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Abstract. The r-Stirling numbers of the first and second kind count restricted permutations and respectively restricted partitions, the restriction being that the first r elements must be in distinct cycles and respectively distinct su bscts. The combinatorial and algebraic properties of these numbers, which in most cases generalize similar properties of the regular Stirling numbers, arc explored starting from the above definition.

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§1 Introduction

The r-Stirling numbers represent a certain generalization of the regular Stirling numbers, which, according Lo Tweedic [26], were so named by Nielscn [18] in honor of James Stirling, who computed them in his "*Methodus Differentialis,*" [24] in 1730. In fact the Stirling nurnbcrs of the first kind wcre known Lo Thomas Hcrriot [15]; in the British Muscurn archive, there is a manuscript [7] of his, dating from around 1600, which contains lhc expansion of the polynomials $\binom{n}{k}$ for $k \leq 7$. Good expositions of the properties of Stirling numbers arc found for example in [4, chap. 5], [9, chap. 4], and [22].

In this paper thc (signless) Stirling numbers of the first kind are dcnotcd $\left[{n \atop m}\right]$; they are defined combinatorially as the number of pcrrnutalions of lhc set $\{1,\ldots,n\}$, having m cycles. Thc Stirling numbers of the second kind, dcnolcd $\left\{{n \atop m}\right\}$, are equal to the number of partitions of thc set $\{1,\ldots,n\}$ into m non-empty disjoin1 sets. Thc notation [] and {} seems to be well suited to formula manipulations. It was introduccd by Knuth in [10, §1.2.6], improving a similar notational idea proposed by I. Marx [20]. Thc r-Stirling numbers count, certain restricted permutations and respectively restricted partitions and are defined, for all positive r, as follows:

$$\left[{n \atop m}\right]_r = \begin{array}{l}\text{The number of permutations of Lhc set } \{1,\ldots,n\} \\ \text{having } m \text{ cycles, such that the numbers } 1,2,\ldots,r \text{ are} \\ \text{in distinct cycles,}\end{array} \tag{1}$$

and

$$\left\{{n \atop m}\right\}_r = \begin{array}{l}\text{Thc number of partitions of the set } \{1,\ldots,n\} \text{ into} \\ m \text{ non-empty disjoint subsets, such that the numbers} \\ 1,2,\ldots,r \text{ are in distinct subsets.}\end{array} \tag{2}$$

There exists a one- to-one correspondence between permutations of n numbers with m cycles, and permutations of n numbers with m left-to-right minima. (This corespondence is itnplicd in [22, chap. 8] and formalized and generalized in [6].) To obtain the image of a given permutation with m cycles put the minimum number within each cycle (called the cycle leader) as the first element of lhc cycle, and list all cycles (including singletons) in decreasing order of their minimum element. After removing parentheses, the result is a permutation with m left-to-right minima. If the nurnbcrs $1,\ldots,r$ arc in distinct cycles in Lhc given permutation, then they arc all cycle leaders and the last r left-to-right minima in lhc image permutation are exactly $r, r-1, \ldots, 1$. Therefore wc have lhc alternative definition

$$\left[{n \atop m}\right]_r = \begin{array}{l}\text{The number of pcrrnutalions of the nurnbcrs } 1,\ldots,n \\ \text{having } m \text{ left-to-right minima such that the numbers} \\ 1,2,\ldots,r \text{ arc all left-to-right minima (or such that} \\ \text{the numbers } 1,2,\ldots,r \text{ occur in decreasing order).}\end{array} \tag{3}$$

Each non-empty subset in a permutation of an ordered set has a tninirnal element; a partition of the set $\{1,\ldots,n\}$ into m non-empty subsets has m associated minimal elements. This terminology allows the alternative definition

$$\left\{{n \atop m}\right\}_r = \begin{array}{l}\text{The number of ways Lo partition the set } \{1,\ldots,n\} \\ \text{into } m \text{ non-empty disjoint subsets, such that the num-} \\ \text{bers } 1,2,\ldots,r \text{ are all tninirnal elements.}\end{array} \tag{4}$$

Note that the regular Stirling numbers can be expressed as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_0, \qquad \begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m \end{Bmatrix}_0 \tag{5}$$

and also as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_1, \qquad \begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m \end{Bmatrix}_1, \qquad n > 0. \tag{6}$$

Another construction that turns out to be equivalent to the r-Stirling numbers was recently discovered by Carlitz [2],[3], who began from an entirely different type of generalization, *weighted Stirling numbers.* Also equivalent are the *non-central Stirling numbers* studied by Koutrns [17] starting from operator calculus definitions (see section 12). The simple approach to be dcvclopcd here leads to further insights about these numbers that appear to be of importance because of their remarkable properties.

§2 Basic recurrences

The r-Stirling numbers satisfy the same recurrence relation as the regular Stirling numbers, except for the initial conditions.

