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THE DINNER TABLE PROBLEM

by

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Abstract

This report contains two papers inspired by the "dinner table problem": If n people are seated randomly around a circular table for two meals, what is the probability that no two people sit together at both meals? We show that this probability approaches e^{-2} as $n \to \infty$, and also give a closed form. We then observe that in many similar problems on permutations with restricted position, the number of permutations satisfying a given number of properties is approximately Poisson distributed. We generalize our asymptotic argument to prove such a limit theorem, and mention applications to the problems of derangements, ménages, and the asymptotic number of Latin rectangles.

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Introduction

The so-called "dinner table problem" is introduced by Bjørstad, Dahlquist, and Grosse [0] as follows:

> "At the most recent Gatlinburg conference on linear algebra, nametags were scattered about the dining room to encourage people to make new acquaintances. One evening a participant remarked: 'This can't be random — I just sat next to this guy at lunch.' Suppose that n people are seated randomly around a circular table for two meals. What is the probability p_n that no one can make such a remark?"

It turns out that versions of this problem were considered as early as 1919 by Poulet [3], who gave a recurrence relation for p_n. Using this recurrence, Bjørstad et. al. obtained a numerical estimate for $\lim_{n\to\infty} p_n$ that was very close to e^{-2}, and they conjectured that this was the exact limit.

In the first part of this report, we verify the above conjecture, and also give a combinatorial derivation of a closed form for p_n. Essentially the same results have been published by Robbins [5].

In the second part, we generalize our argument to explain why many probabilities arising in similar combinatorial problems tend to a value $e^{-\lambda}$ for some small integer λ. In so doing, we attempt to formalize a notion of "nearly independent" events that behave independently in the limit. For example, in the dinner table problem, we may argue that the probability that two people who sat next to each other at lunch do not sit next to each other at dinner is approximately $(n-2)/n$. There are n such pairs, so the probability p_n is roughly $((n-2)/n)^n$, which tends to e^{-2} as $n \to \infty$.

Using the principle of inclusion and exclusion, these events correspond to "properties" that may be satisfied by a permutation. We will show that when there is limited "interference" between properties, the number of properties satisfied by a random permutation approaches a Poisson distribution, as would be expected if the properties were truly independent.

Part 1: Asymptotic Analysis of the Dinner Table Problem

We can restate the dinner table problem **as** follows: Consider permutations of $\{1,\ldots,n\}$ around a circle; there are $(n-1)!/2$ different such permutations (not counting rotation and reflexion). Let q_n be the number of permutations such that the difference between two adjacent elements is not 1 or $n-1$; we have $q_n = p_n(n-1)!/2$.

In 1919, Poulet [3] gave the following recurrence relation for q_n, $n \geq 8$:

$$\begin{aligned}(n^2 - 7n + 9)q_n = (n^3 - 8n^2 + 18n - 21)q_{n-1} + 4n(n-5)q_{n-2} \\ - 2(n-6)(n^2 - 5n + 3)q_{n-3} + (n^2 - 7n + 9)q_{n-4} \\ + (n-5)(n^2 - 5n + 3)q_{n-5}.\end{aligned}$$

We give the first few values of q_n below.

n	3	4	5	6	7	8	9	10	11
q_n	0	0	1	3	23	177	1553	14963	157931

From the recurrence relation, we can derive a differential equation for the generating function $P(z) = \sum_n p_n z^n$. Unfortunately, this does not seem to be of much help because of its complicated form.

We will use the principle of inclusion and exclusion in order to find the asymptotic behaviour of p_n. Let $N = (n-1)!/2$ be the number of distinct permutations of $\{1,\ldots,n\}$ around the circle. Let π_i, for $1 \leq i \leq n$, be the property of i being adjacent to $i+1$ (mod n) along the circle, and let N_i be the number of permutations that have property π_i (they may have other properties as well). Similarly, let N_{ij} be the number of permutations that have *both* property π_i and property π_j, for $1 \leq i < j \leq n$, etc. Thus $N_{12\ldots n}$ is the number of permutations that have all n properties (and in our case this number is one). From the inclusion-exclusion formula, we have

$$q_n = N - \sum_{1\leq i\leq n} N_i + \sum_{1\leq i<j\leq n} N_{ij} - \sum_{1\leq i<j<k\leq n} N_{ijk} + \ldots + (-1)^n N_{12\ldots n}.$$

