+1} \Rightarrow \sum_{i \leq k} a_i = d_k(P)$ *or* $\sum_{i<k} b_i = d_k(Q)$.

Also, equality holds on the right if and only if

(4) $\Delta_k(P) > \Delta_{k+1}(P) \Rightarrow \mathcal{D}$ *is* k*-saturated, and* C *is* ℓ*-saturated* whenever $\mathcal{D}$ *has a chain* of *size* ℓ.

Equality ***on the right is also equivalent to the*** statement obtained ***by*** exchanging $\mathcal{D}$ ***with C and Q for P in (4).***

Proof. The first inequality holds by the same argument that made the greedy product the best way to match up antichains. Increasing a_k (beginning with $k =$

1, then 2, etc.) by shifting units from smaller a_i can only increase g, since those units will be paired with larger b_i than before. We must find an upper bound on this process.

The union of k antichains forms a k-family, so $\langle a_i \rangle$ is a non-increasing sequence with $\sum_{i \leq k} a_i \leq d_k(P) = \sum_{1 \leq i \leq k} \Delta_k$. Similarly for b_i. So, we increase a_1 to $\Delta_1(P)$ and b_1 to $\Delta_1(P)$, then increase a_2 and b_2, etc., until $a_k = \Delta_k(P)$ and $b_k = \Delta_k(Q)$. It is important to note that $\Delta_k \geq \Delta_{k+1}$, a non-trivial result proved in [12]. This guarantees that the non-increasing character of the sequences will be preserved by the process. If we begin with an actual partition $(\mathcal{A}, \mathcal{B})$, we end with $\Delta(P) \cdot \Delta(Q)$ without decreasing the value of g.

When will equality hold? If $\langle a_i \rangle, \langle b_i \rangle$ are the sequences for $\mathcal{A}$ and $\mathcal{B}$ and $\sum_{i \leq k} a_i$ is less than $d_k(P)$ for some k with $b_k > b_{k+1}$, we can increase g by increasing a_k at the expense of the smallest a_i. (Technically, we increase d_j for the smallest j such that $a_j = a_k$.) If $a_k = a_{k+1}$ and $b_k = b_{k+1}$, but both initial segments sum to less than the respective d_k, there will be room to gain by making such a change in both sequences simultaneously. On the other hand, if (1)–(3) are never violated, all the (legal) switches made to reach $\Delta^P \cdot \Delta^Q$ will leave them satisfied and produce no gain, so equality holds.

The second inequality is more subtle. We need more notation. Let $\alpha_k(C)$ be the number of chains in partition C which have at least k elements. If a partition of P is simultaneously $(k-1)$- and k-saturated, by definition $m_{k-1}(C) = d_{k-1}(P)$ and $m_k(C) = d_k(P)$. Subtracting the first from the second yields $\alpha_k(C) = \Delta_k(P)$. So, if a completely saturated partition exists, the number of chains with exactly k elements will always be $\Delta_k(P) - \Delta_{k+1}(P)$. Let $\beta_k(P) = \Delta_k(P) - \Delta_{k+1}(P)$.

Next we cite the discrete analogue of integration by parts. Assuming the boundary terms vanish,

$$\sum_k u_k(v_k - v_{k+1}) = \sum_k (u_k - u_{k-1})v_k$$

for u_k plug in d_k of one partial order, and for v_k use Δ_k of the other. Since $\beta_k = \Delta_k - \Delta_{k+1}$, we have

$$\sum d_k(P)\beta_k(Q) = \sum \Delta_k(P)\Delta_k(Q) = \sum \beta_k(P)d_k(Q). \tag{5}$$

