The Average Number of Stable Matchings

by

Boris Pittel

Department of Computer Science

Stanford University

Stanford, California 94305

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Compu ter Science Department

Stanford University

Abstract. The probable behavior of an instance of size n of the stable marriage problem, chosen uniformly at random, is studied. The expected number of stable matchings is shown to be asymptotic to $e^{-1}n \ln n$ for n — ∞. The total rank of women by men in the male optimal (pessimal) matching is proven to be close to $n \ln n$ (resp. $n^2/ \ln n$), with high probability.

1. Introduction.

In the usual formulation, an instance of size n of the stable marriage problem involves n men and n women, with each person ranking representatives of the opposite sex in order of individual preference for a marriage partner. A complete matching, i.e., a set of n marriages, is called stable if no man and woman, who are not married to each other, would prefer each other to their actual partners under the matching. Gale and Shapley, who introduced the problem, showed that at least one stable matching does exist; in fact, they provided an iterative procedure that finds a stable set of marriages [4]. Later, McVitie and Wilson [11] developed an alternative ("fundamental") algorithm; its work is described by a sequence of proposals of men to women made one at a time, while the Gale-Shapley algorithm used rounds of simultaneous proposals. Both algorithms yield the same matching which is male-optimal compared to any other stable matching, simultaneously for all men. Using a reduction to a classic urn scheme, Wilson [13] proved that the expected running time of the fundamental algorithm for the random instance of the problem is bounded by nH_n, $(H_n = 1 + \cdots + 1/n.)$. In the course of a detailed study of the stable marriages problem, Knuth [10] found a better upper bound $(n-1)H_n + 1$ and established a lower bound $nH_n - O(\log^4 n)$. Among several open problems, Knuth [10] posed the question of estimating the expected number of stable matchings. He indicated that the answer to this question might be found via an integral formula for the probability that a given matching is stable.

A primary purpose of this paper is to establish-by using Knuth's formula-that the expected number of stable matchings is asymptotic to $e^{-1} n \ln n$ for $n \to \infty$. Curiously, it is of the same order as the average number of proposals in the fundamental algorithm. This should be compared with the fact that the **minimum** number of stable matchings for any problem instance of size n is 1, while the *maximum* number grows **at** least exponentially with n, Knuth [10], Irving and Leather [6]. (For other deterministic results on the structure of the set of stable matchings, we refer the reader to Irving [7], Irving et al [8], and Gusfield et al [5].)

Another purpose of this paper is to show that, almost surely (as.) for a random problem instance, the maximum (minimum) total rank of women by men for a stable matching is asymptotic to nlnn (resp. $n^2/\ln n$). Since the minimum rank of women by men coincides with the number of proposals by men in the fundamental algorithm, the statement shows that this number is a.s. close to $n \ln n$. On the other hand, the maximum rank of women by men coincides, in distribution, with the total rank of men by women in the male-optimal stable matching, which this algorithm determines. So, the latter rank is a.s. close to $n^2/\ln n$, and far exceeds $n \ln n$. The stable matching in question heavily favors men, at the expense of women. The situation is just the opposite in the female-optimal stable matching.

The rest of the paper is organized as follows. In Section 2, we derive a general formula for the probability that a given matching is stable, and that its rank has a specified value. This is a generalization of Knuth's formula for the probability that a matching is stable. The latter is used in Section 3 to obtain asymptotics of the expected number of stable matchings. The general formula is applied then in Section 4 to study the a.s. asymptotic behavior of the minimum rank and the maximum rank for a stable matching. In the appendix, we prove some auxiliary results for a random partition of the unit interval.

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2. Basic formulas.

By symmetry, each one of n! matchings (pairings) of n men and n women has the same probability P_n of being stable. Knuth [10] proved that

$$(2.1) \qquad P_n = \overbrace{\int \cdots \int}^{2n} \prod_{1 \le i \ne j \le n} (1 - x_i y_j) \, dx \, dy,$$

where $dx = dx_1 \ldots dx_n$, $dy = dy_1 \ldots dy_n$, $0 \le x_i \le 1$, $0 \le y_j \le 1$ $(1 \le i, j \le n)$.

Define the (men-oriented) rank of a stable matching as the sum of the ranks of women by men in this matching. The rank lies between n and n^2; it equals n (resp. n^2) if each man happens to be matched with a woman whom he ranks first (resp. last). Define P_{nk} as the probability that a given matching is stable and that its rank equals k $(n \le k \le n^2)$. We want to show that

$$(2.2) \qquad P_{nk} = \int \cdots \int [z^{k-n}] \prod_{1 \le i \ne j \le n} (1 - x_i(1 - z + z y_j)) \, dx \, dy;$$

here the integrand equals the coefficient of z^{k-n} in the product. Notice that this relation implies (2.1) since the sum of the integrands over k equals the integrand in (2.1).

Proof of (2.2). (a) Let $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n)$ be the set of men and the set of women. Each man $u \in U$ (resp. woman $v \in V$) ranks women (resp. men) uniformly at random, independently of all other men and women. A way to generate such a random ranking system is as follows. Let us assume that there are given two n x n matrices $X = [X_{ij}]$, $Y = [Y_{ij}]$ whose entries are all independent, each uniformly distributed on the interval [0, I]. For each man u_i (woman v_j) we define a permutation, i.e., ordering, π_i (resp. ω_j) of the set $\{1, \ldots, n\}$ such that

$$X_{i,\pi_i(1)} < X_{i,\pi_i(2)} < \cdots < X_{i,\pi_i(n)},$$
$$\left(\text{resp. } Y_{\omega_j(1),j} < Y_{\omega_j(2),j} < \cdots < Y_{\omega_j(n),j}\right).$$

We postulate that the woman $v_{\pi_i(j)}$ is the j-th best choice for the man u_i, and that the man $u_{\omega_j(i)}$ is the i-th best choice for the woman v_j. By the definition of X and Y, the $2n$ random permutations are independent, of one another, and each is distributed uniformly. (The cases when two elements of one row of X, or one column of Y, coincide have total probability zero, and thus can be neglected.)

(b) We may, and shall, consider the particular matching $M = \{(u_i, v_i) : 1 \le i \le n\}$. The rank Q_n of this matching equals $n + \sum_{i=1}^{n} |\{j : X_{ij} < X_{ii}\}|$, and we need to evaluate P_{nk}, the probability of the event. $A = \{M$ is stable and $Q_n = k\}$.