Theorem 1. *The r-Stirling numbers of the first kind obey the "triangular" recurrence*

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix}_r &= 0, && n < r, \\ \begin{bmatrix} n \\ m \end{bmatrix}_r &= \delta_{m,r}, && n = r, \\ \begin{bmatrix} n \\ m \end{bmatrix}_r &= (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}_r + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_r, && n > r. \end{aligned} \tag{7}$$

Proof: A permutation of the numbers 1, . . . , n with m left-to-right minima can be formed from a permutation of the numbers 1, . . . , n − 1 with m left-to-right minima by inserting the number n after any number, or from a permutation of the numbers 1,. . . , n − 1 with $m-1$ left-to-right minima by inserting the number n before all the other numbers. For n > r this process does not change the last r left-to-right minima. ∎

Theorem 2. *The r-Stirling numbers of the second kind obey the "triangular" recurrence*

$$\begin{aligned} \begin{Bmatrix} n \\ m \end{Bmatrix}_r &= 0, && n < r, \\ \begin{Bmatrix} n \\ m \end{Bmatrix}_r &= \delta_{m,r}, && n = r, \\ \begin{Bmatrix} n \\ m \end{Bmatrix}_r &= m\begin{Bmatrix} n-1 \\ m \end{Bmatrix}_r + \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}_r, && n > r. \end{aligned} \tag{8}$$

Proof: A partition of the set $\{1,\ldots,n\}$ into m non-empty subsets can be formed from a partition of the set $\{1,\ldots,n-1\}$ into m non-empty subsets, by adding the number n to any of the m subsets, or from a partition of the set $\{1,\ldots,n-1\}$ into $m-1$ non-empty subsets, by adding the subset $\{n\}$. Obviously, for $n > r$ this process does not influence the distribution of the numbers $1,\ldots,r$ into different subsets. ∎

The following special values can be easily computed:

$$\begin{bmatrix} n \\ n \end{bmatrix}_r = \begin{Bmatrix} n \\ n \end{Bmatrix}_r = 1, \qquad n \geq r; \tag{9}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \begin{Bmatrix} n \\ m \end{Bmatrix}_r = 0, \qquad m > n; \tag{10}$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_r = (n-1)(n-2)\ldots r = r^{\overline{n-r}}, \qquad n \geq r; \tag{11}$$

$$\begin{Bmatrix} n \\ r \end{Bmatrix}_r = r^{n-r}, \qquad n > r. \tag{12}$$

The r-Stirling numbers form a natural basis for all sets of numbers $\{a_{n,k}\}$ that satisfy the Stirling recurrence except for $a_{n,n}$. That is, the solution of the Stirling recurrence of the first kind

$$\begin{aligned} a_{n,k} &= 0, && n < 0, \\ a_{n,k} &= (n-1)a_{n-1,k} + a_{n-1,k-1}, && k \neq n,\ n \geq 0, \end{aligned} \tag{13}$$

is

$$a_{n,k} = \sum_r \begin{bmatrix} n \\ k \end{bmatrix}_r (a_{r,r} - a_{r-1,r-1}). \tag{14}$$

Similarly, the solution of

$$\begin{aligned} b_{n,k} &= 0, && n < 0, \\ b_{n,k} &= kb_{n-1,k} + b_{n-1,k-1}, && k \neq n,\ n \geq 0, \end{aligned} \tag{15}$$

is

$$b_{n,k} = \sum_r \begin{Bmatrix} n \\ k \end{Bmatrix}_r (b_{r,r} - b_{r-1,r-1}). \tag{16}$$

For concreteness, the following tables were computed using the recurrences (7) and (8).

$\left[{n \atop k}\right]_1$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=1$	1					
$n=2$	1	1				
$n=3$	2	3	1			
$n=4$	6	11	6	1		
$n=5$	24	50	35	10	1	
$n=6$	120	274	225	85	15	1

$\left\{{n \atop k}\right\}_1$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=1$	1					
$n=2$	1	1				
$n=3$	1	3	1			
$n=4$	1	7	6	1		
$n=5$	1	15	25	10	1	
$n=6$	1	31	90	65	15	1

Table 1. $r = 1$

$\left[{n \atop k}\right]_2$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	1					
$n=3$	2	1				
$n=4$	6	5	1			
$n=5$	24	26	9	1		
$n=6$	120	154	71	14	1	
$n=7$	720	1044	580	155	20	1

$\left\{{n \atop k}\right\}_2$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	1					
$n=3$	2	1				
$n=4$	4	5	1			
$n=5$	8	19	9	1		
$n=6$	16	65	55	14	1	
$n=7$	32	211	285	125	20	1

Table 2. $r = 2$

$\left[{n \atop k}\right]_3$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=3$	1					
$n=4$	3	1				
$n=5$	12	7	1			
$n=6$	60	47	12	1		
$n=7$	360	342	119	18	1	
$n=8$	2520	2754	1175	245	25	1

$\left\{{n \atop k}\right\}_3$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=3$	1					
$n=4$	3	1				
$n=5$	9	7	1			
$n=6$	27	37	12	1		
$n=7$	81	175	97	18	1	
$n=8$	243	781	660	205	25	1

Table 3. $r = 3$

§3 "Cross" recurrences

The "cross" recurrences relate r-Stirling numbers with different r.

Theorem 3. *The r-Stirling numbers of the first kind satisfy*

$$\left[{n \atop m}\right]_r = \frac{1}{r-1}\left(\left[{n \atop m-1}\right]_{r-1} - \left[{n \atop m-1}\right]_r\right), \qquad n \geq r > 1. \tag{17}$$

Proof: An alternative formulation is

$$(r-1)\left[{n \atop m}\right]_r = \left[{n \atop m-1}\right]_{r-1} - \left[{n \atop m-1}\right]_r.$$

The right side counts the number of permutations having $m-1$ cycles such that $1, \ldots, r-1$ are cycle leaders but r is not. This is equal to $(r-1)\left[{n \atop m}\right]_r$ since such permutations can be obtained in $r-1$ ways from permutations having m cycles, with $1, \ldots, r$ being cycle leaders, by appending the cycle led by r at the end of a cycle having a smaller cycle leader. ∎

Theorem 4. *The r-Stirling numbers of the second kind satisfy*

$$\left\{{n \atop m}\right\}_r = \left\{{n \atop m}\right\}_{r-1} - (r-1)\left\{{n-1 \atop m}\right\}_{r-1}, \qquad n \geq r \geq 1. \tag{18}$$