By grouping pairs of chains appropriately, it is easy to see

$$\sum_i m_{|C_i|}(D) = m(C, D) = \sum_j m_{|D_j|}(C). \tag{6}$$

Now let C^* be a collection of chains with $\beta_k(P)$ of size k and D^* a collection with $\beta_k(Q)$ chains of size k. C^* and D^* may not exist as chain decompositions of P and Q, but as we did with antichains we can still apply the function m to those collections of chain sizes. In particular, C^* and D^* behave like completely saturated partitions, with $m_k(C^*) = d_k(P)$ and $m_k(D^*) = d_k(Q)$. Applying this to (6), we get

$$\sum d_k(P)\beta_k(Q) = m(C^*, D^*) = \sum \beta_k(P) d_k(Q)\ . \tag{7}$$

When we use D rather than D^*, the first half of *(6)* gives

$$m(C^*, D) = \sum \beta_k(P) m_k(D) \geq \sum \beta_k(P) d_k(Q) \tag{8}$$

with equality if and only if D is k-saturated whenever $\beta_k(P) > 0$, i.e., when $\Delta_k(P) > \Delta_{k+1}(P)$. Similarly $m(C, D') \geq m(C^*, D^*)$.

Now, if $\gamma_k(D)$ is the number of chains in D of size k, the other half of (6) gives

$$m(C^*, D) = \sum m_k(C^*)\gamma_k(D) \geq \sum d_k(P)\gamma_k(D)\ . \tag{9}$$

Replacing C^* by an actual partition C gives

$$m(C, D) = \sum m_k(C)\gamma_k(D) \geq \sum d_k(P)\gamma_k(D)\ , \tag{10}$$

equations (5)–(10) combine to give

$$m(C, D) \geq m(C^*, D) \geq m(C^*, D') = \sum \Delta_k(P) . \Delta_k(Q)\ . \tag{11}$$

For equality to hold every step of the way, the conditions are as stated in the theorem, i.e., saturation requirements when β_k and γ_k are non-zero. Note that passing through $m(C, D^*)$ gives us the other set of conditions. The two are equivalent. □

Of course, if equality holds on both sides of *(0)* the desired duality holds. It has not been shown that the conditions for equality hold when the extremal semiantichain and unichain covering are both decomposable. Even if they do, the extremal packing and covering are not always decomposable, although there always exists a maximal decomposable semiantichain (i.e., no larger semiantichain contains it). Furthermore, the size of the optimal semiantichain and unichain covering may be strictly greater or strictly less than $\Delta^P . \Delta^Q$. The first example of a direct product with no decomposable maximum-sized semiantichain was found by Saks [28]. Pictured in Figure 1, it has $\Delta^P \cdot \Delta^Q = 13$, but the largest semiantichain has 14 elements, as indicated. The smallest example we know of is the product in Figure 2(a). The largest decomposable semiantichains have 9 elements, but it is not hard to find one of size 10, namely $\{1a, 1b, 1c, 2d, 2e, 2f, 3d, 3e, 3b, 3c\}$, indicated by large dots. Meanwhile, $m(21, 2211) = 10$. The unichain covering is indicated by heavy lines. However, when a slight change is made to reach Figure 2(b) (adding the relation $3 > 1$), the semiantichain of size 10 disappears. *Now* the largest semiantichain is decomposable ($g(21, 411) = 9$), but the smallest unichain covering is not.

[Figure 1]

[Figure 2]

The usefulness of decomposable objects is that the extremal value among such objects can be computed quickly. For unichain coverings we can consider the broader class of *quasi-decomposable* coverings. These fix a partition of only one of the partial orders, then match each chain in that partition with some partition of the other order. We do best by providing a k-saturated partition for each k-chain. Then, if Q had the fixed partition, the size of the induced covering is $\sum d_k(P)\gamma_k(D)$. In the proof above, this is $m(C^*, D)$, so such coverings are also bounded by $\Delta^P \cdot \Delta^Q$.

