April 1987

Report No. STAN-CS-87-1153

Optimum Grip of a Polygon

by

Xanthippi Markenscoff

Christos H. Yapadimitriou

Department of Computer Science

Stanford University
Stanford, CA 94305

OPTIMUM GRIP OF A POLYGON

Xanthippi Markeuscoff † and Christos H. Papadimitriou‡

ABSTRACT: It has *been shown by Baker, Fortune, and Grosse that any two-dimensional polygonal object can be prehended stably with three fingers, so that its weight (along the third dimension) is* balanced. Besides, in *this paper* we *show* that *form* closure of a *polygonal* object *can be* achieved *by four fingers (previous proofs were not complete). We formulate* and *solve the problem of finding the optimum stable grip* or *form* closure of *any given polygon.* For sta*ble grip it* is *most* natural to *minimize* the *forces* needed to balance through *friction the object's weight along* the third dimension. *For form closure, we minimize the* worst-case *forces needed to balance any* unit force acting on the center of *gravity* of *the object.* The *mathematical techniques used in the two instances are an interest mix of Optimization and Euclidean geometry.* Our *results lead to algorithms for the efficient computation of the optimum grip in* each case.

1. INTRODUCTION

Suppose that a. robot hand with three fingers must prehend the two-dimensional object shown in Figure 1. By "prehending" we mean that the fingers should apply compressional forces on three points at the perimeter of the object in directions normal to the perimeter so that (a) the three forces should be in equilibrium on the plane of the object, and (b) friction forces along the third dimension should balance the weight of the object. In other words, the object to be grasped is a thin plaque with the indicated cross-section: its weight, applied to the center of gravity G, is assumed to be also along the third dimension. It has been shown [BFG] that, for polygonal objects, such a grip always exists that is stable in terms of the potential energy of the fingers (considered as springs); in Figure 1(a) through (e) we show several distinct possibilities.

It. is perhaps intuitively clear to the reader that the grip in Figure 1(a) is somehow better than that of Figure 1(e). But which of these five grips is "the best"? And what does this mean exactly? In this paper we formalize this intuition by defining a. measure of the *quality* of a grip. Our notion of quality is related to the *compressional forces required in order to balance* the *weight* of the *object*. The smaller the forces the robot hand must exert on the object, the better the grip. Minimizing the necessary forces seems the natural criterion, and was first. suggested by [BFG]. Large forces would mean unnecessary stress and deformation of both the object and the robot, hand, and higher deformation energy; minimum force prehension seems to coincide with our intuitive idea of a. "firm" or "good" grip of an object.

† Department of Mechanical and Environmental Engineering, University of California. Santa. Barbara.

‡ Depts. of Computer Science and Operations Research, Stanford University.

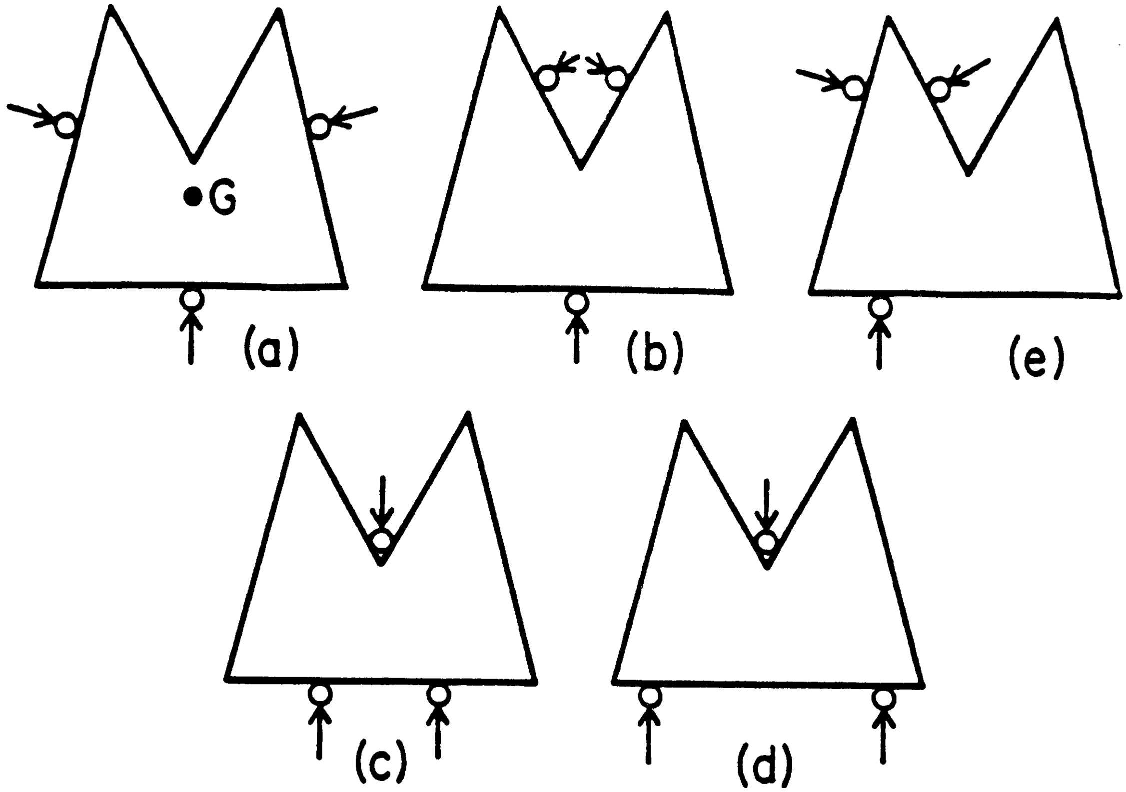

Figure 1: Which grip is the best?

In the first part of this paper we present an analytical method for calculating the quality of a. grip of a. polygonal object. by three fingers. Our method leads to a realistic algori t hm, whereby we can analyze any reasonably complex object from this point of view rather rapidly and compute the optimum grip by examining a finite number of cases (growing as the cube of the number of sides of the polygon).

In the second part we attack the more difficult problem of optimizing the form closure of an object [La, SR]. A grip is said to be a form closure if compressional forces along the fingers can balance *any* wrench acting anywhere on the object,. It is claimed in [La] that form closure of any polygon can be achieved with four fingers, but it is not shown in that paper how to actually realize this by compressional forces normal to the perimeter. We show that, indeed, four fingers can achieve form closure of any polygon (in a separate paper [MNP] we actually show that form closure by four fingers is possible for any compact object in two dimensions except for the circle, and that a similar result holds for seven fingers in three dimensions). We also define form *semiclosure* to be a grip that can balance any force through the center of gravity of the object. Four fingers are necessary and sufficient for form closure and semiclosure of polygons; the polygons for which three fingers are enough have a. unique form semiclosure, and thus optimizing the grip is not meaningful. Thus, we concentrate on finding the optimum form semiclosure of any polygon using four fingers. This is an involved optimization problem, which we solve by an interesting sequence of reductions using linear programming duality and some intricate plane geometry.

The rest of this paper is organized as follows: In Section 2 we analyse the problem of optimum grip by three fingers in the special case in which the object is a triangle; since we assume that three fingers are available, this is the basic case to be resolved. We derive a formula for the quality of a given grip, discuss the problem of optimizing the grip, and illustrate the process by examples. Optimum grip of any convex polygon can be reduced to the triangle case. We also study certain subproblems that arise when the object has concave angles and parallel sides, and give a complete procedure for computing the optimum grip of a polygon by three fingers; we illustrate this algorithm by analysing the example of Figure 1. In Section 3 we discuss the notions of form closure and semiclosure of polygons, and state and solve the corresponding optimization problem.

