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AN "ALMOST-EXACT" SOLUTION TO THE N-PROCESSOR, M-MEMORY BANDWIDTH PROBLEM

by

B. Ramakrishna Rau

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ABSTRACT

A closed-form expression is derived for the memory bandwidth obtained when N processors are permitted to generate requests to M memory modules. Use of generating functions is made, in a rather unusual fashion, to obtain this expression. The one approximation involved is shown to result in only a very small error--and that, too, only for small values of M and N. This expression, which is asymptotically exact, is shown to be more accurate than existing closed form approximations. Lastly, a family of asymptotically exact solutions are presented which are easier to evaluate than is the first one. Although these expressions are less accurate than the previously derived closed-form solution, they are, nevertheless, better than existing solutions. This family of solutions is shown to include a couple of existing solutions.

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1. INTRODUCTION

Interleaved memories are employed in most medium and large scale computer systems. More recently, a certain amount of interest has been shown in multiprocessor configurations served by a common interleaved memory, an example being the C.mmp [6]. It is of practical interest to have available analytic tools capable of predicting the performance of such systems.

A model of such a system was discussed by Skinner and Asher [4]. A slightly simpler model considered by Strecker [5] will be used in this study. The model makes the following assumptions:

1) The system consists of N identical processors and M identical memory modules interconnected so that any processor is able to make a request to any memory.

2) The system operates synchronously. All the processors generate their requests at the beginning of a memory cycle. Each memory, which has at least one request outstanding, services exactly one of those during the cycle, at the end of which every processor whose request was serviced during the cycle is released to make a new request. Processors which were not serviced remain queued at the appropriate module until serviced.

3) The system is memory limited, i.e., a processor which has been serviced during one cycle immediately submits a new request the very next cycle.

4) Processors make requests with equal probability to all memory modules.

5) An enqueued processor cannot generate a new request until the previous one has been serviced.

In [5] an approximate solution was found for the bandwidth observed in such a model. Bhandarkar,[2], describes the exact analysis of this model by constructing a Markov chain and solving for the steady state probabilities. Though this result is an exact solution, it is computationally expensive since the analysis has to be repeated for every pair M, N. Baskett and Smith, [1], have obtained two asymptotically exact solutions to the same model. In addition, they give some evidence to show that this abstract model produces results which agree fairly well with those observed in practice.

As in [1] and [2] we shall represent the state of the system by a Markov chain. Since the processors are identical, the state of the system is fully described by an M-tuple which lists the number of processors queued or in service at each memory module. The set of all feasible states constitutes a Markov chain since the next state depends only on the current state. Furthermore, this Markov chain is aperiodic since it is possible to make a transition from a state to itself, and it is irreducible since it is possible to reach one state from another one in a finite number of transitions. Consequently, the Markov chain has a unique steady state solution. Our approach shall be to represent this steady state solution by a generating function, which we shall then use to calculate the steady state probabilities of interest. The advantage of this technique derives from the fact that it is unnecessary to enumerate all the states. Hence it is possible to obtain a solution to the model which is not specific to a particular choice of the pair M, N.

2. THE GENERATING FUNCTION TECHNIQUE

In this section we shall illustrate with a trivial example the technique used in the next section to solve the bandwidth problem.

Assume we have an M/D/l server whose service time is 1 cycle. The arrival process constitutes a sequence of Bernoulli trials, i.e., an arrival occurs at the beginning of a cycle with probability p and does not occur with probability (1-P). Let the probability during cycle t of there being n customers waiting or receiving service be P(n;t). We can then represent the state at t by the generating function,

$$H(x;t) = \sum_{n=0}^{\infty} P(n;t)x^n$$

Similarly, the generating function which represents the number of arriving customers at the beginning of a cycle is,

$$Q(x) = (1-p)+px$$

At the end of cycle t, the server completes the service of a customer and releases it if the server was not idle. Accordingly, if we represent the state of the server at the end of the cycle by H(x;t+) then we have

$$H(x;t+) = P(0;t) + \sum_{n=1}^{\infty} P(n;t)x^{n-1}$$

$$= \frac{H(x;t) - H(0;t)}{x} + H(0;t) = \frac{1}{x} H(x;t) + (1-\frac{1}{x})H(0;t)$$

since $H(0;t) = P(0;t)$.

Consequently, $H(x;t+1)$ is given by

$H(x;t+1) = Q(x)H(x;t+)$, since the arrivals are independent of the state

$$= Q(x)\left[\frac{1}{x}H(x;t) + (1-\frac{1}{x}) H(0;t)\right]$$

In steady state we have $H(x;t+1) = H(x;t)$ and, therefore,

$$H(x) = \frac{1}{x} H(x)Q(x) + (1-\frac{1}{x})H(0)Q(x)$$

where H(x) represents the steady state value of H(x;t).

therefore,
$$H(x)\left[1-\frac{1}{x} Q(x)\right] = (1-\frac{1}{x})H(0)Q(x)$$

and
$$H(0) = \frac{x-Q(x)}{(x-1)Q(x)} H(x)$$

Now $H(1) = \sum_{n=0}^{\infty} P(n) = 1$, $Q(1) = (1-p) + p = 1$ (1)

and $H(0) = P(0)$ = probability the server is idle.