In this broader class less is required for equality. In particular, if one of the orders has a completely saturated partition, D becomes D^* and there is a quasi-decomposable unichain covering of size $\Delta^P . \Delta^Q$. Unfortunately Figure 2b shows that even when both P and Q have completely saturated partitions, there need not be a semiantichain of this size. Here duality still holds, though, because the minimum covering is not even quasi-decomposable, but is smaller yet. As with decomposable coverings, the optimum quasi-decomposable covering is easily computed. Not all chain partitions D of Q need be considered; chain partitions whose sequences are refinements of others are always dominated by the latter. In general, any covering by disjoint unichains can be expressed by partitioning the direct product into suitable sub-products such that the covering is the union of decomposable coverings of the sub-products. However, this formulation is unwieldy. Quasi-decomposable coverings give a quick near-optimal value which can help reduce the search for the optimal.

As for the usefulness of decomposability, we see that products of posets with completely saturated partitions will have unichain coverings of size $\Delta^P.\Delta^Q$. Note also that when the partial orders can be decomposed into antichains of sizes Δ_k, there will be a decomposable semiantichain of size $\Delta^P.\Delta^Q$. This condition says the largest k-families are obtained by uniting the first k of some sequence of antichains. (This is not always true; in the poset of Figure 3 no largest 2-family contains a largest 1-family.)

In particular, strongly Sperner posets satisfy the the latter condition. Sufficient to imply the strong Sperner property is the LYM property. (For a survey of results on LYM orders, see [13] once again.) The question of whether LYM orders always have completely saturated partitions remains open (see [14], [16]). If so, then products of LYM orders would have this "two-part Dilworth property". In any case equality certainly holds for products of symmetric or skew chain orders, etc., which are strongly Sperner and have completely saturated partitions.

3. Magic triples.

We now discuss the analogue of a "simultaneously k and $(k+1)$-saturated partition" for direct products.

We define a *magic triple** $(S, \mathcal{U}, x)$ in a direct product $P \times Q$ to be a maximum-sized semiantichain S, a minimum-sized unichain covering $\mathcal{U}$, and an element x in P or Q satisfying the following properties.

(1) x is the fixed element of unichains in $\mathcal{U}$ the same number of times it is a component of elements in S.

(2) When x is deleted, the restrictions of S and $\mathcal{U}$ to the smaller direct product are still extremal.

Conjecture. A magic triple exists for every $P \times Q$, and hence the duality conjecture follows by induction on $|P| + |Q|$.

*Such a triple was originally called a "Catholic cucumber" due to late-night slurring of "the element is Q-crossed as many times as it is Q-covered."

Of course, if the duality conjecture is true in general, then property (2) above will hold whenever property (1) holds. Showing that implication holds without assuming the duality conjecture would make it easier to show magic triples always exist

The magic triple conjecture is particularly satisfying because, although inductive, it is symmetric in P and Q. In their proof, Greene and Kleitman had to consider two cases, corresponding to whether the element x belongs to P or to Q. The conjecture also explains the peculiarity in their result of guaranteeing simultaneous k and $(k+1)$-saturation but being unable to guarantee more at one time. (The usual example that more cannot be guaranteed simultaneously is "little H" in Figure 1.)

If Q is a $(k+1)$-chain, then any semiantichain in $P \times Q$ "projects down" to a $(k+1)$-family in P of the same size, since it uses $k+1$ disjoint antichains of P in the $k+1$ "copies" of P. Conversely, any $(k+1)$-family in P gives rise to (several) semiantichains of that size, so $d_{k+1}(P) = d(P, Q)$. A unichain covering of $P \times Q$ collapses to a partition C of P by collapsing the unichains that vary in Q to their fixed elements in P. Since Q can be covered by a single chain, such an element of P need not appear in any other unichain. If the unichain covering is minimal, the same chain decomposition of the remaining elements of P will be used in each of the $k+1$ copies of P in $P \times Q$, and all the P-unichains used will have at least $k+1$ elements. So, the bound $m_{k+1}(C)$ given by the corresponding partition C of P has the same size as the unichain covering.