Proof: The above equation can be written as

$$(r-1)\left\{{n-1 \atop m}\right\}_{r-1} = \left\{{n \atop m}\right\}_{r-1} - \left\{{n \atop m}\right\}_r.$$

The right side of the equation counts the number of partitions of $\{1, \ldots, n\}$ into m non-empty subsets such that $1, \ldots, r-1$ are minimal elements but r is not. But this number is equal to $(r-1)\left\{{n-1 \atop m}\right\}_{r-1}$ because such partitions can be obtained in $r-1$ ways from partitions of $\{1, \ldots, n\} - \{r\}$ into m non-empty subsets, such that, $1, \ldots, r-1$ are minimal, by including r in any of the $r-1$ subsets containing a smaller element. ∎

§4 Orthogonality

The orthogonality relation between Stirling numbers generalizes to similar relations for r-Stirling numbers.

Theorem 5. *The r-Stirling numbers satisfy* [2, eq. 6.1]

$$\sum_k \left[{n \atop k}\right]_r \left\{{k \atop m}\right\}_r (-1)^k = \begin{cases} (-1)^n \delta_{m,n}, & n \geq r; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Proof: By induction on n. For $n < r$ the equality is obvious. For $n = r$

$$\sum_k \left[{r \atop k}\right]_r \left\{{k \atop m}\right\}_r (-1)^k = (-1)^r \left\{{r \atop m}\right\}_r = (-1)^r \delta_{m,r}.$$

For n > r, using Theorem 1 and the induction hypothesis

$$\sum_k \left[{n \atop k}\right]_r \left\{{k \atop m}\right\}_r (-1)^k = (n-1)\delta_{n-1,m}(-1)^{n-1} + \sum_k \left[{n-1 \atop k-1}\right]_r \left\{{k \atop m}\right\}_r (-1)^k,$$

and (assuming $m \geq r$) by Theorem 2 applied to the right sum, and the induction hypothesis

$$\begin{aligned} \sum_k \left[{n \atop k}\right]_r \left\{{k \atop m}\right\}_r (-1)^k &= (n-1)\delta_{n-1,m}(-1)^{n-1} - m\delta_{n-1,m}(-1)^{n-1} - \delta_{n-1,m-1}(-1)^{n-1} \\ &= \delta_{n,m}(-1)^n. \end{aligned}$$

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Hence for each r, the r-Stirling numbers form two infinite lower triangular matrices satisfying

$$\left\| \left[{i \atop j}\right]_r (-1)^j \right\| \times \left\| \left\{{i \atop j}\right\}_r \right\| = \left\| \delta_{i \geq r}\, \delta_{i,j} (-1)^i \right\|, \tag{20}$$

where

$$\delta_{i \geq j} = \begin{cases} 1, & i \geq j; \\ 0, & i < j. \end{cases}$$

and we also have

Theorem 6.

$$\sum_k \left[{k \atop n}\right]_r \left\{{m \atop k}\right\}_r (-1)^k = \begin{cases} (-1)^n \delta_{m,n}, & n \geq r; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

∎

These orthogonality relations generalize as shown in section 11.

§5 Relations with symmetric functions

The Stirling numbers of the first kind, $\left[{n \atop m}\right]$, for fixed n, are the elementary symmetric functions of the numbers $1, \ldots, n$ (see, e.g., [4] or [5]). The r-Stirling numbers of the first kind are the elementary symmetric functions of the numbers $r, \ldots,$ n.

Theorem 7. *The r-Stirling numbers **of** the first kind satisfy*

$$\left[{n \atop n-m}\right]_r = \sum_{r\le i_1<i_2\cdots<i_m<n} i_1 i_2 \dots i_m, \qquad n, m \geq 0. \tag{22}$$

Proof: Consider a permutation of the nurnbers $1, \dots, n$ having $n-m$ left-to-right minima. How many such permutations arc there that have a given set of minirna? Denote the numbers that are *not* rninima by $i_1, i_2, \dots, i_m$, where $i_1 < i_2 < \dots < i_m \leq n$. A perrnutation with the prescribed set of left-to-right minima can be constructed as follows: write all the minima in decreasing order; insert i_1 after any of the $i_1 - 1$ minima less than i_1; insert i_2 after any of the $i_2 - 2$ minima less than i_2, or after i_1; etc. Clearly there arc $i_1 - 1$ ways of inserting i_1, $i_2 - 1$ ways of inserting i_2, and so on. Hence the total nurnber of permutations with the given minima is $(i_1-1)(i_2-1)\dots(i_m-1)$. If the numbers $1, \dots, r$ arc minima, then $i_1 > r$. Surnming over all possible sets of left-Lo-right minirna we get

$$\begin{aligned}\left[{n \atop n-m}\right]_r &= \sum_{r<i_1<i_2\cdots<i_m\le n} (i_1-1)(i_2-1)\dots(i_m-1) \qquad n, m \geq 0\\ &= \sum_{r\le i_1<i_2\cdots<i_m<n} i_1 i_2 \dots i_m.\end{aligned}$$

∎

The above theorem can also be proved by induction, but, it is more interesting Lo see the combinatorial meaning of each term in the sum. Tts counterpart for r-Stirling numbers of the second kind is

Theorem 8. *The r-Stirling numbers of the second kind satisfy*

$$\left\{{n+m \atop n}\right\}_r = \sum_{r\le i_1\le\cdots\le i_m\le n} i_1 i_2 \dots i_m, \qquad n, m \geq 0. \tag{23}$$

Proof: Count the number of partitions of the set $\{1, \dots, n+m\}$ into n non-empty subsets, when the n minimal elements are fixed. Denote the elements that are *not* minimal by $x_1, \dots, x_m$, whore $x_1 < \dots < x_m$. If we let i_j be the number of minimal elements less than x_j, then $i_1 \leq i_2 \leq \dots \leq i_m \leq n$. Clearly x_j can belong only to subsets having a minimal element less than it, so that there are i_j ways Lo place it. Hence the total number of partitions with a given set of minimal elements is $i_1 i_2 \dots i_m$. If the numbers $1, \dots, r$ arc all minimal elements, then $i_1 \geq r$. Summing up over all possible sets of minimal elements completes Lhe proof. ∎

Therefore Lhe r-Stirling numbers of the second kind, $\left\{{n+m \atop n}\right\}_r$, arc the monomial symmetric functions of degree m of the integers $r, \dots, n$.