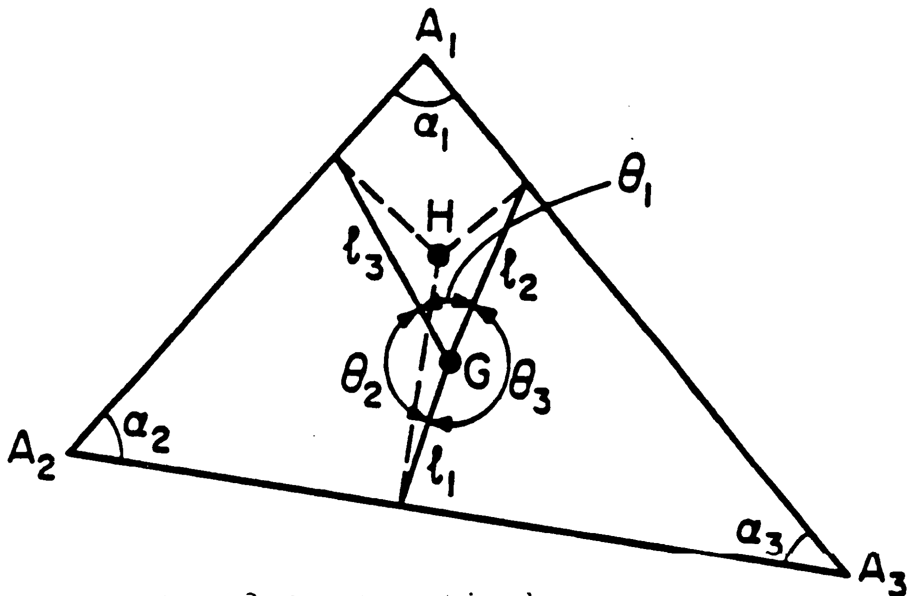

Figure 2: Grasping a triangle.

2. OPTIMUM GRIP OF A POLYGON

2.1 Prehending a Triangle

Prehending a triangle (or any convex polygon) by three fingers entails choosing three points in the interiors of the edges of the triangle. and applying normal compressional forces on the points of contact so that the forces are in equilibrium. Suppose that we choose any single point H on the plane of the triangle (see Figure 2) and project it on the edges: If the projections lie in the interiors of the edges, they constitute a stable grip. As the authors of [BFG] observed, such a choice is always possible, by taking H to be the center of the inscribed circle of the polygon (we shall see that this is in genera.1 not optimum). We shall identify a grip with the corresponding point H. For a triangle, the space of all possible grips H is defined as the intersection of three open strips, each perpendicular to an edge of the triangle (Figure 3).

Now fix such a grip (which we shall identify with the point H). The condition

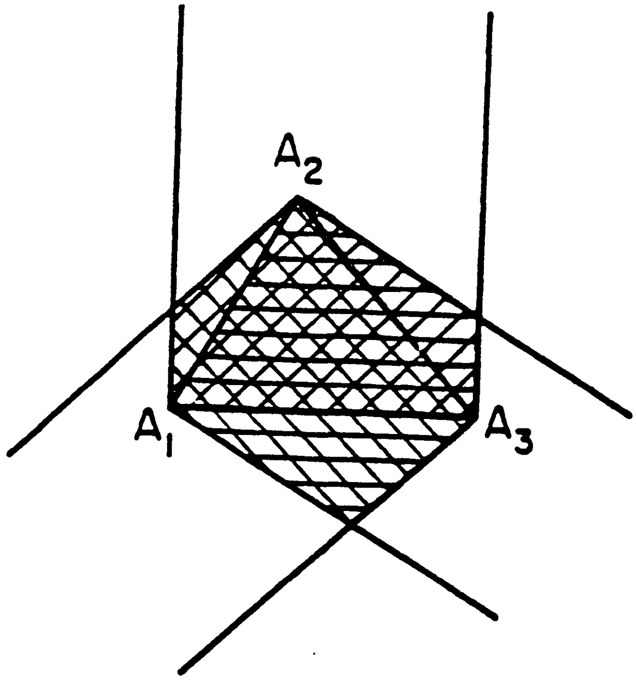

Figure 3: The domain of H.

for equilibrium of the three forces on the plane is the following:

$$\frac{F_1}{\sin\alpha_1} = \frac{F_2}{\sin\alpha_2} = \frac{F_3}{\sin\alpha_3} = f, \tag{1}$$

where f is the constant of proportionality.

We assume that the weight of the triangle (applied to the center G of gravity, which a given point in the interior of the triangle, in general not necessarily its geometric center) is a force along the third dimension. This weight must be balanced by frictional forces along the third dimension. Such forces are modeled as usual as being of magnitude bounded by the applied force times a. friction coefficient μ. Thus, we can always achieve equilibrium in the third dimension by appropriately scaling the forces.

$$\rho_1 F_1 + \rho_2 F_2 + \rho_3 F_3 = 1 \tag{2}$$

$$|\rho_1|. |\rho_2|, |\rho_3| \leq \mu \tag{3}$$

Notice that we have taken, without loss of generality, the weight to be one. The equilibrium of moments in the third direction yields the final equation:

$$\frac{\rho_1 F_1 l_1}{\sin\theta_1} = \frac{\rho_2 F_2 l_2}{\sin\theta_2} = \frac{\rho_3 F_3 l_3}{\sin\theta_3} = m \tag{4}$$

Notice that the forces are completely determined by (1) when f is; thus our task is to minimize the absolute value of f subject to these equations and inequalities. By minimizing f we actually minimize the largest force, the sum of all three, and in fact any nondecreasing function of the F's. Substituting (4) into (2) we obtain

$$m = \frac{1}{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\sin\theta_3}{l_3}},$$

and hence

$$f = \frac{1}{\rho_i}\,\frac{\frac{\sin\theta_i}{l_i\sin\alpha_i}}{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\sin\theta_3}{l_3}},\ i = 1,2,3. \tag{5}$$

One brief parenthesis on the possibility of the denominator of (5) being zero: This cannot happen if all projections are in the interior of the edges, as required by our definition of a. grip. This can be shown as follows: Suppose that, indeed, the denominator is zero. This means that one of the sines, say that of θ_1, is negative (see Figure 4). If we define x to be the part of l_1 up to the point that GH_1 meets H_3H_2, it is well-known that $\sin\theta_1/l_1 + \sin\theta_2/l_2 = \sin(\theta_1 + \theta_2)/x$. Comparing this equation with the denominator of (5) we conclude that $l_1 = x$, or H_1 must lie on the line H_3H_2. We claim however that this cannot happen if a.11 H_i's are in the interiors of the edges of the triangle (our claim finally establishes that the denominator is never zero). To prove our claim, consider any such grip H. If two of the projections H_1, H_2, H_3 coincide, then H coincides with them and a. vertex of the triangle, so we can assume the projections are distinct. Suppose that the order in which the projections occur on the line is H_1, H_2, H_3, and consider the angle HH_2H_1 (Figure 4). If this angle is obtuse then H_3 is outside the triangle; if it is acute, then H_1 is outside; and if it is a right angle, then H_1 and H_3 coincide with vertices. Hence the denominator of (5) is always nonzero (and thus is alwavs positive, bv continuity).