For x $= (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ $(0 \le x_i, y_j \le 1, 1 \le i, j \le n)$, define $P_{nk}(x, y)$ to be the *conditional* probability of the event A given that $X_{ii} = x_i$, $Y_{jj} = y_j$ $(1 \le i \le n, 1 \le j \le n)$; in short $P_{nk}(x, y) = \Pr(A|\bullet)$. Since all $X_{\alpha\beta}, Y_{\alpha\beta}$ are independent, by the Fubini theorem it will suffice to show that $P_{nk}(x, y)$ equals the integrand in (2.2). To this end, we first observe that

$$(2.3) \qquad P_{nk}(x, y) = [z^k] E\left(\chi(M) z^{Q_n} | \bullet\right),$$

where $\chi(M)$ is the indicator of the event $\{M$ is stable$\}$, and the expected value is conditioned on $X_{ii} = x_i$, $Y_{jj} = y_j$ $(1 \leq i \leq n, 1 \leq j \leq n)$. **To** evaluate this expectation, it is convenient to introduce a "marking" procedure: Fix $z \in (0, 1)$; scan the pairs (i, j) and, whenever $X_{ij} < X_{ii} (= x_i)$, mark the pair with probability z, independently of all other pairs. Then, setting $B = \{M$ is stable *and* a.11 the pairs (i, j) such that $X_{ij} < X_{ii}$ are marked$\}$, we can write

(2.4)
$$E\big(\chi(M)z^{Q_n} \,|\bullet\big) = z^n \Pr(B|\bullet).$$

Let $C = \{(i, j) : i \neq j\}$, and let B_{ij} $((i, j) \in C)$ be the event "$X_{ii} < X_{ij}$, or $(X_{ii} > X_{ij}, Y_{jj} < Y_{ij}$ and (i, j) is marked)." A little reflection shows that

$$B = \bigcap_{(i,j) \in C} B_{ij} .$$

Besides, conditioned on $X_\alpha = x_\alpha$, $Y_{\beta\beta} = y_\beta$ $(1 \leq \alpha \leq n, 1 \leq \beta \leq n)$, the events B_{ij} are independent, and

$$\Pr(B_{ij}|\bullet) = (1 - x_i) + x_i(1 - y_j)z, \qquad (i, j) \in C.$$

Therefore

$$\Pr(B|\bullet) = \prod_{1 \leq i \neq j \leq n} (1 - x_i(1 - 3 + zy_j)),$$

so (see (2.3)**, (2.4))**

$$P_{nk}(x, y) = [z^{k-n}] \prod_{1 \leq i \neq j \leq n} (1 - x_i(1 - z + zy_j)). \qquad \blacksquare$$

Note: In order to obtain (2.1) directly, rather than from (2.2), one can use a similar argument, setting the marking probabiity $z = 1$, so that, $\Pr(B_{ij}|\bullet) = 1 - x_i y_j$. The original proof of (2.1) given by Knuth [10] did not use the random matrices X, Y, but relied instead on an inclusion-exclusion formula, and interpretation of each term as the value of an $2n$-dimensional integral with the integrand equal to the corresponding term in the expansion of $\prod_{1 \leq i \neq j \leq n}(1 - x_i y_j)$.

3. Expected Number of Stable Matchings for Large n.

We shall prove in this section that

(3.1)
$$P_n = (1 + o(1))e^{-1} n \ln n/n! .$$

Since there are $n!$ matchings, the formula (3.1) implies immediately

Theorem 1. *The expected number of stable matchings is asymptotic to $e^{-1} n \ln n$.*

Proof of (3.1). In the course of the argument, and in the nest section as well, we will use the following facts.

Let $X_1, \ldots, X_n$ be independent random variables each distributed uniformly on $[0, 1]$. Set

$$S_n = \sum_{j=1}^{n} X_j, \qquad T_n = \left(\sum_{j=1}^{n} X_j^2 \right) \bigg/ S_n^2 .$$

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Introduce also the random variables $L_1, \ldots, L_n$ which are the lengths of the consecutive subintervals of $[0, 1]$ obtained by selecting independently $n - 1$ points, each uniformly distributed on $[0, 1]$ (in particular, the L_i sum up to 1). Set

$$U_n = \sum_{j=1}^{n} L_j^2, \qquad M_n = \max_{1 \leq j \leq n} L_j .$$

Lemma 1. *Let* $f_n(\cdot)$, $f_n(\cdot, \cdot)$, $g_n(\cdot)$ *be the density of* S_n, (S_n, T_n), **and** U_n *respectively. Then*

$$(3.2) \qquad f_n(s) = \frac{s^{n-1}}{(n-1)!} \Pr(M_n \leq s^{-1}) ,$$

so, in particular,

$$(3.3) \qquad f_n(s) \leq \frac{s^{n-1}}{(n-1)!} .$$

Also

$$(3.4) \qquad f_n(s, t) \leq \frac{s^{n-1}}{(n-1)!} g_n(t) .$$

Lemma 2. *(Asymptotic behavior of M_n, U_n). In probability, as $n \to \infty$,*

$$M_n / n^{-1} \ln n \to 1 ,$$

$$n U_n \to 2 .$$

Besides, for every $\rho > 0$,

$$\Pr\big(M_n \geq n^{-1}(\ln n - \ln\ln n - \rho)\big) \geq 1 - O(n^{-d}) , \qquad \forall\, d \in (0, e^\rho - 1)$$

Note. The relation (3.2) is well known (see Feller [3, Ch. 1], for instance), but (3.4) appears to be new. We prove both relations in the Appendix by using the fact that the joint density of $L_1, \,,\,, L_{n-1}$ equals $(n-1)!$, whenever this density is not zero. As for Lemma 2 (also proven iu the Appendix), the argument is based on a classic result which states the following: $\{L_j : 1 \leq j \leq n\}$ has the same distribution as $\{W_j / \sum_{k=1}^{n} W_k : 1 \leq j \leq n\}$, where $W_1, W_2 \ldots$ are independent, exponential, with parameter one (Breiman [I], Karlin and Taylor [9], Rényi [12]). (Sam Karlin has informed me that, in a course which he taught in 1986, he used this connection for asymptotic study of M_n and other extremal characteristics of $\{L_j : 1 \leq j \leq n\}$.)

(a) Let us begin with an upper bound for P_n. Since $1 - \alpha \leq \exp(-\alpha - \alpha^2/2)$, we get from (2. I):

$$(3.5) \qquad P_n \leq \overbrace{\int \cdots \int} \left(\prod_{j=1}^{n} \int_0^1 \exp(-y s_j - y^2 t_j / 2)\, dy \right) dx_1 \ldots dx_n ,$$

where

$$s_j = \sum_{i \neq j} x_i , \qquad t_j = \sum_{i \neq j} x_i^2 ,$$

and integration is taken over $0 \leq x_i \leq 1$ $(1 \leq i \leq n)$.

