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BIFURCATION PROBLEMS FOR DISCRETE VARIATIONAL INEQUITIES

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Bifurcation problems for discrete variational inequalities

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The buckling of a beam or a plate which are subject to obstacles is typical for the variational inequalities that are considered here. Bifurcation is known to occur from the first eigenvalue of the linearized problem. For a discretization the bifurcation point and the bifurcating branches may be obtained by solving a constrained optimization problem. An algorithm is proposed and its convergence is proved. The buckling of achmped beam subject to point obstacles is considered in the continuous case and some numerical results for this problem are presented.

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1. Introduction

In this work we are concerned with the numerical solution of nonlinear variational problems of the form

$$(1.1)\qquad g(u_o) - kf(u_o) = \min_K (g(u) - kf(u)),\ k > 0$$

where f and g are functionals on a Hilbert space H and $K \subset H$ is a closed convex cone $\{0\} \subset K \neq \{0\}$. A solution of (1.1) under suitable assumptions satisfies

$$(1.2)\qquad \lambda(g'(u_o), u - u_o) \geq (f'(u_o), u - u_o),\ \forall u \in K,\ \lambda = k^{-1}$$

where $(.,.)$ denotes the inner product in H, i. e. a nonlinear variational inequality and bifurcation may occur for (1.2).

Instead of considering (1.1), (1.2) in an abstract setting we shall use here and in the sequel a typical example from elasticity theory. Assume a beam is clamped at the points $x = 0$, $x = 1$ and is supported from below respectively

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from above in the sets C, D $\subset$ (o,l). We define

$$(1.3a) \qquad K = \{u \in H_o^2[o,1],\ u(C) \geq 0,\ u(D) \leq 0\}$$

where H_o^2 is the usual Sobolev space including the zero boundary conditions for u and u'.

Let further be

$$(1.3b) \qquad f(u) = \int_0^1 (\sqrt{1+u'^2}-1)\,dx, \quad g(u) = \frac{1}{2}\int_0^1 u''^2 dx .$$

Then (1.1), (1.3) describes the displacement u of the beam under the action of an axial force k = P. In the case C = D = Ø this is Euler's famous beam-buckling problem. It has been considered very frequently in the literature and for more recent work on the numerical solution of this problem we refer to [5,17].

The above formulation is only one of the possible ways to treat the beam-buckling problem. We have chosen this problem for the sake of simplicity. It is possible without essential difficulties to treat other boundary conditions in (1.3) or other problems as e. g. the buckling of plates.

The approximate solution of (1.2) was listed as an open problem in [12] since no numerical literature on this subject was known to the authors. A bifurcation theory, however, for problems of this form and particularly for (1.3) was given in [6] and considerably generalized in [18]; for other related work cf. e. g. [1, 2, 7, 9, lo, 13, 15, 16].

In the following we shall look at discretizations of (1.1), (1.2) and give a convergence proof for an algorithm solving these problems as well as the corresponding linear eigenvalue problem. We thus propose and investigate a numerical method for the computation of the bifurcation points and of the bifurcating branches. Numerical results are finally given for a finite element discretization of (1.3).

The contents of the following sections are

2. The discrete bifurcation problem
3. The numerical procedure
4. Convergence proof
5. The beam problem
6. Discretization and results.

2. The discrete bifurcation problem

In the following we assume that (1.1) is reduced to a finite-dimensional problem by a discretization method characterized by a parameter $h > 0$. A finite difference method with mesh width h or a finite element method with intervals of maximal diameter h yields

$$(2.1)\qquad g_h(x_o) - kf_h(x_o) = \min_{K_h}(g_h(x) - kf_h(x)),\ k > 0$$

where now x_o, $x \in H_h$, a finite-dimensional Hilbert space and again $K_h \subset H_h$ a closed convex cone with vertex 0.