We can solve for $H(0)$ by putting $x=1$. However, since both numerator and the denominator on the right hand side go to zero we must apply L'Hospital's rule. Doing so, we get,

$$H(0) = \left. \frac{1-Q'(x)}{Q(x)+(x-1)Q'(x)} \right|_{x=1} H(1)$$

since $Q'(1) = p$ we have

$$H(0) = \frac{1-p}{1+(0)(1-p)} \cdot 1 = 1-p$$

Therefore, the probability that the server is idle = $(1-p)$ and

$$H(x) = \frac{(x-1)[(1-p)+px]}{x-[(1-p)+px]} (1-p) = Q(x)$$

This trivial result demonstrates the manner in which we can employ generating functions to solve Markov chains. The method used in the next section relies essentially on the same technique.

3. A RECURRENCE RELATION

In this section we shall use the generating function technique to obtain a recurrence relation. As before, the steady state distribution of the Markov chain corresponding to the N-processor, M-memory model is represented by a generating function, but one involving M variables. We have

$$H(x_1,\ldots,x_M) = \sum_{j \in \mathcal{J}} P(S_j) \prod_{i=1}^{M} x_i^{n_i}$$

where $\mathcal{J}$ is the index set into the states of the Markov chain, S_j is the j-th state of the Markov chain, $P(S_j)$ is the steady probability of being in state S_j, n_i is the number of processors queued up on the i-th memory module when the system is in state S_j,

and
$$\sum_{i=1}^{M} n_i = N$$

At the end of each cycle, every memory which was not idle will release a processor, i.e., every $n_i \geq 1$ will be decremented by 1. However, each released processor queues up during the next cycle with equal probability on one of the memory modules. The generating function for a released processor is given by

$$Q(x_1,\ldots,x_M) = \sum_{i=1}^{M} \frac{1}{M} x_i$$

It is convenient to define an operator R_i which performs the above operations on module i, viz,

$$\{R_i\} H(x_1,\ldots,x_M) = H(x_1,\ldots,x_M)\Big|_{x_i=0} + \left[H(x_1,\ldots,x_M) - H(x_1,\ldots,x_M)\Big|_{x_i=0} \right] \cdot \frac{1}{x_i} \cdot Q(x_1,\ldots,x_M)$$

$$= \frac{Q(x_1,\ldots,x_M)}{x_i} H(x_1,\ldots,x_M) + \left(\frac{1-Q(x_1,\ldots,x_M)}{x_1}\right) \{U_i\} H(x_1,\ldots,x_M)$$

where U_i is defined as an operator such that

$$\{U_i\} H(x_1,\ldots,x_M) = H(x_1,\ldots,x_M)\Big|_{x_i=0}$$

Accordingly, the generating function at the end of a cycle is given by

$$\left\{\prod_{i=1}^{M} R_i\right\} H(x_1,\ldots,x_M) = \left\{\prod_{i=1}^{M}\left[\frac{Q(x_1,\ldots,x_M)}{x_1} + \frac{1-Q(x_1,\ldots,x_M)}{x_1} U_1\right]\right\} H(x_1,\ldots,x_M)$$

where $\left\{\prod_{i=1}^{M} R_i\right\}$ denotes the concatenation of M operators. But since the Markov chain is in steady state by assumption, we have

$$H(x_1,\ldots,x_M) = \left\{\prod_{i=1}^{M}\left[\frac{Q(x_1,\ldots,x_M)}{x_1} + \left(\frac{1-Q(x_1,\ldots,X_M)}{x_i}\right) U_i\right]\right\} H(x_1,\ldots,x_M)$$

We can obtain an expression involving the generating function for the steady state marginal distribution of the queue size of module 1 by putting $x_2 = x_3 = \ldots = x_M = 1$. Noting that

$$Q(x_1,1,\ldots,1) = \left(\frac{x_1}{M} + \frac{M-1}{M}\right) = V(x_1) \text{ by definition}$$

and that

$\{U_i\}\ H(x_1,1,\ldots,1)$ will have a 0 in the i-th position

we get

$$H(x_1,1,\ldots,1) = \left\{\left[\frac{V(x_1)}{x_1} + \left(\frac{1-V(x_1)}{x_1}\right)U_1\right] \prod_{i=2}^{M}\left[V(x_1)+(1-V(x_1))\,U_i\right]\right\}$$

$$\times\ H(x_1,1,\ldots,1)$$

where the operator Ui, $i \neq 1$, now has the effect of replacing the 1 in the i-th position by a 0. However, by the symmetry of the model, the probability of a certain configuration of queue sizes is the same irrespective of the module that each of those queues is associated with. Consequently, $H(x_1,\ldots,x_M)$ is symmetric in all the x_i's. In particular,

$$\{U_i\}\ H(x1,1,1,\ldots,1) = \{U_j\}\ H(x_1,1,\ldots,1) \quad \text{for } i \neq 1,\ j \neq 1,\ i \neq j.$$

It is, therefore, of no consequence which position i, $i \neq 1$ has the zero. All that matters is the total number of positions which contain a zero. We can also replace the operator U_i, $i \neq 1$ by the single operator U, and the operator U^k may be defined as one which replaces k 1's by 0's.