§6 Ordinary generating functions

Corollary 9. *The r-Stirling numbers of the first kind have the "horizontal" generating function*

$$\sum_k \left[{n \atop k}\right]_r z^k = \begin{cases} z^r(z+r)(z+r+1)\ldots(z+n-1), & n \geq r \geq 0; \\ 0, & \textit{otherwise}, \end{cases} \tag{24}$$

∎

Corollary 10. *The r-Stirling numbers of the second kind have the "vertical" generating function* [2, eq. 3.10]

$$\sum_k \left\{{k \atop m}\right\}_r z^k = \begin{cases} \dfrac{z^m}{(1-rz)(1-(r+1)z)\ldots(1-mz)}, & m \geq r \geq 0; \\ 0, & \textit{otherwise}. \end{cases} \tag{25}$$

∎

The above identities follow immediately from equations (22) and (23).

§7 Combinatorial identities

Lemma 11.

$$\left[{n \atop m}\right]_r = \sum_k \binom{n-r}{k} \left[{n-p-k \atop m-p}\right]_{r-p} p^{\overline{k}}, \qquad r \geq p \geq 0. \tag{26}$$

Proof: To form a permutation with m cycles such that $1, \ldots, r$ are cycle leaders first choose k numbers to be in the cycles led by $1, \ldots, p$ and construct these cycles; this can be done in (“;‘>[“a”], ways. The remaining n-p-k numbers must form m-p cycles such that $p+1, \ldots, r$ are cycle leaders, which can be done in $\left[{n-p-k \atop m-p}\right]_{r-p}$ ways. Using equation (11) and summing for all k completes the proof. ∎

In particular for $p = r$ we obtain a definition of r-Stirling numbers of the first kind in terms of regular Stirling numbers of the first kind [2, eq. 5.3],

$$\left[{n \atop m}\right]_r = \sum_k \binom{n-r}{k} \left[{n-r-k \atop m-r}\right] r^{\overline{k}} = \sum_k \binom{n-r}{k} \left[{k \atop m-r}\right] r^{\overline{n-r-k}}. \tag{27}$$

This shows that $\left[{n+r \atop m+r}\right]_r$, for $m, n \geq 0$ is a polynomial of degree $n-m$ in r with leading coefficient $\binom{n}{m}$ and Lemma 11 can be generalized to a polynomial identitity in p and r:

Theorem 12.

$$\left[{n+r \atop m+r}\right]_r = \sum_k \binom{n}{k} \left[{n-k+p \atop m+p}\right]_p (r-p)^{\overline{k}}. \tag{28}$$

∎

For $p = r - 1$ we get another "cross" recurrence

$$\left[{n \atop m}\right]_r = \sum_k \binom{n-r}{k} \left[{n-1-k \atop m-1}\right]_{r-1} k!. \tag{29}$$

Recall that $\left[{n \atop m}\right]_1 = \left[{n \atop m}\right]_0$ for $n > 0$, so that

$$\left[{n \atop m}\right] = \sum_k \binom{n-1}{k} \left[{n-1-k \atop m-1}\right] k!, \qquad n > 0, \tag{30}$$

an identity that appears in Comtet [4, eq. 5.6c], and also in Knuth [10, eq. 1.2.6(52a)].

Lemma 13.

$$\left\{{n \atop m}\right\}_r = \sum_k \binom{n-r}{k} \left\{{n-p-k \atop m-p}\right\}_{r-p} p^k, \qquad r \geq p \geq 0. \tag{31}$$

Proof: By combinatorial arguments analogous to the proof of Lemma 11. ∎

The counterpart of equation (27) is [2, eq. 3.2]

$$\left\{{n \atop m}\right\}_r = \sum_k \binom{n-r}{k} \left\{{k \atop m-r}\right\} r^{n-r-k}, \tag{32}$$

which shows that $\left\{{n+r \atop m+r}\right\}_r$, for $m, n \geq 0$ is also a polynomial of degree n-m in r, whose leading coefficient is $\binom{n}{m}$. As before this implies a generalization of Lemma 13:

Theorem 14.

$$\left\{{n+r \atop m+r}\right\}_r = \sum_k \binom{n}{k} \left\{{n-k+p \atop m+p}\right\}_p (r-p)^k. \tag{33}$$

∎

The counterparts of equations (29) and (30) are

$$\left\{{n \atop m}\right\}_r = \sum_k \binom{n-r}{k} \left\{{n-1-k \atop m-1}\right\}_{r-1} = \sum_k \binom{n-r}{k} \left\{{r-1+k \atop m-1}\right\}_{r-1}; \tag{34}$$

and

$$\left\{{n \atop m}\right\} = \sum_k \binom{n-1}{k} \left\{{k \atop m-1}\right\}, \qquad n > 0, \tag{35}$$

which is a well known expansion.