In the following we shall assume that H_h may be identified with Euclidean n-space and shall omit the subscript h. The corresponding variational inequality is therefore of the form (1.2)

$$(2.2)\qquad \lambda(g'(x_o),x-x_o) \geq (f'(x_o),x-x_o),\ \lambda = k^{-1},\ \forall x \in K,$$

Here and in the following we use $g'(x) = \nabla g^T(x)$, $g''(x) = \nabla^2 g(x)$.
From now on we assume that $f(o) = g(o) = 0$ and $f'(o) = g'(o) = 0$. For all $\lambda > 0$ thus (2.2) has the trivial solution.

Definition 2.1 $\lambda_o > 0$ is a bifurcation point of (2.2), if there are sequences $\{\lambda_n\}$, $\{x_n\}$, $n = 1,2,\ldots$, solutions of (2.21, with $\lambda_n > 0$, $x_n \in K-\{o\}$ and $\lambda_n \to \lambda_0$, $x_n \to 0$ for $n \to \infty$.

The following results are easy consequences of the theory for the infinite-dimensional case in [6,18].

Theorem 2.2 Assume that $f, g \in C^2(U(o))$, $U(o) \subset \mathbb{R}^n$ an open neighborhood of 0, $f(0) = g(0) = 0$, $f'(o) = g'(o) = 0$, $(g''(o)x, x) \geq \gamma \|x\|^2$, $\gamma > 0$, $\forall x \in \mathbb{R}^n$ and that there exists a $y \in K$ such that $(f''(o)y, y) > 0$. Then the linearized variational inequality

$$(2.3) \qquad \lambda(g''(o)x_o, x-x_o) \geq (f''(o)x_0, x-x_o), \forall x \in K$$

has a solution $x_o \in K - \{o\}$, $\lambda_o > 0$. λ_o is the largest eigenvalue of (2.3) and the largest bifurcation point of (2.2).

Theorem 2.3 In addition to the assumptions of Theorem 2.2 let $(f'(x),x) > 0$, $(g'(x),x) > 0$, $\forall x \in K - \{o\}$ andletthere exist strictly increasing functions $\delta_i(t)$, continuous on $[o,\infty)$ with $\lim_{t\to o} \delta_i(t) = 0$ and $\lim_{t\to+\infty} \delta_i(t) = +\infty$, $i = 1,2$ such that $\delta_1(\|x\|) \leq g(x) \leq \delta_2(\|x\|)$, $\forall x \in K$.
Then for every p, $0 < \rho < \infty$, the problem

$$(2.4) \qquad f(x_\rho) = \max_{K\cap S_\rho} f(x), \quad S_\rho = \{x \in \mathbb{R}^n, \ g(x) \leq \frac{1}{2}\rho^2\}$$

has a solution $x_\rho \neq 0$ which also solves (2.2) with $A = \lambda(x_\rho) = \frac{(f'(x_\rho),x_\rho)}{(g'(x_\rho),x_\rho)} > 0$ and further holds $\lim_{\rho\to o} x_\rho = 0$, $\lim_{\rho\to\infty} \|x_\rho\| = +\infty$, $\lim_{\rho\to o} \lambda(x_\rho) = \lambda_0$, λ_o as in Theorem 2.2, $g(x_\rho) = \frac{1}{2}\rho^2$.

There is a subsequence $\{x_\rho\}$ with

$$(2.5) \qquad \lim_{\rho\to o} \left\| \frac{x_\rho}{\sqrt{(g''(o)x_\rho, x_\rho)}} - x_o \right\| = 0,$$

x_o as in Theorem 2.2.

If (2.4) is uniquely solvable for every $\rho > 0$ then $\rho \to \{(x_\rho, \lambda(x_\rho)) : 0 \leq \rho < \infty\}$, $x_0 = 0$, $\lambda(x_o) = a_0$ is a continuous curve in $\mathbb{R}^n \times \mathbb{R}$, which extends to infinity.