Lastly, if we define $G_M(x_1;k)$ to mean $H(x_1,1,\ldots,1)$ with k 1's replaced by 0's and where the subscript M refers to the dimensionality of $H(x_1,\ldots,x_M)$, we have

$$U^k G_M(x_1;0) = G_M(x_1;k) \text{ if } k < M$$

therefore, $G_M(x_1;0) = H(x_1,1\ldots,1)$

$$= \left\{\left[\frac{V}{x_1} + \frac{1-V}{x_1} U_1\right]\left[V+(1-V)U\right]^{M-1}\right\} G_M(x_1;0)$$

where $V \equiv V(x_1)$

$$= \left\{\left[\frac{V}{x_1} + \left(1-\frac{V}{x_1}\right) U_1\right]\left[\sum_{i=0}^{M-1} V^{M-1-i}(1-V)^i U^i \binom{M-1}{i}\right]\right\} G_M(x_1;0)$$

$$= \left\{\left[\frac{V}{x_1} + \left(1-\frac{V}{x_1}\right) U_1\right]\right\} \sum_{i=0}^{M-1} \binom{M-1}{i} V^{M-1-i}(i-V)^i \; G_M(X_1;i)$$

therefore,

$$G_M(x_1;0) = \sum_{i=0}^{M-1}\binom{M-1}{i} V^{M-i}(i-V)^i \frac{G_M(x_1;i)}{x_1} + \sum_{i=0}^{M-1}\binom{M-1}{i}\left(1-\frac{V}{x_1}\right)V^{M-1-i}(1-V)^i G_M(0;i)$$

Now, $G_M(0;i) = G_M(1;i+1)$, $i \neq M-1$, by the definition of $G_M(.;.)$

and $G_M(0;M-1) = 0$ since the probability of the queue size at all the modules being 0 simultaneously is 0 for $N > 0$.

therefore, $\displaystyle\sum_{i=0}^{M-1}\binom{M-1}{i}\left(1-\frac{V}{x_1}\right)V^{M-1-i}(1-V)^i G_M(0;i)$

$$= \sum_{i=0}^{M-2}\binom{M-1}{i}\left(1-\frac{V}{x_1}\right)V^{M-1-i}(1-V)^i G_M(1;i+1)$$

$$= \sum_{i=1}^{M-1}\binom{M-1}{i-1}\left(1-\frac{V}{x_1}\right)V^{M-i}(1-V)^{i-1} G_M(1;i)$$

therefore,

$$G_M(x_1;0) = V^M \frac{G_M(x_1;0)}{x_1} + \sum_{i=1}^{M-1}\binom{M-1}{i} V^{M-i}(1-V)^i \frac{G_M(x_1;i)}{x_1}$$

$$+\sum_{i=1}^{M-1}\binom{M-1}{i-1}V^{M-i}(i-V)^{i-1}(1-\frac{V}{x_1})G_M(1;i)$$

Replacing x_1 by x and multiplying on both sides by x we get

$$xG_M(x;0) - V^M G_M(x;0) = \sum_{i=0}^{M-1}\binom{M-1}{i}V^{M-i}(1-V)^i G_M(x;i)$$

$$+\sum_{i-1}^{M-1}\binom{M-1}{i-1}V^{M-i}(1-V)^{i-1}(x-V)G_M(1;i)$$

therefore

$$G_M(x;0) = \sum_{i-1}^{M-1}\binom{M-1}{i}\frac{V^{M-i}(1-V)^i}{(x-V^M)}G_M(x;i) + \sum_{i=1}^{M-1}\binom{M-1}{i-1}\frac{V^{M-i}(1-V)^{i-1}(x-V)}{(x-V^M)}G_M(1;i) \qquad (2)$$

We now make the one approximation in the entire analysis.

$\frac{G_M(x;j)}{G_M(1;j)}$ is the conditional marginal distribution of one queue given that j other queues are empty. To see that this is so we first observe that

$\left\{\prod_{i=M-j+1}^{M} U_i\right\} H(x_1,\ldots,x_M)$, $j < M$, i.e.,

$H(x_1,..,x_M)$ with the last j x_i's put to 0, is the generating function representing those states for which the queue sizes of modules $(M-j+1)$ through M are 0. And $\left\{\prod_{i=M-j+1}^{M} U_i\right\} H(x_1,\ldots,x_M)\Big|_{x_1=..=x_M=1}$ is the total probability of being in one of these states, and it is identical to $G_M(1;j)$. Therefore,

$\frac{\left\{\prod_{i=M-j+1}^{M} U_i\right\} H(x_1,\ldots,x_M)}{G_M(1;j)}$ = the conditional generating function for modules 1 through M-j given that modules $M-j+1$ through M are idle (0 queue size)

And if we put $x_2=\ldots=x_{M-j}=1$ we get

$\frac{G_M(x_1;j)}{G_M(1;j)}$ = the conditional marginal generating function for module 1 given that j other modules are idle.

The approximation used is to equate this conditional marginal generating function to the marginal generating function obtained if there were only M-j modules (N being held constant), i.e. $\frac{G_M(x;j)}{G_M(1;j)} = G_{M-j}(x;0)$, $\quad j < M$

the rationale being that since we are given that j modules are idle, they might as well have not been present at all! From the above equation we get the relation

$$G_M(x;j) = G_{M-j}(x;0)\ G_M(1;j)\ ,\ j < M \tag{3}$$

If in Equation (2) we put $x=1$, we shall obtain the recurrence relation we desire. On the left hand side we have

$$G_M(1;0) = H(1,\ldots,1) = 1 \tag{4}$$

On the right hand side, both the numerator and the denominator go to zero as x approaches 1. By applying L'Hospital's rule twice and then putting $x=1$ we get the following results:

As $x \to 1$,

$$\sum_{i=1}^{M-1}\binom{M-1}{i}\frac{V^{M-i}(1-V)^i}{(x-V^M)}\ G_{M-i}(x;0)G_M(1;i) \longrightarrow$$

$$\frac{2(M-1)}{M}G_{M-1}(1;0)G_M(1;1) + 2G'_{M-1}(1;0)G_M(1;1) - \frac{(M-2)}{M}G_{M-2}(1;0)G_M(1;2)$$