§8 Exponential generating functions

Theorem 15. *The r-Stirling numbers of the first kind have the following "vertical" exponential generating function*

$$\sum_k \begin{bmatrix} k+r \\ m+r \end{bmatrix}_r \frac{z^k}{k!} = \begin{cases} \frac{1}{m!}\left(\frac{1}{1-z}\right)^r \left(\ln\left(\frac{1}{1-z}\right)\right)^m, & m \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Proof: The above exponential generating function can be decomposed into the product of two exponential generating functions, namely

$$\frac{1}{m!}\left(\ln\left(\frac{1}{1-z}\right)\right)^m = \sum_k \begin{bmatrix} k \\ m \end{bmatrix} \frac{z^k}{k!},$$

and

$$\left(\frac{1}{1-z}\right)^r = \sum_k \binom{k+r-1}{k} z^k = \sum_{k\geq 0} r^{\overline{k}} \frac{z^k}{k!}.$$

Their product is

$$\sum_n \frac{z^n}{n!} \sum_k \binom{n}{k} \begin{bmatrix} k \\ m \end{bmatrix} r^{\overline{n-k}} = \sum_n \frac{z^n}{n!} \begin{bmatrix} n+r \\ m+r \end{bmatrix}_r$$

by equation (27). ∎

The above theorem'irnplics the double generating function [2, eq. 5.3]

$$\sum_{k,m} \begin{bmatrix} k+r \\ m+r \end{bmatrix}_r \frac{z^k}{k!} t^m = \left(\frac{1}{1-z}\right)^{r+t}. \tag{37}$$

Theorem 16. *The r-Stirling numbers of the second kind have the following exponential generating function* [2, eq. 3.9]

$$\sum_k \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \frac{z^k}{k!} = \begin{cases} \frac{1}{m!} e^{rz}(e^z - 1)^m, & m \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

Proof: Similar to the the proof of Theorem 11, using the expansions

$$e^{rz} = \sum_k r^k \frac{z^k}{k!},$$

and

$$\frac{1}{m!}(e^z - 1)^m = \sum_k \begin{Bmatrix} k \\ m \end{Bmatrix} \frac{z^k}{k!},$$

together with equation (32). ∎

The double generating function for r-Stirling numbers of the second kind is

$$\sum_{k,m} \left\{ {k+r \atop m+r} \right\}_r \frac{z^k}{k!} t^m = \exp\bigl(t(e^z - 1) + rz\bigr). \tag{39}$$

§9 Identities from ordinary generating functions

Theorem 17. *The r-Stirling numbers of the first kind satisfy*

$$\left[{n \atop m} \right]_r = \sum_k \left[{p \atop p-k} \right]_r \left[{n \atop m+k} \right]_p, \qquad r \le p \le n. \tag{40}$$

Proof: From equation (24)

$$z^{r-p}(z+r)\ldots(z+p-1) = \sum_k \left[{p \atop p-k} \right]_r z^{-k}.$$

Express the product

$$z^{r-p}(z+r)\ldots(z+p-1)z^{p-n}(z+p)\ldots(z+n-1) = \sum_k \left[{n \atop n-k} \right]_r z^{-k}$$

as the convolution of the two generating functions and equate the coefficient of z^{m-n} on both sides. ∎

Theorem 18. *The r-Stirling numbers of the second kind satisfy*

$$\left\{ {n \atop m} \right\}_r = \sum_k \left\{ {p+k \atop p} \right\}_r \left\{ {n-k \atop m} \right\}_{p+1}, \qquad r \le p < n. \tag{41}$$

Proof: From *(25)*

$$\frac{1}{(1-rz)\ldots(1-pz)(1-(p+1)z)\ldots(1-nz)} = \sum_k \left\{ {n+k \atop k} \right\}_r z^k.$$

Expressing this product as a convolution we obtain

$$\left\{ {n+m \atop n} \right\}_r = \sum_k \left\{ {p+k \atop p} \right\}_r \left\{ {n+m-k \atop n} \right\}_{p+1},$$

and the theorem follows by suitable changes of variable.

Theorem 19. *The r-Stirling numbers of the first kind satisfy*

$$(-1)^r \begin{bmatrix} n \\ m \end{bmatrix}_r = \sum_k \begin{bmatrix} n \\ m-r+k \end{bmatrix}_p \begin{Bmatrix} k-1 \\ r-1 \end{Bmatrix}_p (-1)^k, \qquad n \geq r > p \geq 0. \tag{42}$$

Proof: From equation (24)

$$\sum_m \begin{bmatrix} n \\ m \end{bmatrix} z^m = z^r(z+r)\ldots(z+n-1) = \frac{z^r(z+p)\ldots(z+n-1)}{(z+p)\ldots(z+r-1)}, \qquad n \geq r > p \geq 0.$$

Let $t = -1/z$. Then

$$\frac{z^{r-p}}{(z+p)\ldots(z+r-1)} = \frac{1}{(1-pt)\ldots(1-(r-1)t)} = (-z)^{r-1}\sum_i \begin{Bmatrix} i \\ r-1 \end{Bmatrix}_p (-z)^{-i}$$

by equation (25). Hence

$$\begin{aligned} \sum_m \begin{bmatrix} n \\ m \end{bmatrix} z^m &= \sum_i \begin{Bmatrix} i \\ r-1 \end{Bmatrix}_p (-z)^{r-1-i} \sum_j \begin{bmatrix} n \\ j \end{bmatrix}_p z^j \\ &= (-1)^{r-1}\sum_m z^m \sum_k \begin{Bmatrix} k \\ r-1 \end{Bmatrix}_p \begin{bmatrix} n \\ m-r+1+k \end{bmatrix}_p (-1)^k. \end{aligned}$$