These theorems show that bifurcation occurs from the maximal eigenvalue λ_o of (2.3) and that points x_ρ on the bifurcating branch may be obtained by solving

$$(2.6) \qquad f(x_\rho) = \max_{K \cap \partial S_\rho} f(x), \rho > 0 \ , \ \partial S_\rho = \{x \in \mathbb{R}^n , g(x) = \tfrac{1}{2}\rho^2\}$$

3. The numerical procedure

In this section we consider the numerical solution of the linear eigenvalue problem

$$(3.1) \qquad \lambda(g''(o)x_o, x-x_o) \geq (f''(o)x_o, x-x_o) \forall x \in K$$

and the computation of the branches bifurcating from the maximal eigenvalue λ_o, i. e. we determine x_ρ such that

$$(3.2) \qquad f(x_\rho) = \max_{K \cap \partial S_\rho} f(x),$$

x_ρ in a neighborhood of x_o for small $\rho > o$.

We restrict ourselves to the case that g is quadratic in x. A further restriction to $g(x) = (x, x)$, however, will not be made here because it would not allow the treatment of the physical-problem (1.3) in the usual setting. Influenced by this example we consider for K the set

$$(3.3) \qquad K = \{x \in \mathbb{R}^n , x_i \geq 0, i \in J_1, x_i \leq 0, i \in J_2\},$$
$$J_1, J_2 \subset \{1,\dots,n\}, J_1 = \{i_1,\dots,i_{n_1}\}, J_2 = \{j_1,\dots,j_{n_2}\}.$$

In the recent paper [3] a gradient method was analysed for the solution of (3.2) in case $K = H$, H an infinite-dimensional Hilbert 'space, and some references where given for earlier work on the approximate solution of this unrestricted problem (in the sense that $K = H$).

We shall consider here the restricted problem but for $\dim(H) < \infty$. We thus prefer here the solution by first discretizing the continuous problem instead of first deriving a sequence of simpler continuous problems (variational equalities respectively unconstrained optimization problems) and then discretizing those. Since we have in mind applications as e. g. (1.3) we do not give a method to compute smaller critical values of the functional f, for theoretical results in this case cf. [8], but we concentrate on the physical relevant value λ_0.

We make the following assumptions. Let f in (3.1) be continuously differentiable on H and let g be of the form

(3.4a) $$g(x) = \frac{1}{2}(Bx, x),$$

where $B : H \to H$ is a linear, symmetric and positive definite operator. Further let there exist $a \cdot M > 0$ such that

(3.4b) $$(f'(x + h) - f'(x), h) \leq M \|h\|^2, \forall x, h \in H$$

(3.4c) $$(f'(x + h) - f'(x), h) > 0, \forall x, h \in H, h \neq 0$$

(3.4d) $$f(0) = 0$$

(3.4e) $$f'(o) = 0$$

The norms used here and in the following are the Euclidean norm for $x \in H$ and the spectral norm for matrices $A \in L(H)$.

We need some further notations. Let $G = (g_1,\dots,g_{n_1+n_2})$, where $g_k = e_{i_k}$, $k = 1,\dots,n_1$ and $g_{n_1+k} = -e_{j_k}$, $e_\ell \in \mathbb{R}^n$ the It-thunitvector. Then K in (3.3) may be rewritten as

$$K = \{x \in \mathbb{R}^n,\ G^T x \geq 0\}.$$

For any $x \in \mathbb{R}^n$ let $I(x) = \{i \in \{1,\dots,2n\},\ g_i^T x = 0\}$ and define $G_I = (g_i)_{i \in I}$, $Q_I = E_n - G_I G_I^T$, E_n the nxn identity matrix. For $x = x_k$ denote $I_k = I(x_k)$, $G_k = G_{I_k}$ and Q_k analogously. Finally we introduce

$$P_k = E_n - \frac{Bx_k x_k^T B}{(x_k, Q_k B x_k)_B}, \tag{3.5}$$

where $(.,.)_B$ denotes the scalar product induced by B and

$$u_k = P_k Q_k r_k,\ r_k = f'(x_k). \tag{3.6}$$

We observe that $g''(o) = B$ and hence with $A = f''(o)$ the maximal eigenvalue λ_o and the corresponding eigenvector may be computed from

$$\lambda_o = \max_{K-\{o\}} \frac{(Ax,\ x)}{(Bx,\ x)} = \max_{K \cap \partial S_1} \frac{1}{2}(Ax,\ x). \tag{3.7}$$

This problem is a special case of (3.2) and it suffices therefore to give an algorithm for that problem.