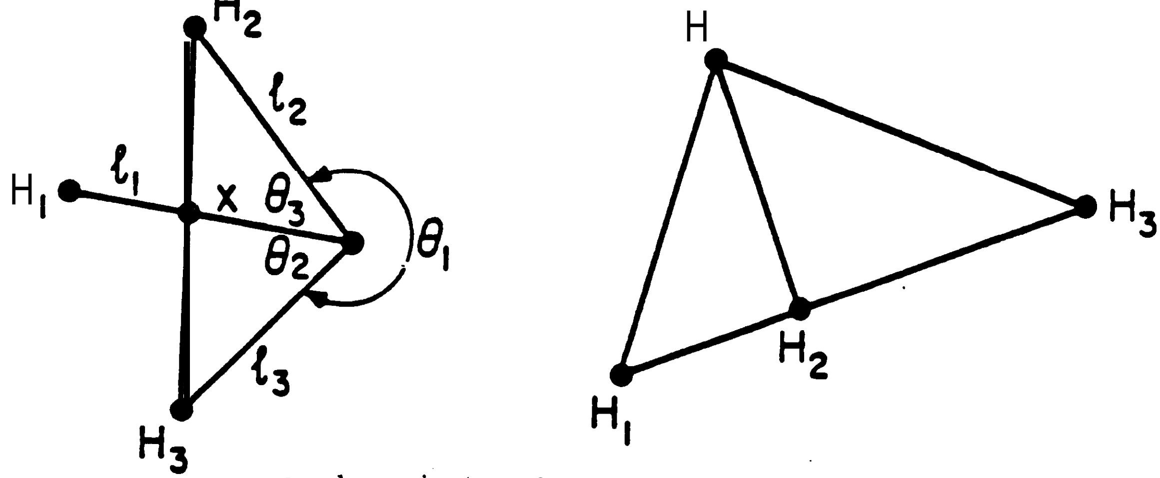

Figure 4: The denominator of (5) is positive.

To minimize f in equation (5) with respect to the ρ_i's and subject to inequalities (3) -assuming the grip H fixed- we take $|\rho_i|$ to be equal to μ, where i is the index for which

$$\left|\frac{\frac{\sin\theta_i}{l_i\sin\alpha_i}}{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\sin\theta_3}{l_3}}\right|$$

is maximum. Hence we define the quality of' the grip H to be the following quantity:

$$q(H) = \min_{i=1,2,3} \sin\alpha_i \frac{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\sin\theta_3}{l_3}}{\frac{|\sin\theta_i|}{l_i}}.$$

Notice that $f = \frac{1}{\mu q(H)}$, and thus a good grip is indeed one with a large q. Our nest. task is to find the point H in the region of Figure 3 such that q(H) is maximized.

Example 1: Consider the isosceles triangle in Figure 5. The optimum three-finger grip of this triangle consists of placing a finger at the middle of the basis, and two other fingers on the two sides, at the same height y; but how large should y be? The answer is, $1 + \sin\alpha$ times the height g of the center of gravity.

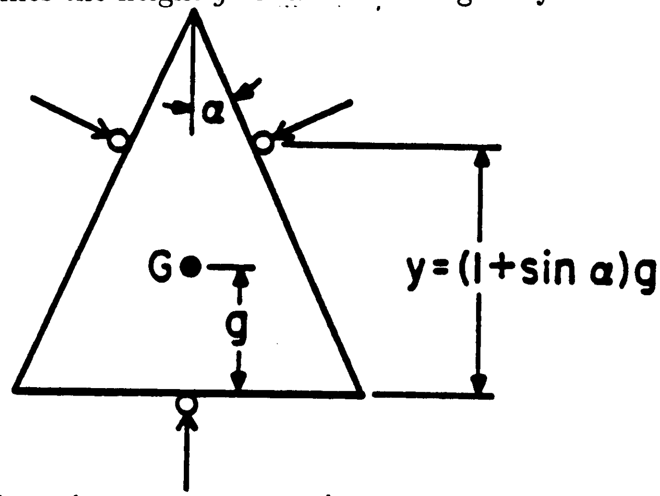

Figure 5: Isosceles triangle example.

To see this, we calculate q(H) from its definition for this triangle and any H on the axis of symmetry:

$$q(H) = \min\{\sin 2\alpha(1 + \frac{2g\sin\theta}{\cos\theta}), \cos\alpha(2 + \frac{l\sin 2\theta}{g\sin\theta})\},$$

and thus

$$q(H) = 2\cos\alpha\min\{\sin\alpha(1 + \frac{g}{l\cos\theta}), 1 + \frac{l\cos\theta}{g}\}.$$

Thus the optimum is attained when $l\cos\theta = g\sin\alpha$, or $y = (1 + \sin\alpha)g$. ■

For a general triangle, $q(H)$ is differentiable except for the boundaries of the areas in which the minimum of (5) is attained by the same term. The partial derivatives of $q(H)$ with respect to the position x and y of H can be calculated from the following formula (refer to Figure 6):

$$\frac{\partial}{\partial x}\frac{\sin\theta_1 l_2}{\sin\theta_2 l_1} = \frac{z_2\sin\beta_2\sin\theta_1}{l_1 l_2\sin\theta_2} - \left(\frac{w_3\sin\beta_3}{l_3^2} - \frac{w_1\sin\beta_1}{l_1^2}\right)\frac{l_2\cos\theta_2\sin\theta_1}{l_1\sin\theta_2}$$

$$-\frac{z_1 l_2 \sin\beta_1 \sin\theta_1}{l_1^3 \sin\theta_2} + \left(\frac{w_3 \sin\beta_3}{l_3^2} - \frac{w_2 \sin\beta_2}{l_2^2}\right)\frac{l_2 \cos\theta_1}{l_1 \sin\theta_2}.$$

Here β_i is the angle formed by the ith side of the triangle with the x axis. For the partial derivative with respect to y we only substitute $\cos\beta_i$ for $\sin\beta_i$ in the above formula. Unfortunately, for a. general triangle there is little hope that we can find a. closed form solution for the optimum H. However, on the basis of an extensive number of experiments, $q(H)$ empirically seems to always be a smooth, convex function on the domain (Figure 3) of H. Hence, q(H) can be optimized by any conves programming algorithm. Computational results using a simple hill-climbing method show that we can find the optimum grip of any triangle within about .05 seconds of CPU time on a. DEC 20.

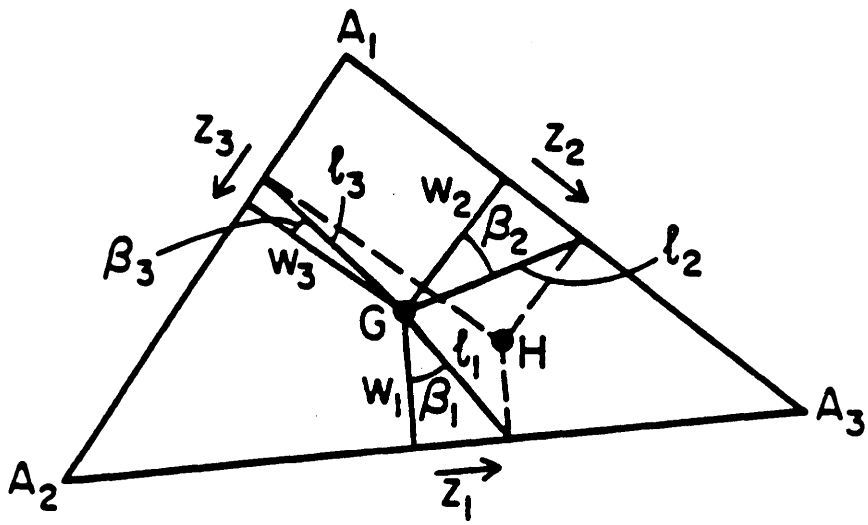

Figure G: Calculation of the partial derivatives.