Let us view the right-hand side as an alternating series $\sum_{0<k<n}(-1)^k s_{kn}$, where

$$s_{kn} = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} N_{i_1 i_2\ldots i_k}.$$

In order to find the asymptotic behaviour of q_n, *we* truncate the alternating series after $l = \lceil \log_2 n \rceil$ terms and show that the truncation error is negligible for large values of n. We then show that s_{kn} is essentially $(n-1)!/2 \times 2^k/k!$, for $1 \le k \le l$, with the main contribution coming from those terms for which there is no summation index i_j such that $i_{j+1} = i_j + 1$.

We start by showing that, for $1 \le k < n$,

$$N_{i_1 i_2 \ldots i_k} \le 2^k \frac{(n-k-1)!}{2}.$$

Let B be the set $\{ i_1+1, i_2+1, \ldots, i_k+1 \}$, i.e., the set of elements with "forced" positions. Ignoring the elements of B, there are $(n-k-1)!/2$ ways to permute the remaining $n-k$ elements around the table. There are then at most two ways to insert each of the elements of B so that the k properties $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ hold. Since this accounts for all permutations having these k properties, we have established the above bound, and thus

$$s_{kn} \le \binom{n}{k} 2^k \frac{(n-k-1)!}{2} = \frac{n}{n-k}\binom{n-1}{k} 2^k \frac{(n-k-1)!}{2} = \frac{n}{n-k}\,\frac{(n-1)!}{2}\,\frac{2^k}{k!}.$$

Let $\alpha > 1$ be any constant. Then, for $l = [\log, n] > 2^{\alpha+4}e$, the absolute value of the truncation error is bounded by

$$\sum_{l \le k \le n} s_{kn} \le n\frac{(n-1)!}{2}\sum_{l \le k \le n}\frac{2^k}{k!} \le n^2\frac{(n-1)!}{2}\frac{2^l}{l!} = n^2\frac{(n-1)!}{2}2^l\sqrt{2\pi l}\left(\frac{e}{l}\right)^l(1+O(l^{-1}))$$

$$= \frac{(n-1)!}{2}\left(\frac{2e}{2^{\alpha+4}e}\right)^l O(n^3) = \frac{(n-1)!}{2}2^{-(\alpha+3)l}O(n^3) = \frac{(n-1)!}{2}O(n^{-\alpha}).$$

Our next step is to show that, for $1 \le k \le l$, *we* **have**

$$s_{kn} = \frac{2^k}{k!}\frac{(n-1)!}{2}(1+O(k^2/n)).$$

We have already established an upper bound on s_{kn} of this form; it remains to derive the lower bound. We do so by considering only those terms of s_{kn} for which no two consecutive properties π_i and π_{i+1} are chosen. That is, we consider partitions of $\{1, \ldots, n\}$ into three disjoint sets A, B, and C such that $A \cup B \cup C = \{1, \ldots, n\}$, $|A| = |B| = k$, and $B = \{ i+1 \pmod{n} \mid i \in A \}$. The set A corresponds to the chosen properties, and the set B to the "forced" elements. Given any such partition, there are $(n-k-1)!/2$ ways to permute the elements in $A \cup C$ around the circle. We can then insert each element of B in exactly two different ways so that the chosen k properties hold. Thus $s_{kn} \ge r_{kn} 2^k (n-k-1)!/2$, where r_{kn} is the number of ways to perform the above partition.