∎

In particular for $p = 0$ we have an alternative expression for the r-Stirling numbers of the first kind in terms of regular Stirling numbers of both kinds,

$$(-1)^r \begin{bmatrix} n \\ m \end{bmatrix}_r = \sum_k \begin{bmatrix} n \\ m-r+k \end{bmatrix} \begin{Bmatrix} k-1 \\ r-1 \end{Bmatrix} (-1)^k, \qquad n \geq r \geq 1. \tag{43}$$

This, combined with (27), gives an identity involving only regular Stirling numbers

$$\sum_k \binom{n}{k} \begin{bmatrix} k \\ m \end{bmatrix} r^{\overline{n-k}} = \sum_k \begin{bmatrix} n+r \\ m+r+k \end{bmatrix} \begin{Bmatrix} k+r-1 \\ r-1 \end{Bmatrix} (-1)^k, \qquad n \geq 0, r \geq 1. \tag{44}$$

The last equation is a polynomial identity in r. For $r = 1$, we obtain equation (30) again.

Theorem 20. *The r-Stirling numbers of the second kind satisfy*

$$(-1)^r \begin{Bmatrix} n \\ m \end{Bmatrix}_r = \sum_k [\ ;I,(" \begin{Bmatrix} -r+k \\ m \end{Bmatrix}_p (-1)^k, \qquad n \geq r \geq p \geq 0. \tag{45}$$

Proof: **The** ordinary generating function of the r-Stirling numbers of the second kind can be **rewritten as**

$$\frac{z^m}{(1-rz)\ldots(1-mz)} = \frac{z^m(1-pz)\ldots(1-(r-1)z)}{(1-pz)\ldots(1-mz)}.$$

Putting $t = -1/z$

$$(1-pz)\ldots(1-(r-1)z) = t^{p-r}(t+p)\ldots(t+r-1) = \sum_i \left[{r \atop i}\right]_p (-z)^{r-i},$$

so that

$$\sum_n \left\{{n \atop m}\right\}_r z^n = \sum_i \left[{r \atop i}\right]_p (-z)^{r-i} \sum_j \left\{{j \atop m}\right\}_p z^j,$$

and the result follows by equating the coefficient of z^n on both sides. ∎

The counterpart of equations (43) and (44) is obtained by making p = 0 in (45). We get

$$(-1)^r \left\{{n \atop m}\right\}_r = \sum_k \left[{r \atop k}\right]\left\{{n-r+k \atop m}\right\}(-1)^k, \qquad n \geq r, \tag{46}$$

the alternate expression for r-Stirling numbers of the second kind in terms of regular Stirling numbers of both kinds. This formula combined with (32), gives an identity in regular Stirling numbers only:

$$\sum_k \binom{n}{k}\left\{{k \atop m}\right\} r^{n-k} = \sum_k \left\{{n+r-k \atop m+r}\right\}\left[{r \atop r-k}\right](-1)^k, \qquad n, r \geq 0, \tag{47}$$

which is a polynomial identity in r. For $r = 1$, this is equation (35).

Theorem 21. *The r-Stirling numbers of the first kind have the "horizontal" generating function (2,* eq. 5.8]

$$(x+r)^{\overline{n}} = \sum_k \left[{n+r \atop k+r}\right]_r x^k, \qquad n \geq 0. \tag{48}$$

Proof: Replacing in equation (24) n by $n+r$ and z by x, we obtain

$$\sum_k \left[{n+r \atop k}\right]_r x^k = x^r (x+r)^{\overline{n}},$$

and the result follows. ∎

Note the equivalent formulation of Theorem 48

$$(x-r)^{\underline{n}} = \sum_k \left[{n+r \atop k+r}\right]_r (-1)^{n-k} x^k, \qquad n \geq 0. \tag{49}$$

Theorem 22. *The r-Stirling numbers of the second kind have the "horizontal" generating function* [2, cq. 3.4]

$$(x+r)^n = \sum_k \left\{ {n+r \atop k+r} \right\}_r x^{\underline{k}}, \qquad n \geq 0. \tag{50}$$

Proof: Usc the identity

$$e^{(x+r)t} = e^{rt}(1+(e^t-1))" = e^{rt} \sum_{k\geq 0} \frac{(e^t-1)^k x^{\underline{k}}}{k!}$$

and Theorem 12, to obtain

$$e^{(x+r)t} = \sum_{n\geq 0} \frac{t^n}{n!} \sum_k \left\{ {n+r \atop k+r} \right\}_r x^{\underline{k}}.$$

∎

The equivalent forrn of Theorem 50 is

$$(x-r)^n = \sum_k \left\{ {n+r \atop k+r} \right\}_r (-1)^{n-k} x^{\overline{k}}, \qquad n \geq 0. \tag{51}$$

§10 Identities from exponential generating functions

The following two theorems are an immediate consequence of the generating functions (36) and (38).