But $G_M(1;2) = G_M(0;1)$ by the definition of $G_M(.;.)$

$= G_M(1;1)G_{M-1}(0;0)$ by (3)

and $G_{M-1}(1;0) = G_{M-2}(1;0) = 1$ by (4)

and so the above e pression, in the limit, reduces to

$$\left[2\frac{(M-1)}{M} + 2G'_{M-1}(1;0) - \frac{M-2}{M}G_{M-1}(0;0)\right]\ G_M(1;1) \tag{5}$$

Similarly, as $x \to 1$

$$\sum_{i=1}^{M-1}\binom{M-1}{i-1}\frac{V^{M-i}(1-V)^{i-1}(x-V)}{(x-V^M)}\ G_M(1;i) \to \left[\frac{2(M-1)}{M} + \frac{2(M-1)}{M}G_{M-1}(0;0)\right]G_M(1;1) \tag{6}$$

Adding up Expressions (5) and (6) we get

$$\lim_{x\to 1} G_M(x;0) = G_M(1;0) = 1 = G_M(1;1)(G_{M-1}(0;0)+2G'_{M-1}(1;0))$$

therefore, $G_M(1;1) = \dfrac{1}{G_{M-1}(0;0)+2G'_{M-1}(1;0)}$

However, $G_M(1;1) = G_M(0;0)$ by the definition of $G_M(.;.)$

and $G'_{M-1}(1;0)$ = average queue size at any one of the (M-1) modules when there are (M-1) modules and N processors in the system

$$= \frac{N}{M-1}$$

therefore,

$$G_M(0;0) = \frac{1}{G_{M-1}(0;0)+\frac{2N}{M-1}}$$

This is the desired recurrence relation in M for $G_M(0;0)$ which is the probability that a module will be idle in steady state. To make the dependence on N more explicit we define

$F(M,N) = G_M(0;0)$ = probability that a memory module is idle in an N-processor, M-memory system

and $T(M,N) = M(1-F(M,N))$ = the average memory bandwidth in an N-processor, M-memory system.

Since $F(M-1,N) = 1 - \frac{T(M-1,N)}{M-1}$

and $F(M,N) = \frac{1}{F(M-1,N)+\frac{2N}{M-1}}$

we have

$$T(M,N) = M(1-F(M,N)) = M\left[1 - \frac{1}{1-\frac{T(M-1,N)}{M-1} + \frac{2N}{M-1}}\right], \tag{7}$$

which is a recurrence relation for T(M,N) in M. Knowing that $T(1,N) = 1$ for all N, we can recursively calculate T(M,N) for any M,N. Table 1 lists the values so computed and Table 2 displays the percentage errors of these results relative to the correct results obtained by an exact solution of the Markov chain as documented in [2]. The agreement is very good.

An examination of Table 1 would lead one to suspect that T(M,N) is symmetric in M and N. To prove this we shall show that T(M,N) also satisfies the following recurrence relation,

$$T(M,N) = N\left[1 - \frac{1}{1-\frac{T(M,N-1)}{N-1} + \frac{2M}{N-1}}\right] \tag{8}$$

We shall do so by two-dimensional induction. Assume that relation (8) is true for all $N \geq 1$, $1 \leq M < m$ and for $1 \leq N \leq n$, M=m. We shall then show that

$$T(m,n+1) = (n+1)\left[1 - \frac{1}{1-\frac{T(m,n)}{n} - \frac{2m}{n}}\right]$$

By (7) we have

$$T(m,n+1) = m\left[1 - \frac{1}{1-\frac{T(m-1,n+1)}{m-1} + \frac{2(n+1)}{m-1}}\right] \tag{9}$$

By the above assumptions and by (8) we have

$$T(m-1,n+1) = (n+1)\left[1 - \frac{1}{1-\frac{T(m-1,n)}{n} + \frac{2(m-1)}{n}}\right] \tag{10}$$

and again by (7) we have

$$T(m,n) = m\left[1 - \frac{1}{1-\frac{T(m-1,n)}{m-1} + \frac{2n}{m-1}}\right]$$

by rewriting which we get

$$T(m-1,n) = (m-1)\left[1 + \frac{2n}{m-1} - \frac{1}{1-\frac{T(m,n)}{m}}\right] \tag{11}$$

Substituting for $T(m-1,n+1)$ in (9) by using (10) and for $T(m-1,n)$ in the resulting expression by using (11) and after some tedious simplification we obtain

$$T(m,n+1) = (n+1)\left[1 - \frac{1}{1-\frac{T(m,n)}{n} + \frac{2m}{n}}\right]$$

which proves the induction step.

The basis step is as follows -- by definition, $T(0,N)=T(M,0)=T(0,0)=0$. Therefore,

$$T(1,N) = 1\left[1 - \frac{(N-1)}{(N-1)-T(1,N-1)+2M}\right] = 1 \text{ for all } N.$$

and $$T(M,1) = 1\left[1 - \frac{(1-1)}{(1-1)-T(M,-0)+2M}\right] = 1 \text{ for all } M$$

irrespective of the value that $T(M,-0)$ assumes.

Both these above results are correct and, therefore, the basis step is valid.

Since the recurrence relations in M and in N are duals and since the boundary conditions for both recursions are identical, $T(a,b)$ when obtained by one recurrence relation must necessarily be indentical with T(b,a) obtained by the other recurrence relation. Hence,

$$T(a,b) = T(b,a) \quad a \geq 0,\ b \geq 0$$

and, therefore, T(M,N) is symmetric in M and N.