Fix $a > 0$ and break the integral into two parts, $\int_1$ for $s := x_1 + \cdots + x_n \le w(n) := a \ln\ln n$ and $\int_2$ for $s > \omega(n)$. Let us show that, for a sufficiently small value of a,

(3.G)
$$\int_1 = o(n \ln n / n!).$$

Dropping the factors $\exp(-y^2 t_j / 2)$, and integrating with respect to y, we have

$$\int_1 \le \int \cdots \int_{s \le \omega(n)} \left(\prod_{j=1}^n \frac{1 - \exp(-s_j)}{s_j} \right) dx \qquad (dx = dx_1 \ldots dx_n).$$

To simplify the last estimate, we observe that for $z > 0$,

$$\left(\ln \frac{1 - e^{-z}}{z} \right)' = -z^{-1} \left(1 - \frac{z}{e^z - 1} \right),$$

which implies existence of a constant c > 0 such that

(3.7)
$$-c \le \left(\ln \frac{1 - e^{-z}}{z} \right)' \le 0;$$

also

(3.8)
$$\left(\ln \frac{1 - e^{-z}}{z} \right)' = -(1 + o(1)) z^{-1}, \quad z \to \infty.$$

By (3.7),

$$\ln \frac{1 - e^{-s_j}}{s_j} = \ln \frac{1 - e}{s} \qquad \int_{s - x_j}^s \left(\ln \frac{1 - e^{-z}}{z} \right)' dz$$

$$\le \ln \frac{1 - e^{-s}}{s} + c\, x_j.$$

Therefore,

$$\sum_{j=1}^n \ln \frac{1 - e^{-s_j}}{s_j} \le n \ln \frac{1 - e^{-s}}{s} + c \sum_{j=1}^n x_j,$$

and, since $\sum_j x_j \le a \ln \ln n$, **we** have

$$\int_1 \le (\ln n)^{ca} \int_{s \le \omega(n)} \left(\frac{1 - e^{-s}}{s} \right)^n dx.$$

The last integral is the expected value of $\left(\frac{1 - \exp(-S_n)}{S_n} \right)^n$ over the event $\{S_n \le u(n)\}$. Then, by (3.3),

$$J \le (\ln n)^{ca} \left((n-1)! \right)^{-1} \int_0^{\omega(n)} \frac{(1 - e^{-s})^n}{s} dx$$

$$\le (\ln n)^{ca} \left((n-1)! \right)^{-1} \omega(n)$$

$$= O(n(\ln n)^{1/2}/n!).$$

provided that $a < (2c)^{-1}$. This proves (3.6).

Turning our attention now to $\int_2$, i.e., to $s > \omega(n)$, the generic factor of the product in (3.5) can be estimated as follows:

$$\int_d^1 \exp(-ys_j - y^2 t_j/2)\, dy \le s_j^{-1} \int_0^\infty \exp(-z - z^2(t_j s_j^{-2})/2)\, dz$$

$$\le s_j^{-1} \int_0^\infty \exp(-z - z^2(t_j s^{-2})/2)\, dz$$

$$= s_j^{-1}\left(1 - t_j s^{-2}\int_0^\infty z\exp\left(-z - z^2(t_j s^{-2})/2\right) dz\right)$$

$$\le s_j^{-1}\left(1 - t_j s^{-2} F(ts^{-2})\right),$$

where

(3.9)
$$t = \sum_{j=1}^n x_j^2, \qquad F(u) = \int_0^\infty z\exp(-z - z^2 u/2)\, dz.$$

Therefore, the integrand in (3.5) is bounded above by

$$\left(\prod_{j=1}^n s_j^{-1}\right)\left(\prod_{k=1}^n (1 - t_k s^{-2} F(ts^{-2}))\right)$$

$$\le \left(\prod_{j=1}^n s_j^{-1}\right)\exp\left(-s^{-2}\left(\sum_{k=1}^n t_k\right) F(ts^{-2})\right).$$

Here

$$\sum_{k=1}^n t_k = (n-1)t,$$

and $(0 \le x_j \le 1,\ s = \sum_{j=1}^n x_j \ge \omega(n))$,

$$\prod_{j=1}^n s_j^{-1} = s^{-n} \prod_{j=1}^n (1 - x_j/s)^{-1}$$

$$= s^{-n} \prod_{j=1}^n \exp\left(x_j/s + O(x_j^2/s^2)\right)$$

$$= s^{-n} \exp\left(1 + O(\omega(n)^{-1})\right).$$

Therefore

$$\int_2 \le \exp\left(1 + O(\omega(n)^{-1})\right) \int_{s\ge\omega(n)} s^{-n}\exp\left(-(n-1)ts^{-2}F(ts^{-2})\right) dx;$$

so, using Lemma 1 (3.4) (and the fact that $s \le n$), we have

(3.10)
$$\int_2 \le \exp\left(1 + O(\omega(n)^{-1})\right)\left((n-1)!\right)^{-1}\int_{\omega(n)}^n s^{-1}\, ds$$

$$\cdot E\left(\exp(-(n-1)U_n F(U_n))\right).$$

Here, by Lemma 2 (see also (3.9)),

$$(P)\lim_{n\to\infty} \exp\left(-(n-1)U_n F(U_n)\right) = \exp(-2F(0))$$

(3.11)
$$= \exp\left(-2\int_0^\infty ze^{-z}\, dz\right) = e^{-2}.$$

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((P) lim designates the limit in probability.) So, invoking the dominated convergence theorem, we can assert that the expected value in (3.10) converges to e^{-2}. Besides,

$$(3.12) \qquad \int_{\omega(n)}^{n} s^{-1} ds = \ln n \left(1 + 0(\ln \ln \ln n / \ln n)\right).$$

Therefore,

$$\int_{J} \leq (1 + o(1)) e^{-1} n \ln n / n!,$$

which implies (see (3.6)) that

$$P_n \leq (1 + o(1)) e^{-1} n \ln n / n!, \qquad \blacksquare$$

(b) It remains to estimate P_n from below. Denote by D the set of a.11 nonnegative x $= (x_1, \ldots, x_n)$ such that

$$(3.13) \qquad 3 \ln n \leq s \leq n / \ln^2 n,$$

$$(3.14) \qquad s^{-1} x_j \leq (1 + \epsilon) \ln n / n, \qquad 1 \leq j \leq n,$$

$$(3.15) \qquad s^{-2} t \leq (1 + \epsilon) 2 / n,$$

$\left(s = \sum_{j=1}^{n} x_j , \ t = \sum_{j=1}^{n} x_j^2\right)$, where $\epsilon > 0$ is fixed. According to (3.13), (3.14),

$$(3.16) \qquad Xj \leq (1 + \epsilon)(\ln n)^{-1} < 1 \qquad (1 \leq j \leq n)$$

for all sufficiently large n, so that

$$D \subset \{x : 0 \leq x_j \leq 1, \ 1 \leq j \leq n\}.$$

We want to show that the dominant part of P_n is contributed by the region D, which should not be very surprising in light of Lemmas 1, 2 and part (a). Set

$$P_n(\epsilon) = \int_{x \in D} \left(\prod_{j=1}^{n} \left(\int_0^1 \prod_{i \neq j} (1 - x_i y_j) \, dy_j \right) \right) dx,$$

so that $P_n \geq P_n(\epsilon)$. Since $1 - \alpha = \exp(-a - \alpha^2(1 + O(\alpha))/2)$, $\alpha \to 0$, using (3.16) we can write for fixed j,