Suppose magic triples exist, and hence duality holds. By the discussion above, the minimum unichain C is $(k+1)$-saturated. If the magic triple for $P \times Q$ has its "element" x in Q, then C is also k-saturated. If x is in P, we use induction on $|P|$. Obtaining a k and $(k+1)$-saturated partition and corresponding k and $(k+1)$-families for $P - x$, we add x to the families and as a single element chain to the partition. The properties of a magic triple guarantee the resulting partition of P is k and $(k+1)$-saturated, and the resulting collections are largest k and $(k+1)$-families.

Note that we have required a triple. It may be that for any extremal semiantichain or unichain covering there exists an example of the other that with it will form a triple. However, it is not true that any pair (S, U) will extend to a triple. For example, when the partial order of Figure 3 is crossed with itself, there are (among others) two largest semiantichains and two largest unichain coverings which extend to triples when paired correctly but not when paired the other way.

[Figure 3]

If a pair (S, U) admits a sequence of elements such that successive restrictions of this pair form magic triples until the partial orders are exhausted, we call them *completely mutually saturated.* Theorem 2 is a sufficient condition for complete mutual saturation which applies to products of partial orders satisfying the conditions for equality in Theorem 1. It would be nice to strengthen this theorem by removing the words "of the same size", i.e., to show that if a maximum-sized semiantichain and minimum-sized unichain covering are both decomposable, then they have the same size.

Also, we note that the converse of the theorem is false, as shown by the examples in Figure 2. The (S, U) pairs shown are not decomposable, but they are completely mutually saturated. The correct sequence of elements to be eliminated starts with $\{3\}$ in Figure 2a and with $\{1\}$ in Figure 2b. Then the reduced pair (S', U') (see proof below)are decomposable, and the theorem can be applied to complete the sequence.

Theorem 2. If a direct product order has a *largest semiantichain* and a smallest unichain covering of the same *size* which are both decomposable, then they are *completely mutually saturated.*

Proof. The element chosen to complete the magic triple can be any element on the chain which is shortest of both partitions. Let S be the semiantichain (induced by A and B), U the unichain covering (induced by C and D), and assume C has the shortest chain so $x \in P$. Then we claim D must be 1-saturated, and x appears in some antichain paired with the maximum-sized antichain of Q in S. We show this will make it a magic triple. When x is removed, what remains of S and U will be extremal and decomposable for $(P - x) \times Q$, so we can repeat this until the orders are exhausted.

Let A' and C' be the reduced antichain and chain partitions of $P - x$, and let $d(P, Q)$ denote the size of the largest semiantichain in $P \times Q$. $d(P - x, Q)$ is bounded from above by the reduced decomposable covering, which gives the first inequality below. The middle equality follows since x lies on the shortest chain. That is, when C and D induce a unichain covering, the elements on the shortest chain always appear as fixed elements crossed with a longer chain in the other order. Removing such an element removes from the count the number of chains in D. So we have

$$d(P-x,Q) \leq m(C',D) = m(C,D) - |D| = |U| - |D|. \tag{12}$$

On the other hand, $d(P-x,Q)$ is bounded from below by the restriction of S, giving the first inequality below. The second follows because in $A \times B = S$, x must be paired with some antichain in B, which has at most $d_1(Q)$ elements. Finally, since D is a partition it has at least $d_1(Q)$ chains, by Dilworth's theorem. This gives us

$$d(P-x,Q) \geq g(A',B) \geq g(A,B) - d_1(Q) \geq |S| - |D|. \tag{13}$$

Since $|S| = |U|$, all the inequalities in (12) and (13) become equalities. In particular, $|D| = d_1(Q)$, and $d_1(Q)$ is the size of the antichain matched with z's antichain. Also, $m(C',D) = g(A',B)$, so (S,U,x) is a magic triple, S' and U' are decomposable and equal, and the argument can be applied to $(P-x) \times Q$ to complete the desired sequence of elements. □