2.2 Concave Vertices and Parallel Edges.

Once the three sides of contact have been fixed, prehending a conves polygon with no parallel sides by three fingers is no harder than grasping a triangle (only it has to be repeated for a.11 possible choices of the three sides of contact, and for each choice H is restricted so that its projection on each of the three sides is in the interior of the side). Thus, the optimum prehension of a. conves polygon with no parallel sides can be determined by fewer than n^3 repetitions of the procedure of the previous section. For general polygons, however, we must take care of two special issues: Concave vertices and parallel sides.

In a non-convex polygon, we can place a finger in a concave vertex, and exert on it any force. *as long as its direction forms obtuse angles with both sides of' the vertex* (i.e.. the force must lie within the shaded area in Figure 7(a)). Thus, with three fingers we may choose any three sides or concave vertices, and a. point. H such that. (a) It projects in the interior of all sides chosen, and (b) It forms obtuse angles with all sides of all concave vertices chosen (an example with two concave vertices and one side is shown in Figure 7(b); the shaded area is the set of a.11 feasible H's). The challenge is again to determine the best choices of sides and concave vertices, and point H, such that the necessary forces are the smallest possible.

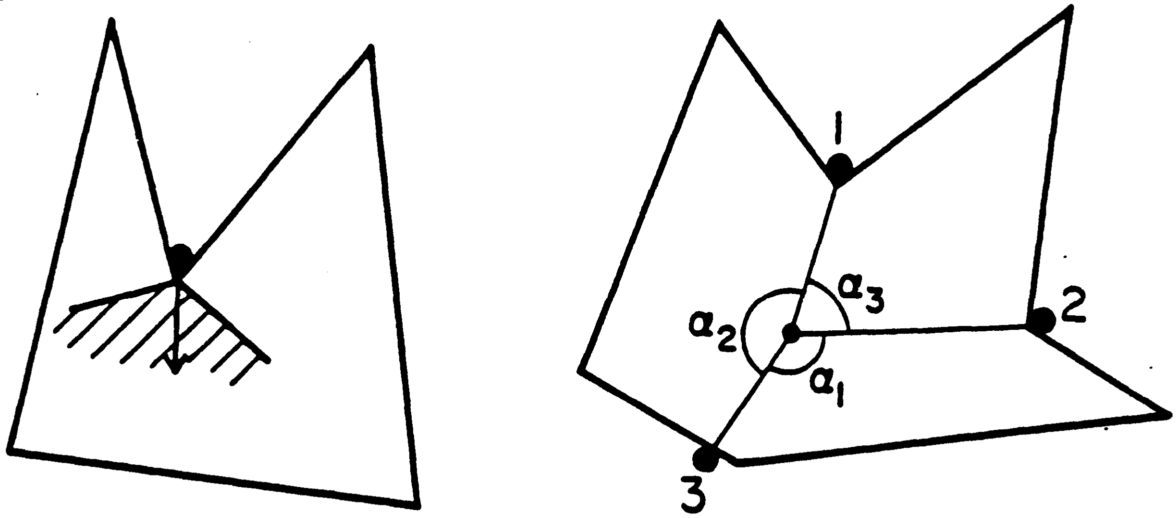

Figure 7: Concave Vertices.

Since there are at most n^3 choices of sides and concave vertices, it remains to see how to optimize the grip for each choice. It turns out that the equations are precisely the same as in the case of a. triangle (equations (1) through (5) in Section 2), with one crucial difference: The angles α_i are no longer fixed, and can vary with H. Accorclingly, it does not suffice to minimize f as before, and we must, fis precisely which nondecreasing function of the forces F_i *we* wish to optimize (their sum, their maximum, or whatever). Let us, for concreteness, choose the sum. The quantity to be maximized is therefore

$$\min_{i=1,2,3} \frac{\sin\alpha_i}{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3} \frac{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\theta_3}{3}}{\frac{|\sin\theta_i|}{l_i}}$$

Thus, a slight modification of the procedure for optimizing the grip of a triangle can be used to optimize over all points H with a fisecl set of three sides and concave vertices.

In general, however, two sides of a polygon may be parallel. In this case, it is possible to prehend the polygon by applying three fingers on the two sides (see Figure 8). If the situation is such that the center of gravity projects in the interiors of both sides, then, by symmetry, the optimum such grip is obtained by applying one force to the projection of the center of gravity on one of the sides, and two opposing forces at the same distance x on either direction from the projection of the center of gravity on the other side (see Figure 8). How large should x be? The answer is that it does *not matter*.

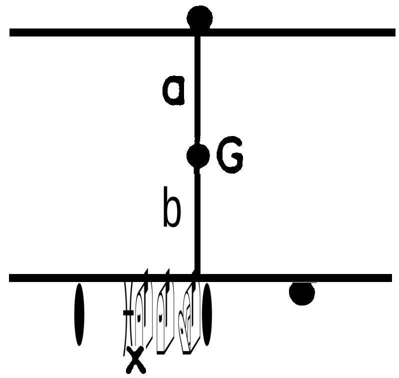

Figure 8: Parallel sides.

Equations (2) through (4) in this situation are the the **same,** only equation (1) becomes the following:

$$\frac{F_1}{2} = F_2 = F_3 = f \tag{1a}$$

Thus equation (5) becomes

$$q(H) = 2 + 2\min\{\frac{a}{b}, \frac{b}{a}\}$$

independent of x.

For the case in which G does not project on one of the two parallel sides, we have a more complicated situation (see Figure 9). Equations (2) through (4) are the same, only equation (1) becomes

$$\frac{F_1}{x_2 + x_3} = \frac{F_2}{x_3} = \frac{F_3}{x_2} = f \tag{1b}$$

and thus we must find the x, x_2, x_3 and $x_1 = x_2 + x_3$ which maximize

$$\min_{i=1,2,3} \frac{x_i}{x_1 + x_2 + x_3} \frac{\frac{\sin\theta_1}{l_1} + \frac{\sin\theta_2}{l_2} + \frac{\sin\theta_3}{l_3}}{\frac{|\sin\theta_i|}{l_i}}.$$

This task is similar (in fact, a. bit easier) than that of optimizing the grip of a triangle, examined in Section 2.

Example 2: Let us return to the polygon of Figure 1 (shown in detail in Figure 10). For the grip of Figure 10(a), our analysis of parallel sides yields sum of forces equal to 1.67, independent of x (where the unit of force is μ^{-1}). For Figure 10(b),

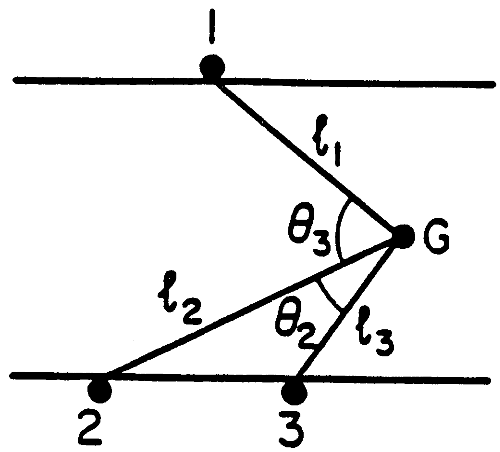

Figure 9: Parallel sides, G does not project on one.

our isosceles triangle example yields a. sum of forces of 1 with y = .629. The grip of Figure 10(c) has exactly the same equations as that of Figure 10(b), and thus a sum of 1 for the same y. Finally, the grip of Figure 10(d) is optimized (using the program of Section 2) when x = .471 y = .631 and y' = .5, with sum of forces 1.365. Thus, the grips of Figures 10(b) and 10(c) are the best with respect to the sum of forces. In fact, the situation does not change if we adopt the maximum force-used as our criterion. The corresponding numbers are .833, .393, .393 and .602. ■

3. OPTIMUM FORM SEMICLOSURE

3.1 Basic Concepts

Consider a. set of fingers (not necessarily three of them) acting on the perimeter of a polygon along the normal at each point of contact. We say that these fingers are a form *closure* of the object [La] if the following is true: Any wrench (force with its application point fixed) acting on the object can be balanced by adjusting the forces on the fingers accordingly, *always* keeping them compressive, i.e. positive. Notice immediately that form closure cannot be achieved by three fingers. In proof, recall that the equilibrium of the object is described by three equations (one for the forces along each of the two dimensions, and one for the moments). Since there are three variables (magnitudes of the forces along the fingers) there is a unique solution for each input wrench. If this solution happens to be positive, then there is an opposite wrench for which the solution is negative, and therefore that wrench cannot be balanced by positive forces. So, four fingers are necessary to achieve form closure (this was first pointed out in [La]). A similar argument establishes that three fingers are not sufficient for form semiclosure, unless the center of gravity projects in the interior of three sides of the polygon (which is not in genera.1 the case).