$$\prod_{i \neq j} (1 - x_i y_j) \geq \exp\left(-y_j s_j - y_j^2 t_j (1 + \sigma_n)/2\right),$$

where

$$\sigma_n = c \, 111^{-1} n, \qquad c = c(\epsilon) > 0.$$

(Recall that $s_j = \sum_{i \neq j} x_i$, $t_j = \sum_{i \neq j} x_i^2$.) Hence, for each j,

$$I_j(x) := \int_0^1 \prod_{i \neq j} (1 - x_i y_j) \, dy_j$$

$$\geq \int_0^1 \exp\left(-y s_j - y^2 t_j (1 + \sigma_n)/2\right) dy$$

$$= s_j^{-1} \int_0^{s_j} e^{-z} \exp\left(-z^2 (t_j s_j^{-2})(1 + \sigma_n)/2\right) dz$$

$$(3.17) \qquad \geq \frac{1 - e^{-s_j}}{s_j} \exp\left(-(t_j s_j^{-2}) \frac{1 + \sigma_n}{2(1 - e^{-s_j})} \int_0^{s_j} e^{-z} z^2 \, dz\right).$$

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(In the last step, we have used Jensen's inequality, namely that

$$\int_{z_1}^{z_2} A(z)B(C(z))\,dz \geq \boldsymbol{B}\left(\int_{z_1}^{z_2} A(z)C(z)dz\right),$$

if $B(\bullet)$ is convex, $A(z) \geq 0$, and $\int_{z_1}^{z_2} A(z)\,dz = 1$.) TO simplify this estimate, we observe Chat, by the definition of D,

$$s_j = s - x_j = s(1 - x_j/s)$$

$$= s\exp\left(-x_j/s + O(x_j^2/s^2)\right) = s\exp\left(-x_j/s + o(x_j/s)\right)$$

$$\geq c'\ln n, \qquad \forall\, c' \in (2,3).$$

uniformly over a: $\in \boldsymbol{D}$, if n is large. Therefore

$$1 - e^{-s_j} = 1 + O(n^{-c'}).$$

$$\int^{s_j} e^{-z} z^2\,dz = 2 - \frac{e^{-s_j}}{2}(1 + s_j + s_j^2/2)$$

$$= 2 + O(n^{-c''}), \qquad c'' = c' - 2 > 0,$$

and (13.1'7) becomes

$$I_j(x) \geq (1 + O(n^{-c'}))s^{-1}\exp\left(x_j/s + o(x_j/s)\right)$$

$$\cdot \exp\left(-(t_j s_j^{-2})(1 + O(\ln^{-1} n))\right).$$

But $c' > 2$, $s_j \leq s = \sum_{j=1}^n x_j$, $\sum_{j=1}^n t_j = (n-1)t$; therefore

(3.18) $$\prod_{j=1}^n I_j(x) \geq (1 + o(1))es^{-n}\exp\left(-n(ts^{-2})(1 + o(1))\right)$$

uniformly over $x \in \boldsymbol{D}$.

Let us switch to new variables, namely

$$u = \sum_{j=1}^n x_j = s,$$

$$v_j = x_j s^{-1}, \qquad 1 \leq j \leq n - 1.$$

Define also $v_n = x_n s^{-1}$. Clearly, $0 \leq v_j \leq 1$ and $\sum_{j=1}^n v_j = 1$. The conditions (3.13)–(3.15), which define $\boldsymbol{D}$, become

(3.19) $$3\ln n \leq u \leq n/\ln^2 n,$$

(3.20) $$v_j \leq (1 + \epsilon)\ln n/n,$$

(3.21) $$\sum_{j=1}^n v_j^2 \leq (1 + \epsilon)2/n.$$

Thus, in new variables, the region is the *direct product* of the interval defined in (3.19) and a region D^* defined in (3.20), (3.21). Besides, the Jacobian of $(x_1, \dots, x_n)$ with respect to $(u, v_1, \dots, v_{n-1})$ equals u^{n-1}. So, it follows from the estimate (3.18) that

$$P_n(\epsilon) \geq (1 + o(1))e((n-1)!)^{-1}\left(\int_{3\ln n}^{n/\ln^2 n} u^{-1}\,du\right).$$

(3.22) $$\cdot \int_{D^*} \exp\left(-n\sum_{j=1}^n v_j^2\right)(n-1)!\,dv, \qquad dv = dv_1 \dots dv_{n-1}.$$

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The first integral here is $\ln n(1 + o(1))$. Let us have a closer look at the second integral. Its integrand is at least $\exp(-2(1 + \epsilon))$ everywhere on D^*. In addition, $(n - 1)!$ is the joint density of the first $(n - 1)$ subintervals among n subintervals $L_1, \ldots, L_n$, introduced in Lemma 1 (see Breiman [1, Ch. 13], for instance). Thus, the inequality (3.21) leads to

$$P_n(\epsilon) \geq (1 + o(1))\exp(-1 - 2\epsilon)(n \ln n)/n!$$

$$\Pr\left(\max_{1 \leq j \leq n} L_j \leq (1 + \epsilon)\ln n/n, \sum_{j=1}^{n} L_j^2 \leq (1 + \epsilon)2/n\right),$$

for every fixed $\epsilon > 0$. But the probability of the event on the right hand side tends to 1 as $n \to \infty$ *(see Lemma 2).* Letting $n \to \infty$ and then $\epsilon \downarrow 0$ we are able to conclude that

$$P_n \geq (1 + o(1))e^{-1}n \ln n/n!. \quad \blacksquare$$

4. Probable Behavior of the Maximum Rank and the Minimum Rank of a Stable Matching.

Let r_n and R_n be the minimum *(male related)* rank and the maximum rank for any stable matching. It is well known that r_n equals the total number of proposals in the fundamental algorithm, in which men propose to women. The resulting stable matching is both male optimal and female pessimal. So, by symmetry, we can assert that, *in distribution,* R_n coincides with the *female related* rank of that particular matching.

Our goal is to prove

Theorem 2. In *probabili ty*

(4.1) $$r_n/n \ln n \to 1,$$

(4.2) $$R_n/n^2 \ln^{-1} n \to 1,$$

as $n \to \infty$.

Notes. According to this theorem, the stable matching reached via proposals made by men to women is a.s. considerably more favorable to men than to women. In short, initiative pays! Also, the relation (4.1) means that the number of proposals in the fundamental algorithm is a.s. close to $n \ln n$.

The core of the proof is the following statement.