4. CLOSED FORM SOLUTION

Using the recurrence relation in M, Equation (7), we can obtain expressions for the bandwidth obtained for a particular value of M and as a function of N.

$$T(1,N) = 1$$

$$T(2,N) = 2\left[1 - \frac{1}{1 - \frac{T(1,N)}{1} + \frac{2N}{1}}\right] = \frac{4N-2}{2N}$$

Similarly $T(3,N) = \frac{12N^2-12N+6}{4N^2+2}$

$$T(4,N) = \frac{32N^3-48N^2+64N-24}{8N^3+16N}$$

$$T(5,N) = \frac{80N^4-160N^3+400N^2-320N+120}{16N^4+80N^2+24}$$

We can use these expressions to make certain observations which can guide us in our quest for the closed form solution. Firstly, we are led to the conjecture that T(M,N) is the ratio of two (M-1)-th order polynomials in N. The closed form solution must, therefore, contain a function which makes the coefficient of higher powers of N go to zero. A likely candidate for T(M,N) is then given by

$$T(M,N) = \frac{\sum_{i=0}^{\infty} \binom{M-1}{i} (\text{polynomial of order } i \text{ in } N)}{\sum_{i=0}^{\infty} \binom{M-1}{i} (\text{another polynomial of order } i \text{ in } N)}$$

For $i \geq M$, $\binom{M-1}{i} = 0$ and, therefore, both the numerator polynomial and the denominator polynomial will be of order (M-1). However, we know that T(M,N) is symmetric in M and N and also that $\binom{N-1}{i}$ is an i-th order polynomial in N. It is tempting to look for a solution of the form

$$T(M,N) = \frac{\sum_{i=0}^{\infty} a_i \binom{M-1}{i}\binom{N-1}{i}}{\sum_{i=0}^{\infty} b_i \binom{M-1}{i}\binom{N-1}{i}} = \frac{A(M,N)}{B(M,N)}$$

If we attempt to write A(M,N) in this form for M = 1,...,5 we get

$$A(1,N) = 1\binom{0}{0}\binom{N-1}{0}$$

$$A(2,N) = 2\binom{1}{0}\binom{N-1}{0} + 4\binom{1}{1}\binom{N-1}{1}$$

$$A(3,N) = 6\binom{2}{0}\binom{N-1}{0} + 12\binom{2}{1}\binom{N-1}{1} + 24\binom{2}{2}\binom{N-1}{2}$$

$$A(4,N) = 24\binom{3}{0}\binom{N-1}{0} + 48\binom{3}{1}\binom{N-1}{1} + 96\binom{3}{2}\binom{N-1}{2} + 192\binom{3}{3}\binom{N-1}{3}$$

$$A(5,N) = 120\binom{4}{0}\binom{N-1}{0} + 240\binom{4}{1}\binom{N-1}{1} + 480\binom{4}{2}\binom{N-1}{2} + 960\binom{4}{3}\binom{N-1}{3} + 1920\binom{4}{4}\binom{N-1}{4}$$

Noting that the coefficients are each twice the previous one and that the first one is M!, we conjecture that

$$A(M,N) = M!\sum_{i=0}^{\infty} 2^i \binom{M-1}{i}\binom{N-1}{i}$$

By a similar procedure we are led to the conjecture that

$$B(M,N) = M!\sum_{i=0}^{\infty} \frac{2^i}{(i+1)}\binom{M-1}{i}\binom{N-1}{i}$$

Accordingly, T(M,N) is given by

$$T(M,N) = \frac{\sum_{i=0}^{\infty} 2^i \binom{M-1}{i}\binom{N-1}{i}}{\sum_{i=0}^{\infty} \frac{2^i}{i+1}\binom{M-1}{i}\binom{N-1}{i}}$$

We shall now prove that this expression satisfies the recurrence relation of F(M,N)

$$F(M,N) = 1 - \frac{1}{M}T(M,N) = 1 - \frac{\sum 2^i \binom{M-1}{i}\binom{N-1}{i}}{M\sum \frac{2^i}{i+1}\binom{M-1}{i}\binom{N-1}{i}}$$

where the limits of summation are assumed to be from 0 to ∞ unless otherwise stated.

Now, $\frac{M}{i+1}\binom{M-1}{i} = \binom{M}{i+1}$

therefore, $F(M,N) = 1 - \dfrac{\sum 2^i \binom{M-1}{i}\binom{N-1}{i}}{\sum 2^i \binom{M}{i+1}\binom{N-1}{i}}$

$$= \frac{\sum 2^i \binom{M}{i+1}\binom{N-1}{i} - \sum 2^i \binom{M-1}{i}\binom{N-1}{i}}{\sum 2^i \binom{M}{i+1}\binom{N-1}{i}}$$

but $\binom{M}{i+1} = \binom{M-1}{i+1} + \binom{M-1}{i}$, refer to [3], (12)

therefore, $F(M,N) = \dfrac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i} + \sum 2^i \binom{M-1}{i}\binom{N-1}{i}}$, (13)

$$= \frac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i-1}\binom{N-1}{i}}$$

by (12)

$$= \frac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + \sum 2^{i+1} \binom{M-2}{i}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i-1}\binom{N-1}{i}}$$