<u>The algorithm</u>

Let $x_1 \in K \cap \partial S_\rho$ be arbitrary. Set $k = 1$ and $\mu_k = 0$, $\mu_k \in \{0,1\}$

Step 1 Determine I_k and terminate the iteration if $G_k^T u_k \leq 0$ and $\|Q_k u_k\|_B = 0$.

Step 2 Compute $|u_{k\ell}| = \max\{|u_{ki}|, (G_k^T u_k)_i > 0\}$.
If $((Q_k u_k, r_k)_B < |u_{k\ell}| \cdot \|Q_k u_k\|_B$ and $\mu_k = 0$ or $\|Q_k u_k\|_B = 0)$ then set $I_k = \tilde{I}_k - \{\ell\}$ and determine $\tilde{Q}_k, \tilde{P}_k, \tilde{u}_k$ otherwise let $\tilde{I}_k = I_k$, $\tilde{P}_k = P_k$, $\tilde{u}_k = u_k$.

Step 3 Compute $p_k = \tilde{Q}_k \tilde{P}_k \tilde{Q}_k r_k$ and determine $\bar{\alpha}_k$ as the maximal admissible steplength in direction p_k.

Step 4 Set $\alpha_k = \min(\tilde{\alpha}_k, \bar{\alpha}_k)$, $\tilde{\alpha}_k \equiv \frac{1}{2M \cdot \mathrm{cond}(B)}$, $x_{k+1} = \frac{x_k + \alpha_k p_k}{\|x_k + \alpha_k p_k\|_B}$,

where $\mathrm{cond}(B) = \|B\| \cdot \|B^{-1}\|$.

Set $\mu_{k+1} = \begin{cases} 1, & \text{if } \alpha_k = \tilde{\alpha}_k \\ 0, & \text{otherwise} \end{cases}$, $k = k + 1$ and go to Step 1.

The following convergence result will be proved for this algorithm.

Theorem 3.1 Let the assumptions (3.4) be satisfied for problem (3.2). Assume that the set

$$\Omega = \{x^* \in K \cap \partial S_\rho,\ G^{*T} u^* \leq 0,\ \|Q^* u^*\| = 0\}$$

is finite and that $G^{*T} u^* < 0$ for all $x^* \in \Omega$. Then the sequence $\{x_k\}, k = 1,2,\ldots,$ generated by the above algorithm converges to a point $x^* \in \Omega$.

The points x^* are Kuhn-Tucker points of the first order of f with respect to the given constraints. In general, of course, we cannot be sure that $\{x_k\}$ converges against the maximizing x_0^*. An easy consequence of Theorem 3.1 guaranteeing this will be stated at the end of the next section.

With the constant stepsize $\tilde{\alpha}_k$ in Step 4 the above algorithm is more of theoretical interest. In the computations presented in the last section the stepsize was chosen by the Goldstein-Armijo rule (cf. e. g. [14]). This still makes it not a very efficient algorithm since it is of gradient-projection type.

In Step 4 we determine then $\tilde{\alpha}_k = 2^{-j}$ where

$$j = \min \{i \in \mathbb{N} \cup \{o\} : f(x_k + 2^{-i}p_k) - f(x_k) \geq 2^{-i-2} p_k^T r_k\} .$$

In order to justify this choice for the constrained case we have to show that

$$\|x_{k+1} - (x_k + \alpha_k p_k)\| = O(\alpha_k^2).$$

If we choose here the norm $\| \cdot \|_B$ then we have

$$\begin{aligned} \|x_{k+1} - (x_k + \alpha_k p_k)\|_B &= \|x_k + \alpha_k p_k\|_B - 1 \\ &\leq \|x_k + \alpha_k p_k\|_B^2 - 1 \\ &= \alpha_k^2 \|p_k\|_B^2 . \end{aligned}$$

But $\|p_k\| \leq \|r_k\|$ and $\|r_k\|$ is obviously uniformly bounded on ∂S_ρ.