Proposition. For every $\rho > 0$ and $\delta \in (0, e^\rho - 1)$,

(4.3) $$\Pr(r_n \geq n(\ln n - \ln \ln n - \rho)) \geq 1 - O(n^{-\delta}),$$

(4.4) $$\Pr(R_n \leq n^2 \ln^{-1} n(1 + (\ln \ln n + \rho)\ln^{-1} n)) \geq 1 - O(n^{-\delta}).$$

Proof of Lemma 3. Notice first that for every k between n and n^2,

$$\Pr(r_n \leq k) \leq n! \sum_{m=n} P_{nm},$$

$$\Pr(R_n \geq k) \leq n! \sum_{m=k}^{n^2} P_{nm}.$$

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Here $P_{n,m}$ is the probability that a fixed matching is stable and its rank equals m. In Section 2, we proved that

$$P_{nm} = \int \int \left([z^{m-n}]\Phi(x, y, z) \right) dx\, dy \, ,$$

(4.5)
$$\Phi(x, y, z) := \prod_{i \neq j} \left(1 - x_i(1 - z + z y_j) \right) .$$

Mimicking an approach due to Chernoff [2] (which allows to estimate the tails of a distribution through its moment generating function), we can write then

(4.6)
$$\Pr(r_n \leq k) \leq n! \int \int \inf_{0 < z \leq 1} \left(z^{n-k} \Phi(x, y, z) \right) dx\, dy \, ,$$

(4.7)
$$\Pr(R_n \geq k) \leq n! \int \int \inf_{z \geq 1} \left(z^{n-k} \Phi(x, y, z) \right) dx\, dy \, .$$

In the argument which follows we will not try to determine the best $z = z(x, y)$; it will be sufficient to choose $z = z(s)$ $(s = \sum_{i=1}^{n} x_i)$.

(I) Consider r_n first. Bounding each factor $1 - x_i(1 - z + z y_j)$ in (4.5) by $\exp\left(-x_i(1 - z + z y_j) \right)$, and integrating with respect to $y = (y_1, \ldots, y_n)$, we obtain from (4.6)

$$\Pr(r_n \leq k) \leq n! \int \inf_{0 < z \leq 1} \left(z^{n-k} \exp\left((z - 1)(n - 1)s \right) \prod_{j=1}^{n} \frac{1 - e^{-z s_j}}{z s_j} \right) dx \, .$$

$(s_j = \sum_{i \neq j} x_i)$. Here, by (3.7), (3.S) and the condition $z \leq 1$,

$$\prod_{j=1}^{n} \frac{1 - e^{-z s_j}}{z s_j} \leq c \left(\frac{1 - e^{-z s}}{z s} \right)^{n} ,$$

where c is an absolute constant. In conjunction with Lemma 1 (3.3), we have then

(4.8)
$$\Pr(r_n \leq k) \leq c n \int_0^n \inf_{0 < z \leq 1} \left(\exp\left(H(s, z) \right) \right) ds \, ,$$

(4.9)
$$H(s, z) := (z - 1)(n - 1)s + n \ln(1 - e^{-z s}) - k \ln z - \ln s \, ,$$

for all $n \leq k \leq n^2$. The relation (4.3) will be proven when we show that the right hand expression in (4.8) goes to 0 as $n^{-\delta}$ $(\delta \in (0, e^p - 1))$, for

(4.10)
$$k = n(\ln n - \ln \ln n - p) \, .$$

To make the best use of (4.8), it is natural to choose $z = z(s)$ which minimizes $H(s, z)$ for $z \in (0, 1]$. But

$$H_z = (n - 1)s + n s (e^{z s} - 1)^{-1} - k z^{-1} = 0$$

if $z s = a$ and a satisfies an equation

(4.11)
$$h(\alpha) = k \, , \qquad h(\alpha) := \alpha \left((n - 1) + n(e^\alpha - 1)^{-1} \right)$$

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Now, $h(0+) = n$, and an elementary (albeit tedious) computation shows that

$$h'(\alpha) \geq h'(0+) = n/2 - 1 \ .$$

Hence, (4.11) does have a unique positive root $a = a(k)$ for all $k > n$, and $a(k)$ is continuously differentiable with $a'(k) > 0$. In particular, if k is given by (4.10) then

$$a = (1 + O(\ln n/n))(k/n)$$

$$= \ln n - \ln \ln n - \rho + O(\ln' n/n)$$

(4.12) $$< \ln n - \ln \ln n - \rho' , \qquad \forall \rho' < \rho ,$$

for n sufficiently large.

Now, we can choose $z = a/s$ if $s > a$, and $z \equiv 1$ for $s \leq a$. Then it follows from (4.9) that

$$\int_0^n \inf_{0 < z < 1} \left(\exp(H(s, z)) \right) ds$$

$$\leq \int_0^a s^{-1}(1 - e^{-s})^n ds + a^{-k}(1 - e^{-a})^n e^{(n-1)a} \int_a^\infty s^{k-1} e^{-(n-1)s} ds$$

$$\leq A_1 + A_2 ,$$

where

$$A_1 = \int_0^a s^{-1}(1 - e^{-s})^n ds ,$$

$$A_2 = \left(a(n-1) \right)^{-k} (1 - e^{-a})^n e^{(n-1)a} (k-1)! .$$

So, if k satisfies (4.10),

$$A_1 \leq a(1 - e^{-a})^{n-1} = O\left(a \exp(-ne^{-a}) \right)$$

(4.13) $$= O(n^{-e^{\rho'}}) , \qquad \forall \rho' \in (0, \rho) .$$

Furthermore, by Stirling's formula for factorials,

(4.14) $$A_2 = O((\ln n/n)^{1/2} \exp\left(\phi_n(k) \right)) ,$$

where

(4.15) $$\phi_n(k) = F_n\left(k, a(k) \right) ,$$

(4.16) $$F_n(\kappa, a) := (n - 1)\alpha + n \ln(1 - e^{-\alpha}) - \kappa \ln \alpha + (\kappa - 1) \ln \frac{\kappa - 1}{e(n-1)} \ .$$

To sharply bound $\phi_n(k)$ from above, we need to look closer at $F_n(\kappa, \alpha)$. First of all,

$$\partial F_n(\kappa, \alpha)/\partial \alpha = (n - 1) + n(e^\alpha - 1)^{-1} - \kappa \alpha^{-1} ,$$

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so that

(4.17)
$$\partial F_n(\kappa, \alpha)/\partial \alpha|_{\alpha = a(\kappa)} = 0,$$

see (4.11), and

(4.18)
$$\partial F_n(\kappa, \alpha)/\partial \kappa = \ln \frac{\kappa - 1}{\alpha(n-1)}.$$

Consequently, (κ_0, α_0) defined by

(4.19)
$$e^{\alpha_0} - 1 = n\alpha_0,$$

(4.20)
$$\kappa_0 = (n-1)\alpha_0 + 1$$

is a stationary point, of $F_n(\kappa, a)$. An easy bootstrapping shows that

(4.21)
$$\alpha_0 = \ln n + \ln\ln n\big(1 + O(\ln^{-1} n)\big),$$

$$\kappa_0 = n\big(\ln n + \ln\ln n(1 + O(\ln^{-1} n))\big),$$

so that $\kappa_0 > k$ (see (4.10)). Then, by (4.15), (4.16), (4.19)–(4.21),

$$\phi_n(\kappa_0) = F_n(\kappa_0, \alpha_0) = n\ln(1 - e^{-\alpha_0}) - \ln \alpha_0$$

$$= -\ln \alpha_0 + O(ne^{-\alpha_0})$$

(4.22)
$$= -\ln \alpha_0 + O(\ln^{-1} n).$$

Let us show that, in fact, $\phi_n(\kappa_0) = \max \phi_n(\kappa)$. Indeed, since $a = a(\kappa)$ is differentiable, using (4.17), (4.18) we obtain

$$\phi'_n(\kappa) = \partial F_n(\kappa, \alpha)/\partial \kappa|_{\alpha = a = a(\kappa)}$$

(4.23)
$$= \ln\left(1 + \frac{1}{a(n-1)}\left(\frac{na}{e^a - 1} - 1\right)\right) \quad > (<)0,$$

for $\kappa < (>)\kappa_0$. ($a/(e^a - 1)$ decreases if α increases, and $a'(\kappa) > 0$.)