Now, $\sum 2^{i+1} \binom{M-2}{i}\binom{N-1}{i} = \sum_{i \geq 1} 2^i \binom{M-2}{i-1}\binom{N-1}{i-1} = \sum 2^i \binom{M-2}{i-1}\binom{N-1}{i-1}$

since $\binom{M-2}{-1} = \binom{N-1}{-1} = 0$

therefore, $F(M,N) = \dfrac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i-1}\binom{N-1}{i-1} + \sum 2^i \binom{M-2}{i-1}\binom{N-1}{i}}$

$$= \frac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + \sum 2^i \binom{M-2}{i-1}\binom{N}{i}} \quad \text{by (12)}$$

again, $\sum 2^i \binom{M-2}{i-1}\binom{N}{i} = \sum_{i \geq 1} 2^i \binom{M-2}{i-1}\binom{N}{i} = \sum 2^{i+1} \binom{M-2}{i}\binom{N}{i+1}$

$$= 2 \sum 2^i \binom{M-2}{i}\binom{N}{i+1}$$

therefore, $F(M,N) = \dfrac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + 2\sum 2^i \binom{M-2}{i}\binom{N}{i+1}}$

but $\binom{M-2}{i} = \binom{M-1}{i+1} \times \left(\frac{i+1}{M-1}\right)$

and $\binom{N}{i+1} = \binom{N-1}{i} \times \left(\frac{N}{i+1}\right)$

therefore, $F(M,N) = \dfrac{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i} + \frac{2N}{M-1}\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}}$

$$= \frac{1}{\dfrac{\sum 2^i \binom{M-2}{i+1}\binom{N-1}{i}}{\sum 2^i \binom{M-1}{i+1}\binom{N-1}{i}} + \dfrac{2N}{M-1}}$$

$$= \frac{1}{F(M-1,\ N) + \frac{2N}{M-1}}$$ by (13) and (12)

therefore, our solution satisfies the recurrence relation.

Also, $T(1,N) = \dfrac{2^0 \binom{0}{0}\binom{N-1}{0}}{\frac{2^0}{1}\binom{0}{0}\binom{N-1}{0}} = 1$ for all N

Then, since the solution is correct for M=l and since it satisfies the recurrence relation, it must be the closed form solution that we are looking for.

Table 1 tabulates T(M,N) for M and N ranging from 1 through 8. Table 2 tabulates the error in T(M,N) when compared with the exact values obtained by the solution of the Markov chain in the conventional manner. These exact values have been tabulated in [2]. It may be observed that the errors, expressed as percentages are extremely small indeed, thereby confirming that our approximation was a good one.

5. ASYMPTOTIC EXPRESSIONS

The closed form solution, though very accurate, has one drawback in that the computation involved in evaluating it becomes extremely large as M and N increase. This is the result of having to evaluate a number of factorials. Consequently, it is desirable to seek approximations which are less burdensome to evaluate. Of particular interest are asymptotically exact approximations, i.e., expressions such that the relative error becomes vanishingly small for large M and N, since this is the region in which evaluation of the closed form solution is troublesome.

Simple Asymptotic Expressions

The asymptotic approximations obtained here, all derive from the basic recurrence relation. We have

$$F(M,N) = \frac{1}{F(M-1,\ N) + \frac{2N}{M-1}}$$

Now, when M is extremely large, the addition of one more memory module may be expected to have a negligible effect on the idle fraction of each module, i.e.,

$$F(M,N) \cong F(M-1,\ N) \text{ for large } M$$

Also, $\frac{2N}{M-1} \cong \frac{2N}{M}$ for large M

we can therefore rewrite the recurrence relation as

$$F(M,N) = \frac{1}{F(M,N) + \frac{2N}{M}} \qquad (14)$$

this leades to the quadratic equation

$$F^2(M,N) + \frac{2N}{M} F(M,N) - 1 = 0$$

which has the solution

$$F(M,N) = -\frac{N}{M} \pm \sqrt{\left(\frac{N}{M}\right)^2 + 1}$$

$$T(M,N) = M(1-F(M,N)) = M+N - \sqrt{M^2 + N^2}$$

the other solution being unacceptable since T(M,N) cannot be greater than M or N.

The derivation of this result relied on M being very large with no assumptions about the value of N. However, since the expression is symmetric in N and M and since the closed form solution is symmetric too, we see that the simple asymptotic solution is equally good for large N as for large M. In other words, the simple asymptotic solution is asymptotically exact when either M or N or both increase. Unfortunately, this solution leads to relatively large percentage errors when both M and N are small and in the range of practical interest. This simple asymptotic solution has been obtained using a different approach in [1].

An Improved Asymptotic Solution

We shall now proceed to obtain a family of asymptotic expressions. We shall see that this family includes an asymptotic solution reported in [1]. However, this is not amongst the better solutions belonging to this family.

Our approach shall be, basically, to perturb Equation (14) in such a fashion as to improve its accuracy for small values of M and N without affecting its asymptotic behavior. By adjusting this perturbation, we can "tune" the behavior of the asymptotic solution for small M and N.

For the remainder of this section, for the purposes of notational convenience, we shall abbreviate F(M,N) to F.

Let us consider equations of the form

$$F = \frac{c}{F + b} \tag{15}$$

where $c = 1 + 0\left(\frac{1}{M}, \frac{1}{N}\right)$

and $b = \frac{2N}{M}\left(1 + 0\left(\frac{1}{M}, \frac{1}{N}\right)\right)$

where $0\left(\frac{1}{M}, \frac{1}{N}\right)$ represents a polynomial such that every term in that has a denominator which consists of a power of M of order greater than or equal to 1 and/or a power of N of order greater than or equal to 1, e.g., $\frac{1}{M}, \frac{1}{N}, \frac{1}{MN}, \frac{1}{M^2N}$, etc.

It is clear then, that as $M,N \to \infty$, $c \to 1$, $b \to \frac{2N}{M}$

and Equation (15) → (14). Accordingly, the asymptotic behavior of the solution is unchanged. However, the behavior at the low end is influenced by our choice of c and b.