Theorem 23. *The r-Stirling numbers of the first kind satisfy*

$$\binom{l+m}{m} \left[{n+r+s \atop l+m+r+s} \right]_{r+s} = \sum_k \binom{n}{k} \left[{k+r \atop l+r} \right]_r \left[{n-k+s \atop m+s} \right]_s. \tag{52}$$

∎

Theorem 24. *The r-Stirling numbers of the second kind satisfy* [2, eq. 3.11]

$$\binom{l+m}{m} \left\{ {n+r+s \atop l+m+r+s} \right\}_{r+s} = \sum_k \binom{n}{k} \left\{ {k+r \atop l+r} \right\}_r \left\{ {n-k+s \atop m+s} \right\}_s. \tag{53}$$

These theorems have also a combinatorial interpretation. For Theorem 23 consider permutations of the set $\{1,\ldots,n+r+s\}$ such that $1,\ldots,r+s$ are in distinct cycles, each cycle is colorcd cither red or green, the cycles containing $1,\ldots,r$ arc all green, and the cycles containing $r+1,\ldots,r+s$ arc all red. The total number of such permutations with $l+r$ green cycles and $m+s$ rcd cycles is $\binom{l+m}{m}\left[{n+r+s \atop l+m+r+s}\right]_{r+s}$ because each permutation with $l+m+r+s$ cycles can be colored in $\binom{l+m}{m}$ ways. On the other hand, we can first decide which k elerncnts, besides $1,\ldots,r$, should be in the $l+r$ grecn cycles; the remaining $n-k+s$ elements must form the $m+s$ red cycles. Theorem 24 has a similar intcrprctation.

§ 11 Generalized or thogonality

Theorem 25. *The r-Stirling numbers satisfy* [2, eq. 6.3]

$$\sum_k \left[{n+r \atop k+r}\right]_r \left\{{k+p \atop m+p}\right\}_p (-1)^k = (-1)^m \binom{n}{m} (r-p)^{\overline{n-m}}, \tag{54}$$

$$\sum_k \left[{k+p \atop m+p}\right]_p \left\{{n+r \atop k+r}\right\}_r (-1)^k = (-1)^m \binom{n}{m} (r-p)^{n-m}. \tag{55}$$

Proof: By (48) and (51)

$$(x-p+r)^{\overline{n}} = \sum_k \left[{n+r \atop k+r}\right]_r (x-p)^k = \sum_k \left[{n+r \atop k+r}\right]_r \sum_i \left\{{k+p \atop i+p}\right\}_p (-1)^{k-i} x^{\overline{i}}$$

Equation (54) is obtained by comparing the coefficient of $x^{\overline{m}}$ on both sides. Similarly, consider the identity (from (50) and (49))

$$(x-p+r)^n = \sum_k \left\{{n+r \atop k+r}\right\}_r (x-p)^{\underline{k}} = \sum_k \left\{{n+r \atop k+r}\right\}_r \sum_i \left[{k+p \atop i+p}\right]_p (-1)^{k-i} x^i$$

and equate the coefficient of x^m on both sides to obtain (55). ∎

§ 12 The r-Stirling polynomials

We have seen that the r-Stirling numbers are polynomials in r. The r-Stirling polynomials are defined for arbitrary x as

$$R_1(n,m,x) = \sum_k \binom{n}{k} \left[{n-k \atop m}\right] x^{\overline{k}} \qquad \text{integer } m,n \geq 0, \tag{56}$$

and

$$R_2(n,m,x) = \sum_k \binom{n}{k} \left\{ {n-k \atop m} \right\} x^k \qquad \text{integer m,n} \geq 0. \tag{57}$$

In particular, by equations (27) and (32), when r is a positive integer, $R_1(n, m, r) = \left[{n+r \atop m+r}\right]$ and $R_2(n,m,r) = \left\{{n+r \atop m+r}\right\}$.

The r-Stirling polynomials have a combinatorial significance given by the following two theorerns.

Theorem 26. *The polynomial $R_1(n, m, x)$ enumerates the permutations of the set $\{1, \ldots, n+1\}$ having $m+1$ left-to-right minima by the number of right-to-left minima different from* 1.

Proof: Expanding raising powers, we get

$$\begin{aligned} R_1(n,m,x) &= \sum_k \binom{n}{k} \left[{n-k \atop k}\right] x^{\overline{k}} = \sum_k \binom{n}{k} \left[{n-k \atop k}\right] \sum_i \left[{k \atop i}\right] x^i \\ &= \sum_i x^i \sum_k \binom{n}{k} \left[{n-k \atop m}\right] \left[{k \atop i}\right] \end{aligned}$$

All the left-to-right minima except 1 must occur at the left of 1, while all right-to-left minima except 1 must occur at the right of 1. Hence the number of permutations having $m+1$ left-to-right minima, $i+1$ right-to-left rninima, and k elements at the right of 1 is $\binom{n}{k}\left[{n-k \atop m}\right]\left[{k \atop i}\right]$.

Note that by Theorern 23 used in the above expansion we obtain

$$R_1(n,m,x) = \sum_i \binom{m+i}{m} \left[{n \atop m+i}\right] x^i. \tag{58}$$

∎

Theorem 27. *The polynomial $R_2(n,m,x)$ enumerates the partitions of the set $\{1, \ldots, n+1\}$ into m non-empty subsets, by the number of elements different from* 1*, in the set containing* 1.

Proof: Obvious, from definition (57). ∎

The r-Stirling polynomials have remarkably simple expressions in operator notation, which generalize the well known formulae for regular Stirling numbers.

Theorem 28.

$$R_1(n,m,x) = \frac{1}{m!} \frac{\partial^m}{\partial x^m} x^{\overline{n}}. \tag{59}$$

Proof: From (48)

$$m! R_1(n,m,x) = \frac{\partial^m}{\partial y^m} (x+y)^{\overline{n}} \Big|_{y=0} = \frac{\partial^m}{\partial x^m} x^{\overline{n}}.$$

∎

Theorem 29.