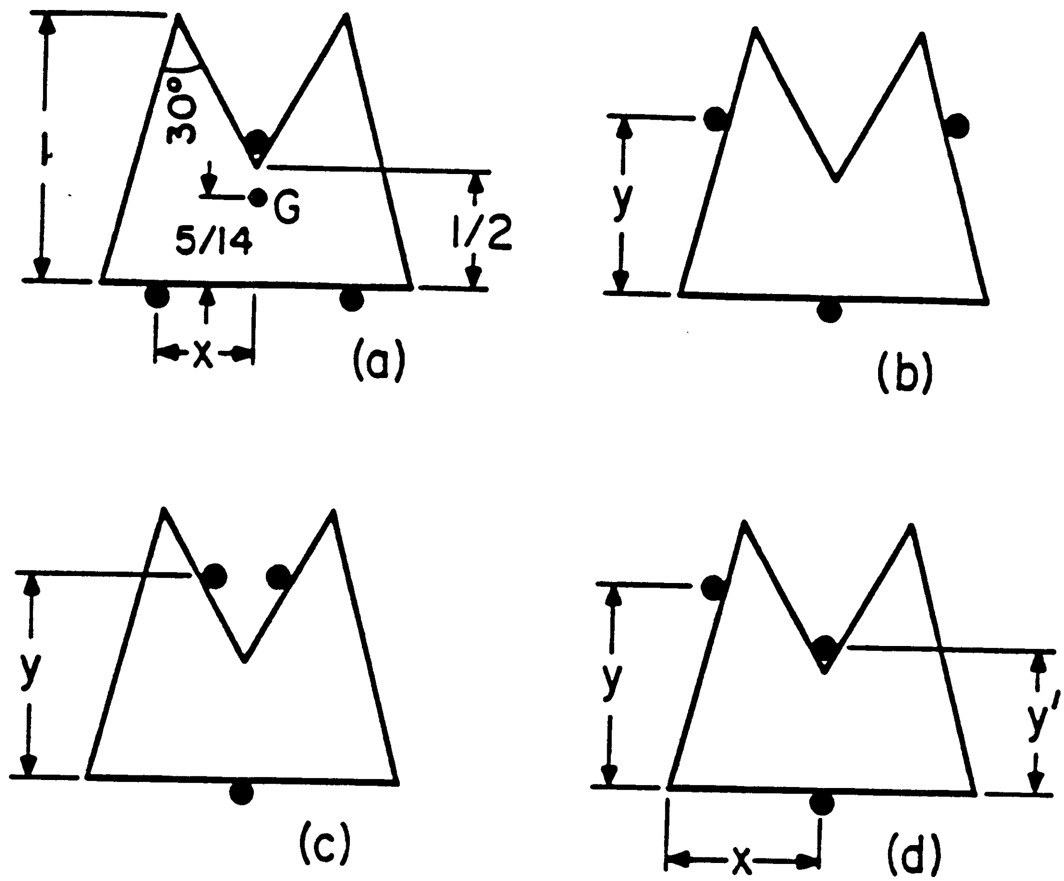

Figure 10: The example of Figure 1 resolved.

Furthermore, four fingers are enough.† In proof, consider the largest circle inscribed in the polygon [BFG]. It either touches the perimeter at three sides (or concave vertices, the argument remains the same), or at two parallel sides. In the first case, we place two fingers on two points of contact, and two more on the third side of contact, close to the point of contact and in the same distance d from it in either direction (Figure 11). The matrix of the three equations of

† Lakshminarayana [La] also argues that four fingers are enough, but does not describe how they can be placed so that they are normal to the perimeter. In another paper [MNP] we show that four fingers are enough for any planar object with piecewise smooth boundaries except for the circle, and extend this to three dimensions.

equilibrium is therefore of the form

$$\begin{array}{cccc} a & b & c & c \\ a' & b' & c' & c' \\ 0 & 0 & d & -d \end{array}$$

We claim that positive combinations of the columns of this matrix can achieve any 3-dimensional vector. To show this, suppose that we replace the last two columns with their sum, and consider the first two rows (the third row is zero). It is easy to see that any 2-dimensional vector can be achieved as a positive combination of these three vectors, since they form three angles none of which is greater than π (recall that they are normals at the three points of contact of the inscribed circle). Thus, to form any given 3-dimensional vector (i.e., to balance any wrench) by the four vectors, we first take care of the third coordinate by a positive multiple of the third or fourth column (clepencling on whether the third coordinate of the wrench is positive or negative), and then balance the remaining wrench by the first two rows as described in the previous sentence. If the circle touches the perimeter on two parallel sides, then a similar argument is possible [MNP].

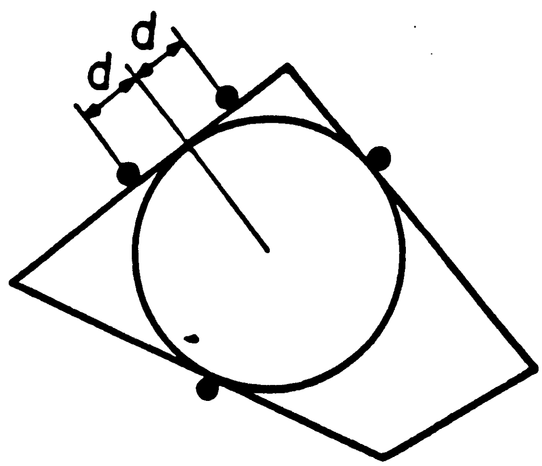

Figure 11: Achieving form closure.

So, four fingers are necessary and sufficient for form closure of a polygon. We next define an interesting variant of form closure. We say that a set of fingers achieves *form semiclosure* of a polygon [MNP] if any wrench *through the center of'* gravity *of* the *polygon* can be balanced by compressive forces normal to the perimeter at the points of contact. 0 bviously, this is a weaker condition when compared to form closure, and is of interest when we anticipate to balance only wrenches due to the weight of the object or to translational acceleration. Four fingers are still sufficient, of course, and there are polygons for which four are necessary (for example, an obtuse triangle whose center of gravity projects in the interior of only two sides, and outside the other one).