Since $\kappa < \kappa_0$, we now know that $\phi_n(b) < \phi_n(\kappa_0)$. Still, we would like to do better, knowing also that, in fact, $\kappa_0 - k$ is close to $2n \ln\ln n$. To this end, choose κ_1 such that $\alpha_1 = a(\kappa_1)$ satisfies

(4.24.)
$$\frac{n\alpha_1}{e^{\alpha_1} - 1} = \ln\ln n,$$

that is (cf. (4.19), (4.21)).

(4.25)
$$\alpha_1 = \ln n + \ln\ln n\big(1 + O(\ln^{-1} n)\big).$$

Then $\kappa_1 \in (k, \kappa_0)$; really, $\kappa_1 < \kappa_0$ since $\alpha_1 < \alpha_0$ (compare (4.21) and (4.25)) and $k < \kappa_1$ (compare (4.12) and (4.25)). Therefore,

$$\phi_n(k) = \phi_n(\kappa_0) - \int_k^{\kappa_0} \phi'_n(\kappa)\, d\kappa$$

(4.26)
$$\leq \phi_n(\kappa_0) - \int_k^{\kappa_1} \phi'_n(\kappa)\, d\kappa.$$

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The derivative $\phi'_n(\kappa)$ is given by (4.23). Since $na/(e^a - 1)$ is at least $\ln \ln n$ (see (4.24)) on $[k, \kappa_1]$, we easily get

$$(4.27) \qquad \phi'_n(\kappa) = (1 + o(1))e^{-a}, \qquad a = a(\kappa),$$

uniformly over $\kappa \in [k, \kappa_1]$. Besides, considering κ as a function of a, we have (see (4.11)) also

$$\frac{d\kappa}{da} = (n - 1) + n(e^a - 1 - ae^a)/(e^a - 1)^2$$
$$(4.28) \qquad = n(1 + O(ae^{-a})) = n(1 + o(1)).$$

Combining (4.26)–(4.28), we estimate

$$\phi_n(k) \le \phi_n(\kappa_0) - (1 + o(1))n(e^{-a(k)} - e^{-\alpha_1}).$$

Here (like in (4.13)))

$$ne^{-a(k)} \ge e^{\rho'} \ln n, \qquad \forall \, \rho' \in (0, \rho),$$

and, by (4.22), (4.25),

$$\phi_n(\kappa_0) \le 0,$$

$$ne^{-\alpha_1} = n \exp(-\ln n - \ln\ln n(1 + O(\ln^{-1} n)))$$

$$= O(\ln^{-1} n).$$

So, we arrive at

$$(4.29) \qquad \phi_n(k) \le -e^{\rho'} \ln n, \qquad \forall \, \rho' \in (0, \rho).$$

Therefore (see (4.14))

$$(4.30) \qquad A_2 = O(n^{-(1/2 + e^{\rho'})}), \qquad \forall \, \rho' \in (0, \rho), .$$

The estimates (4.8), (4.13), and (4.30) show that

$$\Pr(r_n \le n(\ln n - \ln\ln n - p)) = O(n^{-(e^{\rho'} - 1)}), \qquad \forall \, \rho' \in (0, \rho). \quad \blacksquare$$

(11) Turn now to R_n, the maximum rank of a stable matching. With the help of Lemma 1 (3.2), (3.7), (3.8), and (4.7), we obtain similarly to (4.8), (4.9),

$$(4.31) \qquad \Pr(R_n \ge k) \le cn \int_0^n \inf_{z \ge 1} (\exp(H_1(s, z)))\theta_n(s) \, ds$$

where

$$\theta_n(s) = \Pr(M_n \le s^{-1}),$$

and

$$(4.32) \qquad H_1(s, z) = \begin{cases} H(s, z) + \gamma sz, & s \le s_0, \\ H(s, z), & s > s_0; \end{cases}$$

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γ, s_0 are absolute constants. (In the part I, we could afford to drop the factor $\theta_n(s)$, but this time we will need it.)

Let us show that the right hand expression in (4.31) goes to 0 as $n^{-\delta}$ $(\delta \in (0, e^\rho - 1))$, for

$$k = n^2 \ln^{-1} n \left(1 + (\ln\ln n + \rho) \ln^{-1} n\right).$$

The root a of the equation (4.11) is this time

$$a = n \ln^{-1} n.(1 + (\ln\ln n. + \rho_n) \ln^{-1} n),$$

where

$$\rho_n = \rho + o(1).$$

We choose $z = a/s$ if $s < a$, and $z \equiv 1$ for $s \geq a$. (Recall that a feasible z has to be at least 1.) Set

(4.33)
$$a_1 = n \ln^{-1} n \left(1 + (\ln \ln n + \rho') \ln^{-1} n\right), \qquad \rho' \in (0, \rho).$$

Breaking $[0, n]$ into $[0, a_1]$, $[a_1, a]$, and $[a, n]$, and using (4.9), (4.32), we can write

$$\int_{J_0}^{n} \inf_{z \geq 1} \left(\exp\left(H_1(s, z)\right)\right) \theta_n(s)\, ds \leq a^{-k}(1 - e^{-a})^n e^{(n-1)a}(B_1 + B_2) + B_3,$$

where

$$B_1 = e^{\gamma a} \int_0^{a_1} s^{k-1} e^{-(n-1)s} \theta_n(s)\, ds,$$

$$B_2 = \int_{a_1}^{a} s^{k-1} e^{-(n-1)s} \theta_n(s)\, ds,$$

$$B_3 = \int_a^{n} s^{-1}(1 - e^{-s})^n @_n(s)\, ds.$$

We estimate B_1, B_2, B_3, moving backward.