To determine r_{kn}, we will use the fact that there are $\binom{n-k+1}{k}$ ways to select k elements from $\{1, 2, \ldots, n\}$ such that no two consecutive elements are selected (here n and 1 are not considered consecutive). For the circular case, suppose that element n is *not* selected. There are then $\binom{n-k}{k}$ ways to choose k non-consecutive from $\{1, 2, \ldots, n-1\}$. To remove the restriction on n, consider the n cyclic shifts of each of the above choices. This overcounts each unrestricted choice n-k times, so we have

$$r_{kn} = \frac{n}{n-k}\binom{n-k}{k}.$$

Let us estimate r_{kn} for $1 \le k \le l = [\log, n]$; *we* **have**

$$r_{kn} = \frac{n}{n-k}\binom{n-k}{k} = \frac{n}{n-k}\binom{n-1}{k}\frac{(n-k)}{(n-1)}\frac{(n-k-1)}{(n-2)}\cdots\frac{(n-2k+1)}{(n-k)}$$

$$= \frac{n}{n-k}\binom{n-1}{k}(1-\frac{k-1}{n-1})(1-\frac{k-1}{n-2})\ldots(1-\frac{k-1}{n-k}).$$

Furthermore,

$$1 \ge (1-\frac{k-1}{n-1})(1-\frac{k-1}{n-2})\ldots(1-\frac{k-1}{n-k}) > (1-\frac{k-1}{n-k})^k = 1 + O(k^2/n),$$

and hence $r_{kn} = \binom{n-1}{k}(1+O(k^2/n))$. Therefore we have the desired lower bound

$$s_{kn} \ge r_{kn}\, 2^k\, \frac{(n-k-1)!}{2} = \frac{(n-1)!}{2}\frac{2^k}{k!}(1 + O(k^2/n)).$$

Using our estimate of s_{kn}, we have

$$q_n = \frac{(n-1)!}{2}\sum_{0\le k<l}\frac{(-2)^k}{k!} + \frac{(n-1)!}{2}\sum_{1\le k<l}\frac{2^k}{k!}\,O(k^2/n) + \frac{(n-1)!}{2}O(n^{-\alpha})$$

$$= \frac{(n-1)!}{2}e^{-2}(1 + O(n^{-1})).$$

Hence $p_n = e^{-2}(1+O(n^{-1}))$. Combining this with a result from [0], we have the following asymptotic formula:

$$p_n = e^{-a}(1 - \frac{4}{n} + \frac{20}{3n^3} + \frac{58}{3n^4} + \frac{736}{15n^5} + \frac{8428}{45n^6} + \frac{40174}{63n^7} + O(n^{-8})).$$

(Except for the constant e^{-2}, this expansion was obtained from the recurrence relation by the method of undetermined coefficients.)

We can in fact get an exact formula for the quantity q_n by counting the terms of s_{kn} more carefully. Recall that s_{kn} is the number of permutations satisfying a given set of k properties, summed over all $\binom{n}{k}$ choices of k properties. For the above analysis, we showed that it was sufficient to consider only the r_{kn} choices of k properties such that no two consecutive properties π_i and π_{i+1} are chosen. To refine this analysis, let r_{jkn} be the number of choices of k out of the n properties such that there are exactly j "blocks", where any group of consecutive properties $\pi_i, \pi_{i+1}, \ldots,$ is considered a block. Thus $r_{kn} = r_{kkn}$.

To evaluate r_{jkn}, consider first the case where property π_n is not chosen. If we now list the properties in increasing order and consider the sizes of the blocks, we get an ordered *partition* of k into j parts. The number of such partitions is exactly $\binom{k-1}{j-1}$. We then need to count the number of choices of k out of $n-1$ properties corresponding to each partition. This is equal to the number of ways of choosing j out of $n-1-k+j$ objects, no two consecutive. This number is

$$\binom{(n-1-k+j)-j+1}{j} = \binom{n-k}{j}.$$

To eliminate the restriction on property π_n, consider the n cyclic shifts of each of the above choices, As before, this overcounts each unrestricted choice $n-k$ times, so we get

$$r_{jkn} = \binom{k-1}{j-1}\binom{n-k}{j}\frac{n}{n-k}.$$

Now given a choice of k properties with j blocks, there are $(n-k-1)!/2 \times 2^j$ permutations satisfying these properties. Therefore we have

$$q_n = \frac{(n-1)!}{2} + \left(\sum_{1\le k<n} (-1)^k \sum_{1\le j\le k} \binom{k-1}{j-1}\binom{n-k}{j}\frac{n}{n-k}\frac{(n-k-1)!}{2}2^j\right) + (-1)^n$$

From this formula, it is possible to obtain the asymptotic expansion given above.