Clearly, we can get an infinite number of solutions depending on our choice of c and b. We shall restrict ourselves to a set of solutions which are symmetric in M and N since this is the case for the closed form solution.

Since $F = \frac{c}{F + b}$

we have, $F^2 + bF - c = 0$

therefore, $F = \frac{-b \pm \sqrt{b^2 + 4c}}{2}$

and $T = M(1-F) = M + M\frac{b}{2} - \sqrt{\frac{M^2b^2}{4} + M^2c}$

T will be symmetric if both $\left(M + M\frac{b}{2}\right)$ and $\left(\frac{M^2b^2}{4} + M^2c\right)$ are symmetric

therefore, we have that b must be of the form

$$b = \frac{2N}{M} + \frac{2x}{M} + 2\frac{Q_1}{M}$$

where Q_1 is a polynomial in $\frac{1}{MN}$ of order greater than 0.

Consequently, for the part under the square root sign to be symmetric we require c to be of the form

$$c = \left(1 + \frac{2x}{M} + \frac{2}{M} \cdot Q_1\right) + \frac{Z}{M^2} + \frac{1}{M^2} Q_2$$

where Q_2, too, is a polynomial in $\frac{1}{MN}$ of order greater than 0.

We then have as the template for our symptotic solutions the following:

$$T = M + N + x + Q_1 - \sqrt{M^2 + N^2 + 2x(M+N) + x^2 + Z + 2(M+N)Q_1 + 2xQ_1 + Q_1^2 + Q_2}$$

The family of solutions which we derive here will be the subset obtained by taking Q_1 and Q_2 to be identically equal to 0. Therefore,

$$T = M + N + x - \sqrt{M^2 + N^2 + 2x(M+N) + x^2 + Z}$$

and the quadratic equation in F which gives rise to this is

$$F^2 + \left(\frac{2N}{M} + \frac{2x}{M}\right) F - \left(1 + \frac{2x}{M} + \frac{Z}{M^2}\right) = 0$$

which can be written in the form

$$\left(\frac{2F}{M} - \frac{2}{M}\right)x - \frac{1}{M^2} Z + \left(F^2 + \frac{2N}{M} F - 1\right) = 0 \qquad (16)$$

In this form, the equation is a linear equation in x and Z. We can solve for x and Z if we have two such simultaneous equations. We obtain these by choosing two pairs of M,N points and substituting the values of M,N and the correct value of F(M,N) into (16). The choice of the two points is arbitrary. However, the choice of M = N = 1 is rather beneficial since this causes T(M,N) to be correct for M = 1 (for all N) and for N = 1 (for all M). If we do so we obtain the relation

$$Z = - (2x + 1)$$

Our final asymptotic solution now has x as a parameter. It was found that a value of about -0.6374, corresponding to the choice of M = N = 4 as the second point, gave about the best results. Tables 3 and 4 display the results obtained using this improved asymptotic expression and the attendent error respectively. In Tables 5 and 6 we compare the performance of the closed form solution, our improved asymptotic solution, the simple asymptotic expression and another asymptotic solution termed the "Binomial Approximation", reported in [1] along the line N = M. It is of interest to note that the Binomial Approximation of [1] though obtained by other means, corresponds to the choice of x = -0.5. In the Tables 7 and 8 we compare the performance of the improved asymptotic solution for different values of x along the line M = N.

The asymptotic solution corresponding to x = -0.6374 is

$$T(M,N) = M + N - 0.6374 - \sqrt{(M+N)^2 - 1.2748(M+N) - 2MN + .6811}$$

This is roughly the optimal asymptotic solution under the constraints that Q_1 and Q_2 be identically zero. If this constraint is removed we can obtain very much more accurate approximations. Another alternative could be to remove the constraint that the solution be symmetric, and obtain a solution which would be very accurate for, say, $M \geq N$. By taking the dual solution we would also have an accurate solution for $M \leq N$.

6. CONCLUSION

We have employed generating functions in a somewhat unusual manner to obtain an "almost exact" closed form solution to the problem of evaluating the memory bandwidth obtained in an often used model of an N-processor, M-memory computer system. This solution is:

$$T(M,N) = \frac{\sum_{i=0}^{\infty} 2^i \binom{M-1}{i}\binom{N-1}{i}}{\sum_{i=0}^{\infty} \frac{2^i}{(i+1)} \binom{M-1}{i}\binom{N-1}{i}}$$

We have also obtained a family of asymptotically exact approximations to the above solution which though somewhat less accurate are computationally preferable. One of the better approximations belonging to this family is:

$$T(M,N) = M + N - 0.6374 - \sqrt{(M+N)^2 - 1.2748(M+N) - 2MN + 0.6811}$$

Both these expressions have been shown to be more accurate than existing solutions.

REFERENCES

[1] Baskett, F., and Smith, A. J., "Interference in Multiprocessor Computer Systems with Interleaved Memory", CACM 19, 6 (June 1976), 327-334.

[2] Bhandarkar, D.P., "Analytic Models for Memory Interference in Multiprocessor Computer Systems", Ph.D. Th., Carnegie-Mellon U., Pittsburg, PA., 1973.

[3] Knuth, D. E., "The Art of Computer Programming, Vol. 1, Fundamental Algorithms", 2nd edition (Addison-Wesley) 1973, Pg. 54.

[4] Skinner, C. and Asher, J., "Effect of Storage Contention on System Performance", IBM Sys. Jour., Vol. 8, No. 4, 1969, pp. 319-333.

[5] Strecker, W. D., "Analysis of the Instruction Execution Rate in Certain Computer Structures", Ph.D. Th., Carnegie-Mellon U., Pittsburgh, PA, 1970.