(1) For $s \in [a, n]$,

$$\theta_n(s) = \Pr(M_n \leq s^{-1}) \leq \Pr(M_n \leq a^{-1}) = \theta_n(a).$$

Since

$$a^{-1} = n^{-1}(\ln n - \ln \ln n - \rho'_n), \qquad \rho'_n = \rho + o(1),$$

we have then by Lemma 2:

(4.34)
$$B_3 \leq \theta_n(a) \int_{J_a}^{n} s^{-1}\, ds = O(n^{-e^{\rho_3}}), \qquad \forall \rho_3 \in (0, \rho).$$

(2) Next,

$$B_2 \leq \theta_n(a_1) \int_{a_1}^{a} s^{k-1} e^{-(n-1)s}\, ds$$

Here

$$\theta_n(a_1) = O(n^{-e^{\rho_2}}), \qquad \forall \rho_2 \in (0, \rho'),$$

because (see (4.33)))

$$a_1^{-1} = n^{-1}(\ln n - \ln \ln n - \rho' + o(1)).$$

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Also

$$\int_{a_1}^{a} s^{k-1} e^{-(n-1)s}\, ds \le \int_0^\infty s^{k-1} e^{-(n-1)s}\, ds$$

$$= (n-1)^{-k}(k-1)! = 0\left(\left(\frac{k-1}{e(n-1)}\right)^{k-1}\right).$$

($k \sim n^2 \ln^{-1} n$.) Therefore,

(4.35)
$$B_2 = O\left(n^{-e^{\rho_2}}\left(\frac{k-1}{e(n-1)}\right)^{k-1}\right).$$

(3) Finally,

$$B_1 \le e^{\gamma a} \int_0^{a_1} s^{k-1} e^{-(n-1)s}\, ds$$

$$= e^{\gamma a} \int_0^{a_1} e^{q(s)}\, ds, \qquad q(s) := (k-1)\ln s - (n-1)s.$$

The function $q(s)$ achieves its maximum at

$$s_* = \frac{k-1}{n-1} = n\,111^{-1}\,n\big(1 + (\mathbf{111111}\,n + \rho_n'')\ln^{-1} n\big), \qquad \rho_n'' = \rho + o(1).$$

By (4.33), $s_* > a_1$, so that $q(s)$ is increasing on $[0, a_1]$. Also

$$s_* = a_1\,(1 + o(1))\,,$$

$$s^* - a_1 = n\,\ln^{-2} n\big(\rho - \rho' + o(1)\big)\,.$$

Therefore, since $q''(s) = -s^{-2}(k-1)$,

$$B_1 \le e^{\gamma a} a_1 e^{q(a_1)} \le a_1 e^{\gamma a}\exp\big(q(s_*) + \tfrac{1}{2}q''(\tilde s)(a_1 - s^*)^2\big)$$

$$(\tilde s \in [a_1, s^*])$$

(4.36)
$$= 0\left(\left(\frac{k-1}{e(n-1)}\right)^{k-1}\exp(\gamma a - An''\ln^{-3} n)\right)$$

$$= 0\left(\left(\frac{k-1}{e(n-1)}\right)^{k-1}\right)\exp(-\lambda n^2 \ln^{-3} n/2), \qquad \lambda > 0\,.$$

(Recall that $a = O(n\,\ln^{-1} n)$, so that $a = o(n^2 \ln^{-3} n)$.)

A combination of (4.31), (4.34)–(4.36) yields at last

$$\Pr(R_n \le k) \le cn\big(n^{-e^{\rho_2}}\exp(\phi_n(k)) + n^{-e^{\rho_3}}\big)\,,$$

for every $\rho_2 \in (0, \rho')$ and every $\rho_3 \in (0, \rho)$, provided that $\rho' < p.$ Here, as in the part I,

$$\phi_n(k) = F_n(k, a)\,, \qquad a = a(k)\,.$$

It remains to recall (see (4.22)) that the maximum value of the function $\phi_n(k)$ is negative.

The proof of Proposition is now complete. ∎

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The rest is short. First, of all, r_n equals the total number of proposals in the fundamental algorithm, and this number is *stochastically* dominated (Knuth [10], Wilson [13]) by the total number of draws in the coupon collector problem with n coupons, and the expected value of the latter is $n\,H_n$ ($H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$).

Given $\epsilon_n > 0$, let A be the event that a certain coupon, say j, has not been drawn in the first $N = (1 + \epsilon_n)n \ln n$ draws. Then

$$\Pr\left(r_n \leq (1 + \epsilon_n)n \ln n\right) \geq 1 - n\,\Pr(A)$$

$$= 1 - n(1 - n^{-1})^N \geq 1 - n\exp(-N/n)$$

$$= 1 - n^{-\epsilon_n} \to 1\,,$$

provided that $\epsilon_n \ln n \to \infty$. Combining this with (4.3), we obtain:

$$\Pr\left(n(\ln n - \ln\ln n - \rho) \leq r_n \leq n(\ln n + w(n))\right) \to 1\,,$$

for every $\rho > 0$, and $\omega(n) \to \infty$ however slowly. Consequently, $r_n/n \ln n \to 1$ in probability.

Furthermore, denote by π_{nj} the total number of proposals made to a woman j in the course of the fundamental algorithm. Let R_{nj} be the rank of her eventual partner, according to her preferences, needless to say. By symmetry,

$$(4.37) \qquad E(\pi_{n1}) = \cdots = E(\pi_{nn}) \leq n^{-1}(n\,H_n) = H_n\,.$$

Besides, given $\pi_{nj} = k$, $R_{nj} - 1$ is binomially distributed with parameters $n - k$ and $p = (k+1)^{-1}$. Therefore

$$E(R_{nj}|\pi_{nj}) = 1 + \frac{n - \pi_{nj}}{\pi_{nj} + 1} = \frac{n + 1}{\pi_{nj} + 1}\,,$$

and, by Jensen's inequality and (4.37),

$$E(R_{nj}) \geq \frac{n + 1}{E(\pi_{nj}) + 1} = \frac{n + 1}{H_n + 1}\,.$$

so,

$$E(R_n) = E\left(\sum_{j=1}^{n} R_{nj}\right) \geq n(n + 1)/(H_n + 1)$$

$$= n^2 \ln^{-1} n\left(1 + O(\ln^{-1} n)\right)\,.$$

A simple argument, which uses (4.4) and the last, relation yields that, in probability, $R_n/n^2 \ln^{-1} n \to 1$.

Theorem 2 is proven.

Appendix

Proof of Lemma 1. For $0 < s_1 < s_2 < n$,

$$\Pr(s_1 \leq S_n \leq s_2) = \int \cdots \int dx_1 \ldots dx_n,$$

where $0 \leq x_j \leq 1$ ($1 \leq j \leq n$), and $s_n = \sum_{j=1}^{n} x_j \in (s_1, s_2)$. We switch to new variables

$$u = \sum_{j=1}^{n} x_j,$$

$$v_j = x_j u^{-1}, \qquad 1 \leq j \leq n - 1$$

Define also $v_n = x_n n^{-1}$, so that $\sum_{j=1}^{n} v_j = 1$. The inverse transformation is

$$x_j = u v_j, \qquad 1 \leq j \leq n,$$

where

$$v_n = 1 - \sum_{j=1}^{n-1} v_j.$$

Its Jacobian is u^{n-1}, whence

$$\Pr(s_1 \leq S_n \leq s_2) = \int \cdots \int u^{n-1} \, du \, dv_1 \ldots dv_{n-1},$$

where $s_1 \leq u \leq s_2$ and $\max_{1 \leq j \leq n} v_j \leq u^{-1}$. Therefore

$$f_n(s) = s^{n-1} \underbrace{\int \cdots \int}_{\substack{\max v_j \leq s^{-1} \\ 1 \leq j \leq n}} dv_1 \ldots dv_{n-1}$$

$$= \frac{s^{n-1}}{(n-1)!} \Pr(M_n \leq s^{-1}),$$

since $(n - 1)!$ is the joint density of the first $(n - 1)$ intervals $L_1, \ldots, L_{n-1}$ in the random partition of $[0, 1]$ by, $(n - 1)$ random points.