3.2 The Optimization Problem

There are usually many possible form semiclosure grips with four fingers, and they may vary drastically in terms of their "quality". One reasonable measure of the quality of a grip is how small forces are necessary in order to counter any unit wrench applied to the center of gravity of the object. In other words, we wish to *find* the *grip* which *performs* best in *the worst* case against an intelligent and *malevolent* opponent who tries to make us use large forces but can only apply a unit force on the object. In this subsection we shall formulate this optimization problem, and reduce it to a. problem in plane geometry. Our analysis applies only to convex polygons, as concave angles become much harder to treat in this case.†

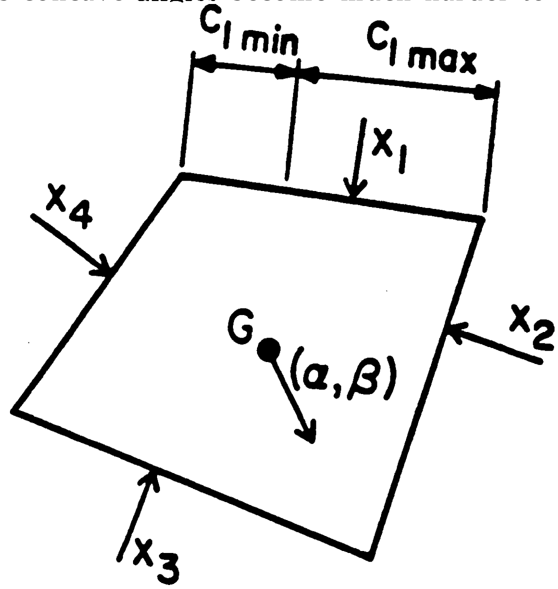

Figure 13: Optimum form semiclosure.

Consider a polygon such as that of Figure 12, and fix the sides of the four contact points (two of these sides may coincide). For any selection of contact points and any unit force through the center of gravity, the equations of equilibrium (with the center of gravity as the origin) are the following:

$$\begin{aligned} a_1x_1+a_2x_2+a_3x_3+a_4x_4 &= \alpha \\ b_1x_1+b_2x_2+b_3x_3+b_4x_4 &= \beta \\ c_1x_1+c_2x_2+c_3x_3+c_4x_4 &= 0 \end{aligned} \tag{6}$$

Here the x_i's are the magnitudes of the forces applied to the fingers, the a_i's are the sines of the directions of the normals, the b_i's the cosines, and the c_i's are the

† Notice that, for the polygons for which form semiclosure is achievable by three fingers, there is a finite number of form semiclosure grips, and thus optimizing form semiclosure by three fingers is not. interesting.

distances of the normals from G, with a sign reflecting the sign of the generated torque (see Figure 12). The unit wrench to be balanced is (a, β), applied on G. We wish to minimize the sum of the x_i's subject to (6).

Once the sides of the four fingers are fixed, the a_i's and the b_i's are also fixed, but the c_i's must vary. What are the constraints on the c_i's? The requirement is that the points of contact must be within the sides of the polygon. This means that

$$c_{i\ \min} \leq c_i \leq c_{i\ \max} \qquad (7),$$

where $c_{i\,\min}$ and $c_{i\,\max}$ are the coordinates of the vertices of the ith side, where the projection of G on that side is taken to be zero. It follows that optimizing the semiform closure of the quardilateral can be formulated as the following optimization problem:

$$\min_{c \text{ satisfies } (7)} \max_{\alpha^2+\beta^2=1} \min_{x\geq 0 \text{ satisfies } (6)} \{x_1 + x_2 + x_3 + x_4\} \qquad (8)$$

Notice that this optimization has the structure of a game: We choose the finger positions c, our opponent picks the wrench (α, β), after which we move again to fix the magnitudes x of the forces on the fingers; we pay our opponent an amount equal to the sum of the forces.

The innermost minimization in (8) is a linear program. Taking its dual [Da, PS] we have the following equivalent problem (equivalent in the sense that the optimal values are the same).

$$\max\{\alpha u + \beta v\}$$

$$\text{subject to } a_i u + b_i v \leq 1 - c_i w,\ i = 1, \ldots, 4.$$

Here u, v, and w are the dual variables. Now that the two innermost optimizations in (8) are both maximizations, their order may be reversed; with u and v fixed, it is clear that the maximum of $\alpha u + \beta v$ is achieved when (α, β) is the unit vector along the direction of (u, v). Thus the innermost two maximizations of (8) (after we have replaced the innermost minimization by its dual maximization) are equivalent to

$$\max\{\sqrt{u^2 + v^2}\}$$

$$\text{subject to } a_i u + b_i v \leq 1 - c_i w, \quad i = 1, \ldots, 4. \qquad (9)$$

Call this maximum $F(c)$ (it could be infinity); our task is to choose c within the bounds (7) so that $F(c)$ is minimized.

This problem has a simple interpretation in plane geometry. For $w = 0$, the constraints in (9) describe a. convex quadrilateral with the same slopes of sides as the four sides of contact, and circumscribed around the unit circle (recall that the vector (a_i, b_i) is a. unit vector, as it represents the sine and cosine of the ith side of contact). Now let w vary; the result is that the ith side of the quadrilateral moves, remaining parallel to itself, with a velocity (positive or negative) equal to c;. The question is, what is the maximum distance from the origin of any vertex of any quadrilateral thus formed? Hopefully, the quadrilaterals will vanish after some "gliding" of this sort, and thus the answer will be finite, but this may not necessarily be the case (for example, if all c_i's are positive). This maximum distance is in fact F(c). Our task is to choose the c_i's in such a way that this distance is minimized.

3.3 The Case of a Triangle

This problem has a surprisingly simple answer when the polygon to be grasped is a triangle. In particular, let α be the smallest angle of the triangle. Then the value *F of (8) equals* $\frac{1}{\sin(\alpha/2)}$. In proof, suppose that F_0 is this value when w is bound to be zero; obviously, $F_0 \geq F$. However, it is easy to see that, when $w = 0$ the equations in (9) describe the triangle with scaled so that the radius of the inscribed circle is one? and drawn so that the center of the inscribed circle is the origin. F_0 is the distance of the furthest vertex of the triangle from the center of the inscribed circle, again measured in radii of the inscribed circle. This distance, however, is exactly $\frac{1}{\sin(\alpha/2)}$.

It remains to see that, for some feasible choice of the c_i's, $F(c) = F_0$. If all projections of the center of gravity on the sides are in the interiors of the sides (i.e., $c_{i\,\min} < 0$ and $c_{i\,\max} > 0$ for each i) this is easy to show: Take $c_1 = c_2 = 0$, and $c_3 = -c_4 = \epsilon$, for some appropriately small $\epsilon > 0$ (recall that fingers 3 and 4 are on the same side). It follows that, for each value of w, the feasible region in (9) is a subset of this with $w = 0$, hence the result.

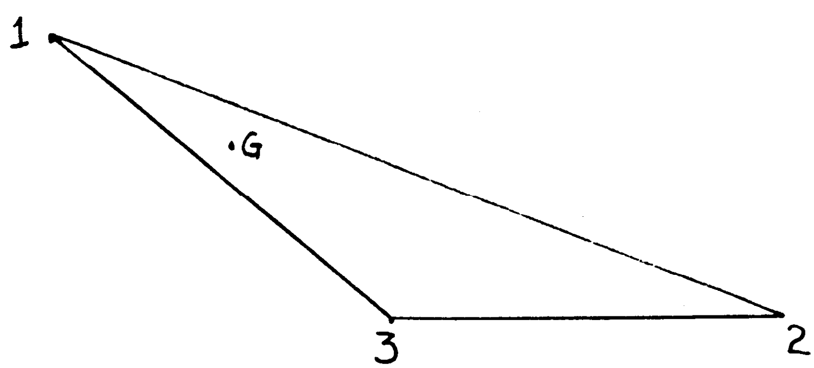

Figure 13: Obtuse triangle.