Part 2: The Poisson Approximation for the Number of Permutations Satisfying k Properties

Suppose that n people leave their hats at the checkroom before going to dinner in a restaurant. When they leave, they are handed hats at random. What is the probability p_k that exactly k people get their own hats back?

This is the well-known *problème* des *rencontres* (matching problem), and it is a rather striking result that the probability p_0 that no person gets his own hat back approaches e^{-1} as $n \to \infty$. In fact, it turns out that for **any** fixed k, *we* have $\lim_{n\to\infty} p_k = e^{-1}/k!$. In other words, the distribution of the number of people who get their own hats back approaches a Poisson distribution

$$p(k;\lambda) = e^{-\lambda}\frac{\lambda^k}{k!},$$

where in our case $\lambda = 1$.

As another example, consider the problem of neighbors remaining neighbors ([2], [3], [5]): If n people are seated at a circular table for two meals, what is the probability that no two people sit next to each other at both meals? It turns out that this probability approaches e^{-2} as $n \to \infty$. Furthermore, the probability that exactly k pairs of people sit next to each other at both meals approaches $e^{-2}2^k/k!$, that is, a Poisson distribution with parameter $\lambda = 2$.

Actually, the relation of these problems to the Poisson distribution should not be all that surprising. In the hat-check problem, we can compare the events that each person gets his own hat back with a sequence of n trials, each with probability l/n of success. It is well known that the Poisson distribution is an excellent approximation to the outcome of a sequence of n independent Bernoulli trials, where the probability p of success in each trial is small, but $\lambda = np$ is "moderate". The probability of exactly k successes in n trials is

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(\frac{n-\lambda}{n}\right)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}\frac{\lambda^k}{k!}.$$

For fixed k, the above expression approaches $e^{-\lambda}\lambda^k/k!$ as $n \to \infty$, since $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$.

In most problems on permutations with restricted position, the "events" are not independent. As a result it can be difficult to determine the exact probability of k successes. However, we will show that the effect of the dependence can be ignored if we are only interested in the asymptotic behavior of the p_k.

Our arguments will be based on the principle of inclusion and exclusion (sieve method). The sieve method is a powerful technique in enumeration problems of this type, and in fact the exact solution to the matching problem is one of the best-known examples of this technique. Our approach will be to combine the inclusion-exclusion formula with simple bounds on the terms involved in order to estimate the p_k.

Let us first restate our examples in terms of permutations with restricted position. Given a permutation π of $\{1, 2, \ldots, n\}$, the matching problem is concerned with the number of i such that $\pi(i) = i$. In the language of inclusion-exclusion, these n conditions are "properties" which may be satisfied by the permutation. Similarly, the neighbors remaining neighbors problem can be thought of as counting the number of j such that j is next to $j + 1$ in **a** random permutation. (For simplicity, we will also count n next to 1, so that there are n properties. Furthermore, we will consider the people as being arranged in a line rather than in a circle.)

So suppose that we have defined n properties of objects in some set, where for now we will be referring to the set of all n! permutations on $\{1, 2, \ldots, n\}$. The inclusion-exclusion formula tells us that the number of permutations which satisfy none of the properties is given by

$$N_0 = \sum_{0 \le k \le n} (-1)^k s_k, \tag{1}$$

where

$$s_k = \sum_{i_1, \ldots, i_k} N(i_1, i_2, \ldots, i_k)$$

is the number of objects satisfying a given set of k properties, summed over all choices of k out of the n properties.

More generally, the number of permutations satisfying exactly m of the properties is given by

$$N_m = \sum_{0 \le k \le n-m} (-1)^k \binom{m+k}{k} s_{m+k}.$$

In the examples we have given, each of the properties can be satisfied in λ "ways", where λ is some small fixed number. More precisely, we have, for $0 \le k \le n$,

$$s_k \le \binom{n}{k} \lambda^k (n-k)! = n! \frac{\lambda^k}{k!}. \tag{2}$$

The interpretation is that there are $\binom{n}{k}$ ways to choose k out of n properties, at most λ ways to satisfy each property for a total of λ^k ways, and finally $(n - k)!$

ways of permuting the remaining elements. Thus in the hat-check problem we have $\lambda = 1$ and in the neighbors remaining neighbors problem $\lambda = 2$ (ignoring the k "forced" elements $j + 1$, there are $(n - k)!$ ways to arrange the other elements, and then at most two ways to insert each of the forced elements $j + 1$ so that it is next to j).