We close with the only example we have yet found where neither the maximum semiantichain nor the minimum unichain covering is decomposable. One factor is the order "Big H" devised by Saks and mentioned previously. The other is an example devised by Griggs [16] to show lack of implication among various poset properties. After much worry, we found the semiantichain and unichain coverings both of size 40 pictured in Figure 4. Again the elements of the semiantichain appear as heavy dots, one on each unichain. Elements labeled 4 or 5 in the direct product are covered by unichains which are copies of 4 or 5 element maximal chains in "Big H". Although not decomposable, this pair still extends to a magic triple by selecting either of the two points of highest degree (marked x) in "the fish". After they are removed, the reduced semiantichain and unichain covering are still extremal but no longer extend to a magic triple. We are left with four disjoint products, including two copies of Saks' example and two selections of a 2-family from “Big H”. By choosing different extremal pairs, we can continue finding magic triples until the orders are exhausted.

[Figure 4]

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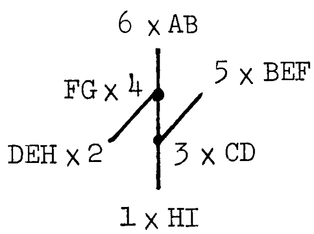
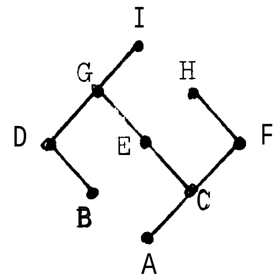
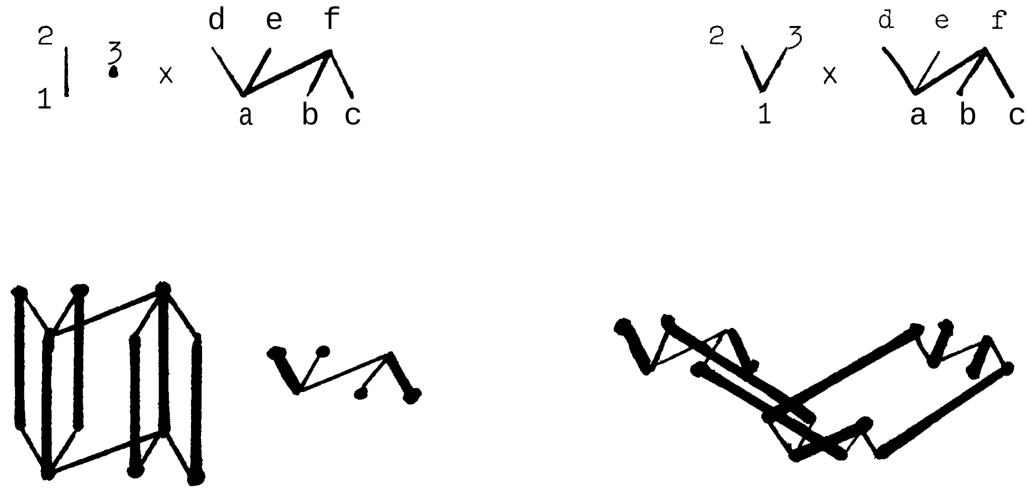




$$g(2211 \times 32211) = 13 = \Delta(P) \cdot \Delta(Q)$$

$$14 = m(33, 522) = m(411, 522) = d(P, Q)$$

Figure 1. "Little H" x "Big H".



$$\Delta^P \cdot \Delta^Q = (21 \cdot 42) = 10$$

$d(P, Q) = m(21, 2211) = 10$ $\qquad$ $d(P, Q) = g(21, 411) = g(21, 33) = 9$

(a) $\qquad$ (b)

Figure 2. Non-decomposability.