$$R_2(n, m, x) = \frac{1}{m!}\Delta^m x^n. \tag{60}$$

Proof: Similar to the proof of Theorem 28. A direct proof is based on combining (8) and (18) to obtain

$$\left\{ {n+r \atop m+r} \right\}_r = (m+1)\left\{ {n+r-1 \atop m+r} \right\}_{r-1} + \left\{ {n+r-1 \atop m+r-1} \right\}_{r-1},$$

which implies

$$\Delta R_2(n, m-1, x) = mR_2(n, m, x)$$

and therefore

$$\Delta^m x^n = \Delta^m R_2(n, 0, x) = m!R_2(n, m, x).$$

∎

Corollary 30 [2, cq. 3.8]

$$R_2(n, m, x) = \frac{1}{m!}\sum_k \binom{m}{k}(-1)^{m-k}(x+k)^n. \tag{61}$$

Proof: Use the formal expansion

$$\Delta^m = (\mathbf{E}-1)^m = \sum_k \binom{m}{k}(-1)^{m-k}\mathbf{E}^k,$$

where $\mathbf{E}$ is the shift operator, $\mathbf{E}f(x) = f(x+1)$. ∎

Because of these properties the r-Stirling polynomials, especially the r-Stirling polynomials of the second kind, were studied in the framework of the calculus of finite differences. Nielsen [19, chap. 12] developped a large number of formulae relating $R_2(n, m, x)$ to the Bernoulli and Euler polynomials. (Nielsen's notation is $A^n_m(x) = m!R_2(n, m, x)$.) Carlitz [3] showed by different means that the r-Stirling polynomials are related to the Bernoulli polynomials of higher order and also studied the representation of $R_1(n, n-k, x)$ and of $R_2(n, n-k, x)$ as polynomials in n. The asymptotics of the numbers $\left\{ {n+r \atop m+r} \right\}_r$ were derived in [8]. Broder [1] obtained several formulas relating r-Stirling polynomials of the second kind to Abelian sums [23, §1.5], for example

$$\sum_k \binom{n}{k}(x+k)^{k+p}(y+n-k)^{n-k} = \sum_k \binom{n}{k} k!(x+y+n)^{n-k}R_2(k+p, k, x), \qquad p \geq 0. \tag{82}$$

§13 T-Stirling numbers of the second kind and Q-series

Knuth defined the Q-series as

$$Q_n(a_1, a_2, \ldots) = \sum_{k \geq 1} \binom{n}{k} k! n^{-k} a_k. \tag{63}$$

For a certain sequence $a_1, a_2, \ldots$, this function depends only on n. In particular, $Q_n(1, 1, 1, \ldots)$ is denoted Q(n).

Q-series arc relevant to many problems in the analysis of algorithms [13], for instance representation of equivalence relations [16], hashing [12, §6.4], interleaved memory [15], labelled trees counting [21], optimal cacheing [13], permutations in *situ* [25], and random mappings [11, §3.1].

It can be shown that the Q-series satisfy the recurrence

$$Q_n(a_1, 2a_2, 3a_3, \ldots) = nQ_n(a_1, a_2 - a_1, a_3 - a_2, \ldots). \tag{64}$$

Theorem 31.

$$Q_n\left(\left\{ {h \atop 1} \right\}_r, 2\left\{ {h+1 \atop 2} \right\}_r, \ldots\right) = n^h \frac{n^{\underline{r}}}{n^r}. \tag{65}$$

Proof: Note that from (8)

$$\left\{ {k+h \atop k} \right\}_r - \left\{ {k+h-1 \atop k-1} \right\}_r = k\left\{ {k+h-1 \atop k} \right\}_r$$

for all $k \geq 0$ if $h > 0$. Applying this together with (64) $h - 1$ times, we obtain

$$\begin{aligned} Q_n\left(\left\{ {h \atop 1} \right\}_r, 2\left\{ {h+1 \atop 2} \right\}_r, \ldots\right) &= n^{h-1} Q_n\left(\left\{ {1 \atop 1} \right\}_r, 2\left\{ {2 \atop 2} \right\}_r, \ldots\right) \\ &= n^{h-1} Q_n(\delta_{1 \geq r}, 2\delta_{2 \geq r}, \ldots) \end{aligned}$$

One more application of (64) for $r > 0$ results in

$$n^h Q_n(\delta_{1,r}, \delta_{2,r}, \ldots) = n^h \frac{n^{\underline{r}}}{n^r}$$

and for $r = 0$ results in

$$n^h Q_n(1, 0, 0, \ldots) = n^h$$

Corollary 32. *Let*

$$f(k) = \sum_r a_r \left\{ {k+h-1 \atop k} \right\}_r ,$$

where a_r *depends only* on r. *Then*

$$Q_n(f(1), 2f(2), 3f(3), \ldots) = n^h\big(Q_n(a_1, a_2, a_3, \ldots) - a_0\big). \tag{66}$$

∎

In [13] Knuth introduced the half integer Stirling numbers $\left\{ {n+1/2 \atop k} \right\}$. These numbers satisfy the recurrence

$$\begin{aligned} \left\{ {n+1/2 \atop k} \right\} &= 0, && n < 0, \\ \left\{ {n+1/2 \atop n} \right\} &= n, && n \geq 0, \\ \left\{ {n+1/2 \atop k} \right\} &= k\left\{ {n-1/2 \atop k} \right\} + \left\{ {n-1/2 \atop k-1} \right\}, && k \neq n, n \geq 0, \end{aligned} \tag{67}$$

which has the form of (15) and therefore has the solution

$$\left\{ {n+1/2 \atop k} \right\} = \sum_{r \geq 1} \left\{ {n \atop k} \right\}_r . \tag{68}$$

Hence, by Corollary 32

$$Q_n\left(\left\{ {h+1/2 \atop 1} \right\}, 2\left\{ {h+3/2 \atop 2} \right\}, \ldots \right) = n^h Q_n(1, 1, \ldots) = n^h Q(n), \tag{69}$$

which is in fact the equation used to define the half-integer Stirling numbers in [13].

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