4. Convergence Proof

The essential tool for proving Theorem 3.1 will be the following lemma.

Lemma 4.1 Let $x_k \in K \cap \partial S_\rho$ be generated by the above algorithm. Then $x_{k+1} \in K \cap \partial S_\rho$ and

$$(4.1) \qquad f(x_{k+1}) - f(x_k) \geq \tilde{c}_k \begin{cases} \|p_k\|^2, & \text{if } \mu_k = 1, \\ \max \{\|p_k\|, |\tilde{u}_{k\ell}|\}^2 & \text{otherwise,} \end{cases}$$

where $\tilde{c}_k \equiv c_1 > 0$ for $\mu_{k+1} = 0$ and $\tilde{c}_k = c_2 \bar{\alpha}_k$, $c_2 > 0$, for $\mu_{k+1} = 1$, $\tilde{u}_k = \tilde{P}_k \tilde{Q}_k r_k$.

Proof $x_{k+1} \in K \cap \partial S_\rho$ is valid by construction of the algorithm. For a suitable $\tau \in (o,1)$ we have from (3.4b)

$$f(x_{k+1}) - f(x_k) = (f'(x_k + \tau(x_{k+1} - x_k)), x_{k+1} - x_k)$$

$$= -\frac{1}{\tau}(-f'(x_k + \tau(x_{k+1} - x_k)) + f'(x_k), \tau(x_{k+1} - x_k)) + (f'(x_k), x_{k+1} - x_k)$$

(4.2) $$\geq - M \|x_{k+1} - x_k\|^2 + (f'(x_k), x_{k+1} - x_k)$$

$$\geq - M \|B\|^{-1} \| x_{k+1} - x_k\|_B^2 + (B^{-1}f'(x_k), x_{k+1} - x_k)_B$$

$$\geq 2M \|B^{-1}\| (x_k + y_k, x_{k+1} - x_k)_B, \; y_k = \frac{B^{-1}r_k}{2M\|B^{-1}\|}$$

Hence we have

$$f(x_{k+1}) - f(x_k) \geq d_k(x_k + y_k, x_k + \alpha_k p_k - \|x_k + \alpha_k p_k\|_B x_k)_B,$$

where $d_k = \frac{2M\|B^{-1}\|}{\|x_k + \alpha_k p_k\|_B} > 0$ and we show next that the second term on the righthand side is nonnegative. Observing that $(x_k, p_k)_B = 0$ this relation may be rewritten as

(4.3) $$(1 - \|x_k + \alpha_k p_k\|_B)(1 + (y_k, x_k)_B) + \alpha_k (y_k, p_k)_B \geq 0.$$

We have

(4.4) $$2M \|B^{-1}\| (y_k, p_k)_B = \|p_k\|^2 ,$$

and

(4.5) $$\|x_k + \alpha_k p_k\|_B^2 = 1 + \alpha_k^2 \|p_k\|_B^2 \leq 1 + \alpha_k^2 \|B\| \; \|p_k\|^2.$$