For the remaining case, suppose that the center of gravity projects outside a side (say, side 1, see Figure 13), that is, $c_{1,\min} > 0$. We take $c_1 = c_{1\,\min}$, $c_2 = 0$, $c_3 = c_{3\,\min}$ and $c_4 = c_{3\,\max}$. It can be argued by an involved case analysis that either (a) the maximum distance from the center of the inscribed circle is the vertex opposite side 1 for all values of w, or (b) for positive values of w the vertex opposite of side 2 writes a straight line segment which brings it closer to the center of the inscribed circle (Figure 13). The result follows. Hence we have:

For any triangle *there* is a way *of* grasping *it with* four *fingers* such that for any unit *force through the* center *of* gravity the *sum of the forces needed to* balance *the unit force equals the* inverse *of the sine of half the smallest angle of the* triangle. Furthermore, *this* worst-case *sum* is *the best* possible.

If we are interested in minimizing the worst-case maximum of the forces needed to balance a unit excitation (as opposed to their sum) a similar result holds. Equation (S) becomes

$$\min_{c \text{ satisfies } (7)} \max_{\alpha^2+\beta^2=1} \min_{x\geq 0 \text{ satisfies } (6)} \max\{x_1, x_2, x_3, x_4\},$$

apparently adding another move to the game. The innermost two optimizations, however, are equivalent to minimizing y subject to, in addition to equations (6), the inequalities $x_i \leq y$, $i = 1, \ldots, 4$. A similar analysis to the one in the previous paragraph yields that the optimum value is the following geometric parameter of the triangle: Consider, for each point p of the triangle, the distance from p of the furthest vertex of the triangle, divided by the sum of the distances from p of the sides- of the triangle, with the largest such distance added twice. Take now the maximum of this quotient over a.11 points p. This is the value of the optimum!

3.4 The General Case.

To solve the genera.1 problem, we must treat the case of a quadrilateral with arbitrary upper and lower bounds on the c_i's. It turns out that the plane geometry becomes quite a. bit more complicated, since the optimum is not determined by the $w = 0$ case, and so we must actually study the "gliding" of the sides of the quadrilateral. No closed form answer is possible here, but finding the optimum of (S) can be reduced to the examination of a small number of cases. The details are rather complicated, so we only sketch the argument below. We shall assume that the four chosen sides on which the fingers are to be applied are distinct: the ideas are similar (in fact, a bit simpler) when two of them coincide.

Recall from the last paragraph of Section 3.2 that, for $w = 0$, the inequalities (9) describe a. convex quadrilateral with the same slopes of sides as the four sides of contact, and circumscribed around the unit circle. If w varies, the ith side of the quadrilateral moves, remaining parallel to itself, with a. velocity (positive or negative) equal to c_i. For each set of c_i's within the bounds (7) the vertices of the quadrilateral move along straight lines, depending on the ratios of the c_i's. Any point on such a line uniquely determines a value of w (positive or negative), and thus a. quadrilateral which is the result of gliding of the original quadrilateral for w units of time. However, such a point may define a quadrilateral that has vanished, because the inequalities (i) are no longer satisfied for this value of w.

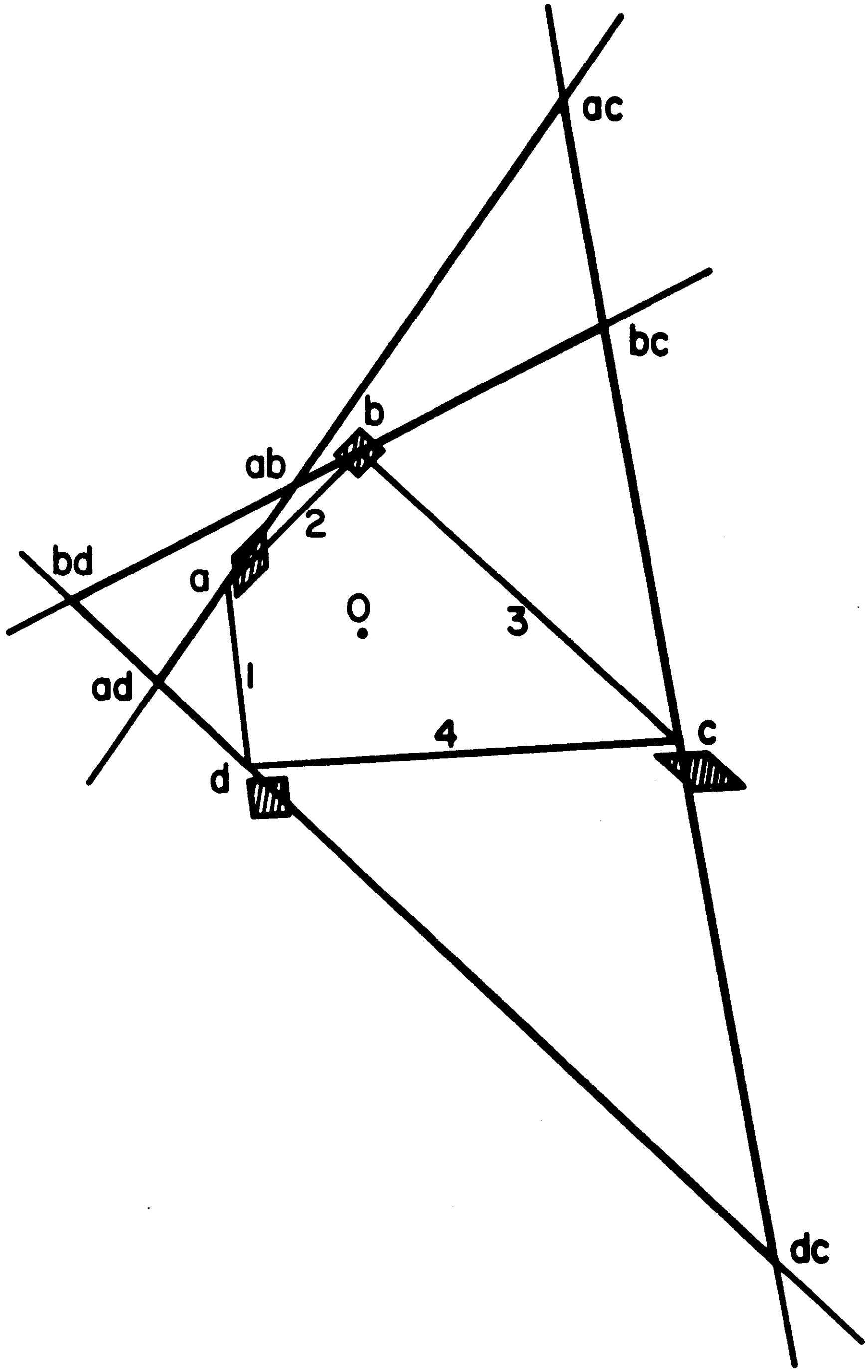

Figure 14: A quadrilateral and an arrangement of gliding lines.

For example, in Figure 14 the shaded areas close to the vertices represent the allowable values of the c_i's, by (7) -they show all possible motions of the vertices within a unit of "time" w. The gliding lines corresponding to a particular choice of the c_i's form the arrangement shown, but only the bold parts represent actual feasible u, v, and w's; the rest represent "vanished" quadrilaterals. We must choose the c_i's in such a way that the maximum distance from the origin of any vertex of a non-vanished quadrilateral (that is, any bold point) is minimized.