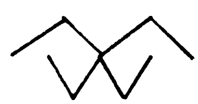
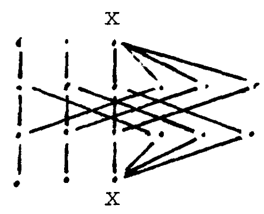
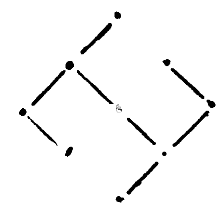


Figure 3. " M over W ".

 x 

Fish × Big H

d(P, Q) = 40

$\Delta \cdot \Delta = g(6633, 32211) = 39$

m(4442211, 522) = 42

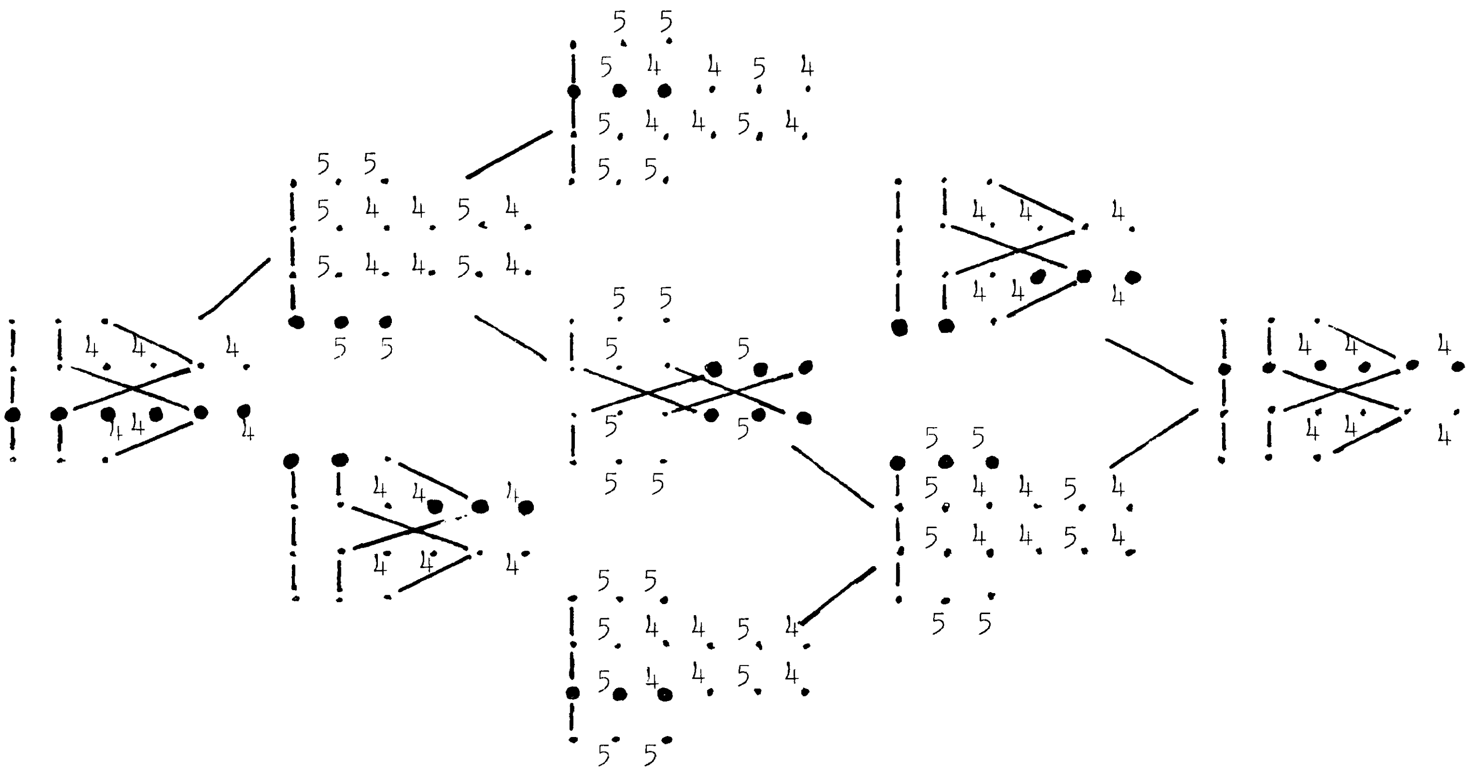


Figure 4