Notice that we say "at most λ ways", because a particular way of satisfying one property may interfere with some other property, that is, it may reduce the number of ways of satisfying the other property. Hence we have only an upper bound on s_k.

However, we cannot allow each property to interfere with too many other properties, or else we will not get a satisfactory estimate for s_k. So suppose that each property can "interfere" with at most d properties, where d is usually a small fixed number. There are then at least $n(n-d)\cdots(n-(k-1)d)/k!$ ways of choosing k properties such that no two of them interfere with each other. Our requirement of limited interference between properties can be expressed as the following lower bound on s_k, assuming $(k-1)d < n$:

$$s_k \geq \frac{n(n-d)\cdots(n-(k-1)d)}{k!}\lambda^k(n-k)!. \tag{3}$$

For example, in the hat-check problem we have $d = 1$, because each property interferes only with itself. In this case the bounds in (2) and (3) are identical, so we have equality. In the neighbors remaining neighbors problem, we have $d = 3$, because property j interferes with properties $j - 1$ and $j + 1$, as well as itself.

Theorem. Suppose that each of the n properties can be satisfied in λ ways, and that there is limited interference between properties. More precisely, suppose that inequalities (2) and (3) are satisfied. Then for any fixed m we have

$$\lim_{n\to\infty} p_m = e^{-\lambda}\frac{\lambda^m}{m!},$$

where $p_m = N_m/n!$ is the probability that a random permutation on $\{1, 2, \ldots, n\}$ satisfies exactly m of the properties.

Proof. The lower bound (3) can be rewritten

$$\begin{aligned}\frac{s_k}{n!} &\geq \frac{n(n-d)\cdots(n-(k-1)d)}{n(n-1)\cdots(n-(k-1))}\frac{\lambda^k}{k!}\\ &= \left(1-\frac{d-1}{n-1}\right)\cdots\left(1-\frac{(k-1)(d-1)}{n-(k-1)}\right)\frac{\lambda^k}{k!}\\ &\geq \left(1-\frac{(k-1)^2(d-1)}{n-(k-1)}\right)\frac{\lambda^k}{k!}.\end{aligned}$$

Combining this with the upper bound (2), we get, for $(k-1)d < n$,

$$\frac{s_k}{n!} = \frac{\lambda^k}{k!}\left(1 + O\left(\frac{k^2 d}{n}\right)\right). \tag{4}$$

(Note that for d = 1 the error term is actually zero.)

In the inclusion-exclusion formula (1), the error made by truncating the sum at any point is bounded by the absolute value of the first term omitted. Then from (2), we have

$$p_0 = \sum_{0 \le k < l} (-1)^k \frac{s_k}{n!} + O\left(\frac{\lambda^l}{l!}\right).$$

Now using (4), we obtain

$$\begin{aligned} p_0 &= \sum_{0 \le k < l} (-1)^k \frac{\lambda^k}{k!} + \sum_{0 \le k < l} O\left(\frac{k^2 d}{n}\right)\frac{\lambda^k}{k!} + O\left(\frac{\lambda^l}{l!}\right) \\ &= \sum_{0 \le k \le \infty} (-1)^k \frac{\lambda^k}{k!} + O\left(\frac{d}{n}\right) \sum_{0 \le k < l} \frac{\lambda^k}{k!} + O\left(\frac{\lambda^l}{l!}\right) \\ &= e^{-\lambda} + O\left(\frac{de^\lambda}{n}\right) + O\left(\frac{\lambda^l}{l!}\right). \end{aligned} \tag{5}$$

If d and λ are fixed, and $l \le n/d$ goes to infinity with n (e.g. $l = \log n$), then we get $\lim_{n\to\infty} p_0 = e^{-\lambda}$.