Similarly, denoting $\sum_{j=1}^{n} v_j^2$ by t,

$$\Pr(s_1 \leq S_n \leq s_2, \, t_1 \leq T_n \leq t_2) = \underbrace{\int \cdots \int}_{\substack{s_1 \leq u \leq s_2 \\ t_1 \leq t \leq t_2 \\ \max v_j \leq u^{-1}}} u^{n-1} du \, dv_1 \ldots dv_{n-1}$$

$$\leq \underbrace{\int \cdots \int}_{\substack{s_1 \leq u \leq s_2 \\ t_1 \leq t \leq t_2}} u^{n-1} du \, dv_1 \ldots dv_{n-1}$$

(in both integrals, $\sum_{j=1}^{n-1} v_j \leq 1$). Since this inequality holds for all $s_1 < s_2$ and $t_1 < t_2$, we have

$$f_n(s,\tau) \leq \frac{s^{n-1}}{(n-1)!} \frac{d}{d\tau} \left(\underbrace{\int \cdots \int}_{t \leq \tau} (n-1)! \, dv_1 \ldots dv_{n-1} \right)$$

$$= \frac{s^{n-1}}{(n-1)!} g_n(\tau),$$

where $g_n(\cdot)$ is the density of $\sum_{j=1}^{n} L_j^2$.

Proof of Lemma 2. The random variables $L_1, \ldots, L_n$ are exchangeable and, for $1 \leq k \leq 12 - 1$, the joint density of $L_1, \ldots, L_k$ is given by

$$(A.1) \qquad f_{L_1 \ldots L_k}(x_1, \ldots, x_k) = (n-1)^{\underline{k}} \left(1 - \sum_{j=1}^{k} x_j \right)^{n-k-1}, \qquad \left(a^{\underline{b}} \underset{\text{def}}{=} a(a-1) \ldots (a-b+1) \right),$$

where $0 \leq x_j \leq 1$, $\sum_{j=1}^{k} x_j \leq 1$. In particular,

$$(A.2) \qquad\qquad \mathrm{Pr}(L_1 \geq x_1, \ldots, L_k \geq x_k) = \left(1 - \sum_{j=1}^{k} x_j \right)^{n},$$

provided that $\sum_{j=1}^{k} x_j \leq 1$. (See E'eller [3, Ch. I].)

(1) Fix z and let $x = (\ln n + z)/n$. Define N_n as the total number of the variables $L_j \geq x$. Then by (A.2), for every $k \geq 1$ and n large enough,

$$E(N_n^{\underline{k}}) = n^{\underline{k}} \mathrm{Pr}(L_1 \geq 2, \ldots, L_k \geq x)$$

$$= n^{\underline{k}}(1 - kx)^n = (1 + o(1))n^k \exp(n \ln(1 - kx))$$

$$= (1 + o(1)) \exp(k \ln n + n(-kx + O(\ln^2 n/n^2)))$$

$$= (1 + o(1))(e^{-z})^k, \qquad n \to \infty.$$

Therefore, N_n converges in distribution to a Poisson distributed random variable N with parameter $\lambda = e^{-z}$. Consequently, denoting $\max_{1 \leq j \leq n} L_j$ by M_n,

$$\mathrm{Pr}(M_n < (\ln n + z)/n) = \mathrm{Pr}(N_n = 0) \to \mathrm{Pr}(N = 0) = e^{-e^{-z}},$$

and

$$M_n = (\ln n + O_p(1))/n,$$

where $O_p(1)$ stands for a random variable bounded in probability as $n \to \infty$. ∎

(2) Let $x = (\ln n - \ln \ln n - \rho)/n$. We want to show that

$$\mathrm{Pr}(M_n \leq x) = O(n^{-e^{\rho'}}), \qquad \forall \rho' \in (0, \rho).$$

To this end, we observe that $(L_1, \ldots, L_n)$ coincides in distribution with $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ where

$$\mathcal{L}_j = W_j \left(\sum_{k=1}^{n} W_k \right)^{-1}$$

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and $W_1, \ldots, W_n$ are independent, exponentially distributed with parameter 1 (Breiman [1, Ch. 13], Karlin and Taylor [9, Ch. 13]). Using the central limit theorem for (moderately) large deviations (Feller [3, Ch. 16]), we have ($E(\mathsf{W}j) = \mathrm{var}(W_j) = 1$) :

$$\Pr\left(\left|\sum_{j=1}^n W_j - n\right| \geq n^{1/7} \cdot n^{1/2}\right) = O\left(\exp(-n^{2/7}/2)\right).$$

so,

$$\Pr(M_n \leq x) \leq \Pr\left(\max_{1 \leq j \leq n} \mathsf{W}j \leq x(n + n^{9/14})\right) + O\left(\exp(-n^{2/7}/2)\right).$$

Here,

$$\Pr\left(\max_{1 \leq j \leq n} W_j \leq x(n + n^{9/14})\right) = \left(1 - \exp\left(-x(n + n^{9/14})\right)\right)^n$$

$$\leq \exp\left(-n\exp\left(-x(n + n^{9/14})\right)\right)$$

$$= \exp\left(-\exp\left(\ln n - (\ln n - \ln\ln 77 - \rho)(1 + n^{-5/14})\right)\right)$$

$$\leq \exp(-\exp(\ln\ln n + \rho')) = n^{-e^{\rho'}}, \qquad \forall \rho' \in (0, \rho). \quad \blacksquare$$

(3) It remains to show that $nU_n \to 2$ in probability, where $U_n = \sum_{j=1}^n L_j^2$. To this end, we notice that U_n coincides, in distribution, with

$$U_n = \left(\sum_{j=1}^n W_j^2\right) \Big/ \left(\sum_{k=1}^n W_k\right)^2 ,$$

and, by the weak law of large numbers,

$$(\mathrm{P}) \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n W_j^2 = \int_0^\infty x^2 e^{-x}\, dx = 2 ,$$

$$(\mathrm{P}) \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n W_j = \int_0^\infty x e^{-x}\, dx = 1 ;$$

so,

$$(\mathrm{P}) \lim_{n \to \infty} n U_n = 2 .$$

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