[6] Wulff, W., and Bell, C. G., "C.mmp, a Multi-mini-processor", AFIPS Conf. Proc., Vol. 41, part II, 1972 FJCC, AFIPS Press, Montvale, NJ, 765-777.

Table 5 Comparison of Various Solutions for N=M

M=N	1	2	3	4	5	6	7	8
Exact	1.0000	1.5000	2.0476	2.6210	3.1996	3.7809	4.3636	4.9471
Closed form	1.0000	1.5000	2.0526	2.6250	3.2056	3.7849	4.3675	4.9510
Improved Asymptotic	1.0000	1.4700	2.0411	2.6210	3.2036	3.7874	4.3718	4.9566
Binomial Approximation	1.0000	1.4384	2.0000	2.5756	3.1557	3.7379	4.3212	4.9052
Simple Asymptotic	0.5858	1.1716	[illegible].7574	2.3431	2.9289	3.5147	4.1005	4.6863

Table 6 Comparison of Percentage Errors for N=M

M=N	1	2	3	4	5	6	7	8
Closed form	0	0	0.2442	0.1526	0.1248	0.1047	0.0901	0.0841
Improved Asymptotic	0	-1.999	-0.317	0	0.126	0.172	0.189	0.193
Binomial Approximation	0	-4.104	-2.325	-1.733	-1.372	-1.137	-0.971	-0.847
Simple Asymptotic	-41.421	-21.895	-14.175	-10.601	-8.4594	-7.040	-6.029	-5.272

Note: All solutions, excepting the Simple Asymptotic Solution, are exact for N=l or for M=l

Table 7 Influence of the Parameter x on the Improved Asymptotic Solution

M=N	1	2	3	4	5	6	7	8
x=-0.7500	1.0000	1.5000	2.0779	2.6606	3.2448	3.8296	4.4147	5.0000
x=-0.6579	1.0000	1.4752	2.0476	2.6281	3.2110	3.7950	4.3795	4.9644
x=-0.6374	1.0000	1.4700	2.0411	2.6210	3.2036	3.7874	4.3718	4.9566
x=-0.6262	1.0000	1.4672	2.0376	2.6172	3.1996	3.7833	4.3676	4.9524

Table 8 Influence of the Parameter x on the Percentage Error of the Improved Asymptotic Solution

M=N	1	2	3	4	5	6	7	8
x=-o.7500	0	0	1.478	1.511	1.412	1.288	1.171	1.069
x=-0.6579	0	-1.655	0	0.269	0.356	0.372	0.365	0.351
x=-0.6374	0	-1.999	-0.317	0	0.126	0.172	0.189	0.193
x=-0.6262	0	-2.184	-0.488	-0.146	0	0.063	0.092	0.106

Table 1 Closed Form Solution

N\M	1	2	3	4	5	6	7	8
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0000	1.500	1.6667	1.7500	1.8000	1.8333	1.8571	1.875
3	1.0000	1.6667	2.0526	2.2727	2.4118	2.5068	2.5758	2.6279
4	1.0000	1.7500	2.2727	2.625	2.8667	3.0395	3.1681	3.2670
5	1.[illegible]00	1.800	2.4118	2.8667	32036	3.4569	3.6516	3.8048
6	1.0000	1.8333	2.5068	3.0395	3.4569	3.7849	4.0454	4.2553
7	1.0000	1.8571	2.5758	3.1681	3.6516	4.0454	4.3675	4.6331
8	1.0000	1.875	2.6279	3.2670	3.8048	4.2553	4.6331	4.9510

Table 2 Percentage Error of Closed Form Solution

N\M	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0.244	0.155	0.094	0.058	0.037	0.027
4	0	0	0.116	0.153	0.128	0.098	0.075	0.057
5	0	0	0.065	0.118	0.125	0.112	0.092	0.075
6	0	0	0.038	0.082	0.103	0.105	0.097	0.083
7	0	0	0.026	0.056	0.081	0.089	0.090	0.084
8	0	0	0.019	0.041	0.062	0.076	0.080	0.080

Table 3 Improved Asymptotic Solution

N\M	1	2	3	4	5	6	7	8
1	1 .0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0000	1.4700	1.6594	1.7526	1.8066	1.8415	1.8658	1.8838
3	1.0000	1.6594	2.0411	2.269	2.4146	2.5138	2.5852	2.6387
4	1.0000	1.7526	2.2690	2.621	2.8664	3.0433	3.1752	3.2766
5	1.0000	1.8066	2.4146	2.8664	3.2036	3.4591	3.6565	3.8122
6	1.0000	1.8415	2.5138	3.0433	3.4591	3.7874	4.0494	4.2613
7	1.0000	1.8658	2.5852	3.1752	3.6565	4.0494	4.3718	4.6385
8	1 .0000	1.8838	2.6387	3.2766	3.8122	4.2613	4.6385	4.9566

Table 4 Percentage Error of Improved Asymptotic Solution

N\M	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	-1.999	-0.436	0.147	0.365	0.447	0.471	0.467
3	0	-0.436	-0.317	-0.008	0.212	0.337	0.404	0.439
4	0	0.147	-0.048	0	0.120	0.225	0.301	0.349
5	0	0.365	0.183	0.109	0.126	0.177	0.228	0.272
6	0	0.447	0.317	0.209	0.168	0.172	0.196	0.223
7	0	0.471	0.392	0.282	0.217	0.189	0.189	0.200
8	0	0.467	0.431	0.334	0.259	0.216	0.196	0.193