The general case of m properties is similar, and after some work we find that

$$p_m = \frac{\lambda^m}{m!}\left[e^{-\lambda} + O\left(\frac{m^2 de^\lambda}{n}\right) + O\left(\frac{\lambda^l}{l!}\right)\right], \tag{6}$$

where $(m + l)d < n$. Then for l = logn, we get $\lim_{n\to\infty} p_m = e^{-\lambda}\lambda^m/m!$, as asserted. ∎

Actually, we can obtain a stronger result than that stated in the theorem, if we remove the restriction that *m* be fixed. From (6) we see that in fact

$$p_m \sim e^{-\lambda}\frac{\lambda^m}{m!}$$

for any $m \le n^{1/2-\epsilon}$.

We may also note that for fixed m, l can be chosen so that (6) yields

$$p_m = e^{-\lambda}\frac{\lambda^m}{m!} + O(n^{-1}).$$

We now mention some other problems to which our result can be applied.

1. Successions. In a random permutation, how many times is j followed immediately by $j+1$? For simplicity, we also count n followed by 1. Now given any k of the above properties, there are $(n-k)!$ ways to permute the unforced elements, and then at most one way to insert each of the forced elements $j+1$. Therefore $\lambda=1$ and d = 3. (A more careful consideration of s_k will show that we can actually take d = 1 in (3), except when $k=n$. However, for our purposes any fixed bound on d will do.)

Note that for both successions and neighbors remaining neighbors there are four possible variants of the problem, depending on whether or not we consider n to be followed by 1 and whether we consider permutations in a line or around a circle. Some of these variants introduce slight complications, but it is not hard to show that they all have the same asymptotic behavior. (For example, see [6].)

2. *Problème des ménages.* In how many ways can we seat n couples around a circular table, men and women alternating, such that no man sits next to his wife? The properties are $\pi(i)=i$ or $i+1$ (mod n), for $1\le i\le n$, and we have $\lambda=2$ and $d=3$.

3. Reverse derangements, A permutation π such that $\pi(i)\neq i$ for any i is also called a derangement. How many permutations are there such that neither π nor its reverse are derangements? That is, the properties are a(i) = i or $n+1-i$, for $1\le i\le n$. We have $\lambda=2$ and d = 2. (Note that n should be even or else property $(n+1)/2$ can only be satisfied in one way. However, again it is not hard to show that this does not affect the asymptotic behavior, because we need only replace n by $n-1$ in (3) and the proof still works.)

4. *Asymptotic* number *of* Latin rectangles ([1], [4, p. 209]). In a $k\times n$ Latin rectangle, each of the rows are permutations of $\{1,\ldots,n\}$ and no element appears twice in the same column. Thus the number of $2\times n$ Latin rectangles is just **n!** times the number of derangements, which we know is asymptotically $n!\,e^{-1}$. *Now* given any $k\times n$ Latin rectangle (a_{ji}), how many ways are there to extend it to a $(k+1)\times n$ Latin rectangle? The properties for row $k+1$ are $\pi(i)=a_{1i},a_{2i},\ldots,$ or a_{ki}, for $1\le i\le n$, and we have $\lambda=k$ and $d=k^2$ (there are k ways to satisfy each property, and each way might interfere with k other properties). Thus we can successfully apply our argument up to $k=(1-\epsilon)\log n$, at which point the e^λ factor in (5) becomes too large, We get that the number of $k\times n$ Latin rectangles is asymptotically

$$\prod_{0\le j<k} n!\,e^{-j}=(n!)^k\,e^{-\sum_{0\le j<k} j}=(n!)^k\,e^{-\binom{k}{2}}$$

5. Successions in a chessboard *permutation.* How many ways are there to place the numbers 1 to n^2 on an $n \times n$ chessboard so that j is never adjacent horizontally or vertically to j + 1? We include the case n^2 adjacent to 1 and also consider the board to be a torus. Here it seems to be easier to state the properties in an alternate way, namely, we consider the properties that $\pi(i)$ and $\pi(i) + 1$ (mod n^2) are adjacent, for $1 \le i \le n^2$. Now given any choice of k properties, there are at most 4 ways to select an adjacent square for each chosen property, and then exactly $(n^2 - k)!$ ways to complete the arrangement. Also, we see that property i interferes with properties $i \pm 2n$, $i \pm (n \pm 1)$, $i \pm n$, $i \pm 2$, and $i \pm 1$, as well as with itself, Thus we have $\lambda = 4$ and $d = 13$.

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