Let us first resolve the simpler problem of determining whether the optimum is finite (that is, the bold part is finite). To put it in other words, we wish to determine whether form semiclosure can be achieved by applying four fingers at the selected sides. It is easy to see that this is possible if and only if there is a selection of c_i's within the bounds (7) so that the equations (6) with $\alpha = \beta = 0$ have a positive solution $x_1, x_2, x_3, x_4 > 0$. This condition, however, is equivalent to saying that the four 3×3 subdeterminants of the matrix of (6) have the same sign. These subdeterminants of are linear forms in the c_i's (with coefficients the sines and cosines of the angles formed between the sides). Thus, the question of feasibility can be reduced to determining whether the intersection of the cone of these linear forms with the hyperrectangle of (7) is non-empty, which is a four-dimensional linear program.

Example 4. In a quadrilateral with the same angles as that in Figure 14 (where all angles are the obvious multiples of $\pi/4$) the equations (6) are:

$$\begin{array}{rrrrl} x_1 & +x_2 & -x_3 & & = 0 \\ & -x_2 & -x_3 & +x_4 & = 0 \\ c_1x_1 & +c_2x_2 & +c_3x_3 & +c_4x_4 & = 0 \end{array}$$

where for simplicity we denote by x_2, x_3 the corresponding magnitudes divided by $\sqrt{2}$. The four subdeterminants are: $-c_2 - c_3$, $-c_1 - c_3 - c_4$, $c_1 + c_2 + c_4$, and $-2c_1 + c_2 - c_3$. Form semiclosure can be achieved only if all these linear forms are positive, or all negative. This defines a. four-dimensional affine cone with center the origin. Its intersection with the box (7) can be determined very easily, either by linear programming, or by the examination of a small number of possible basic feasible solutions. ■

We can restate this algebraic feasibility condition in purely geometric terms. The c_i's (in fact, their ratios) completely determine an arrangement of lines on which the vertices of the quadrilateral "glide," and the bold part of the arrangement represents all possible positions of vertices of a non-vanishing quadrilateral [Figure 14). The question is, which of the many possible arrangements have a bounded bold part? Consider an arrangement α; $C(\alpha)$ denotes the convex hull of the six vertices of α (without loss of generality, bounded). It is easy to see that α has a finite bold part if and only if the following two conditions hold: (a) The vertices a, b, c, d are in the interior of $C(\alpha)$, and (b) The vertices a b and ad (the intersections of pairs of lines representing adjacent vertices with sum of angles greater than π, of which there are two in any convex quadrilateral) must also be in the interior. The intuitive reason is that ab and ad are the only vertices of α which correspond to a non-vanishing quadrilateral. It follows that there are two kinds of feasible arrangements, that is, arrangements with finite bold part: The ones in which the order of the points are exactly as in Figure 14, and another similar class, in which dc is between bc and ac.

Our optimization problem can be restated as that of choosing α among the two classes of feasible arrangements so as to minimize the distance from 0 of the furthest point among bd, dc, and ac --the vertices of $C(\alpha)$. We shall show that this optimization can be reduced to a small number of easy geometric constructions; we shall do this for the class of arrangements in Figure 14, the task for the other class being very similar.

Suppose that dc is the furthest point from 0 among $\{ bd, dc,$ UC}. Since its distance from 0 would be improved if line c were to rotate clockwise, or d counterclockwise, we must conclude that neither motion is possible. There are two reasons why c cannot move: Either the ratio of c_3 and c_4 is at an extreme value (line c intersects the shaded parallelogram of feasible c_i's at a vertex) or the distance of ac from 0 equals that of dc, and thus a rotation would deteriorate the optimum. Similarly for cl. Continuing like this we arrive at the conclusion that at the optimum one of the following situations must hold:

(1) The optimum is determined by the intersection of two fixed lines (such as the most clockwise feasible direction of c and the most counterclockwise of d in the case discussed above).

(2) Two lines are fisecl (say, a and d), and the optimum is determined by two points ac and dc , one on each of these lines, having equal distance from 0, and such that the line defined by the two points passes through c.

(3) The distances of ac, dc, and bd from 0 are equal and determine the optimum.

In each of these cases the optimum can be computed in any desired accuracy in constant time, essentially by binary search. For example, in (2) we perform binary search on the value of the optimum. For any such value R we draw the circle with center 0 and radius R thus determining ac and dc, and we then check whether c is on the largest of the two parts in which chord UC, dc divides the circle; if so, R is too high; otherwise it is too low. Repeating, we can determine the optimum value R within any desired accuracy.

From all of the above, the algorithm for achieving optimum semiclosure grip on a convex polygonal object is the following: We repeat the following process for all quadruples of sides, two of which may coincide: We first determine quickly by the algebraic test whether there is a feasible grip, and if not we abandon the quadruple. In the case that a feasible grip exists, we find the optimum one by resolving be binary search a finite set of cases. The answer is the optimum *optimorum* among all quadruples. The c_i's of choice and the corresponding worst-case value are readily available.

4. DISCUSSION

In this paper we have demonstrated the applicability of elementary optimization techniques to the problem of optimizing the grip of an object; to our knowledge, this is the first such effort.

There is considerable progress to be made in several fronts, if the development started here is to lead to a comprehensive methodology and powerful practical tools. First, there are some technical problems left open in our method. For example, we would like to have a less heuristic procedure for optimizing q(H) in Section 2.1, but the non-convex nature of this criterion leaves little hope. In the problem of optimum form semiclosure, we would like to extend our method to objects with concave angles, and if possible to derive a. direct linear programming approach to the problem of optimizing line arrangements. Also, it would be interesting to consider the case of general form closure, in which the adversary's force can be applied anywhere on the object, and not only at the center of gravity. Our results seem to be extensible in this direction, since the application point of choice for the adversary will be a vertex of the object, and thus most of our methodology would still apply.

However, much more must be done in the direction of incorporating in our model some basic aspects of the mechanics and pragmatics of prehension, which we are ignoring at present. A most important such issue is, of course, friction. Friction is known to have a profound beneficial effect on the problem of prehension, making the solution space significantly richer, but at the same time more interesting and intricate. For example, optimum semiform closure in three dimensions requires *seven* fingers [MNP], although friction is empirically known to reduce this number substantially. It would be rather premature to embark on a formulation of the problem of optimizing three-dimensional frictionless prehension, if the very number of fingers involved depends so crucially on the absence of friction.

Another interesting issue is that of taking into account the structural constraints of robot hands, Here we have assumed that any triple or quadruple of points on the perimeter of the object is a possible grip. In fact, the geometry and structure of the hand may restrict substantially the possible grips. It would be interesting to study the interaction of these constraints with those handled in our optimization methods. Finally, our results simply suggest a desired final grip, but not how to reach it. Typically, an object must be picked up from rest, turned and toppled carefully, so that the desired grip is finally achieved. All intermediate grips may not be form closures, but must still satisfy some minimum requirements (for example, the object must not fall by its own weight). How to achieve this is an interesting problem.

References

[BFG] B. S. Baker, S. Fortune, E. Grosse "Stable Prehension with Three Fingers", *Proc. 1985* Symposium on the Theory of *Computing*, pp.114-120.

[Da] G. B. Dantzig Linear Programming and Extensions, Princeton University Press, 1959.

[La] K. Lakshminarayana "The Mechanics of Form Closure", ASME 78-DET-32, 1978.

[MNP] X. Markenscoff, Luqun Ni, C. H. Papadimitriou, "The Geometry of Form Closure", submitted for publication, 1986.

[PS] C. H. Papadimitriou and K. Steiglitz Combinatorial Optimization: Algorithms and Complexity, Prentice-Hall, 1983.

[SR] J. K. Salisbury, B. Roth "Kinematic and Force Analysis of Articulated Mechanical Hands", 3. of *Mechanisms*, Transmissions, *and* Automation in *Deasign*,, ASME-105, 1983, pp. 35-41.