From **(3.4b)** and the fact that $\|x_k\|_B = 1$ we conclude that

$$(y_k, x_k)_B \leq \frac{\|x_k\|^2}{2\|B^{-1}\|} \leq \frac{1}{2} \; .$$

Hence

$$(1 + (y_k,x_k)_B)^2 \leq 2(1 + (y_k,x_k)_B) + \frac{\|p_k\|^2}{2M^2 \| B\| \cdot \|B^{-1}\|^2}$$

the last **term** being nonnegative and thus

$$\alpha_k \| B\| (1 + (y_k,x_k)_B)^2 \leq \frac{1 + (y_k,x_k)_B}{M\|B^{-1}\|} + \frac{\alpha_k \|p_k\|^2}{4M^2 \|B^{-1}\|^2}$$

which gives

$$\left(1 + \alpha_k^2 \| p_k\|^2 \| B\|\right) (1 + (y_k,x_k)_B)^2 \leq \left(1 + (y_k,x_k)_B + \frac{\alpha_k \|p_k\|^2}{2M\| B^{-1}\|}\right)^2 .$$

Taking the square root on both sides and using **(4.4), (4.5)** we finally arrive at

$$\|x_k + \alpha_k p_k\|_B (1 + (y_k,x_k)_B) \leq 1 + (y_k,x_k)_B + \alpha_k (y_k,p_k)_B$$

which proves (4.3).

Combining (4.2) and (4.3) we obtain with **(3.4c)**

(4.6) $$f(x_{k+1}) - f(x_k) \geq (f'(x_k), x_{k+1} - x_k)$$

$$\geq M\| x_{k+1} - x_k\|^2 \geq 0 .$$

In order to show (4.1) we estimate using (4.5)

$$(r_k, x_{k+1} - x_k) = \frac{1}{\|x_k + \alpha_k p_k\|_B} [(r_k, x_k)(1 - \|x_k + \alpha_k p_k\|_B) + \alpha_k (r_k, p_k)]$$

$$\geq \frac{\alpha_k \|p_k\|^2 (1 - \alpha_k M \text{ cond } (B))}{\|x_k + \alpha_k p_k\|_B}$$

For $\| x_k + \alpha_k p_k\|_B$ **a uniform** upper bound L is easy to obtain, hence $\alpha_k \leq \tilde{\alpha}_k$ yields

$$f(x_{k+1}) - f(x_k) > \frac{\alpha_k \|p_k\|^2}{2L}$$

which proves (4.1) in the case $\mu_k = 1$.

If $\mu_k = 0$ and $\tilde{I}_k \neq I_k$ then $|p_{k\ell}| = |\tilde{u}_{k\ell}|$ and thus $\|p_k\| \geq |\tilde{u}_{k\ell}|$. If $\mu_k = 0$ and $\tilde{I}_k = I_k$ then $p_k = Q_k u_k$ and because of the strategy in Step 2 we must have

$$\|p_k\|_B^2 = (p_k, r_k)_B \geq |u_{k\ell}| \, \|p_k\|_B$$

and hence $\|p_k\| \geq \|B\|^{-1/2} |u_{k\ell}|$ which completes the proof of the lemma.

Proof of Theorem 3.1 Let $\{x_k\}, \{\mu_k\}, \{I_k\}$ be generated by the above algorithm. As in [11] we distinguish two cases. Ass&e that there is an infite subset $J \subset N$ with $\mu_k = \mu_{k+1} = 0$ for $k \in J$. Because of the compactness of ∂S_ρ and the finite number of constraints in K an infinite subset $J_o \subset J$ may be chosen such that $x_k \to \bar{x} \in K \cap \partial S_\rho$ and $I_k = I \subset \bar{I} = I(\bar{x})$ for $k \in J_o$. Since $f(x_{k+1}) - f(x_k) \geq 0$ from (4.6) for $k \geq 1$ and $f(x_{k+1}) - f(x_k) \geq c_1 \max\{\|p_k\|, |\tilde{u}_{k\ell}|\}^2$ for $k \in J_o$, we have.

$$\|\tilde{Q}_k \tilde{u}_k\| \to 0, \; |\tilde{u}_{k\ell}| \to 0, \; k \to \infty, \; k \in J.$$

$\tilde{Q}_k \tilde{u}_k$ is a continuous function of x_k for fixed index set I and hence $Q_I \bar{u} = 0$, $\bar{u}_\ell = 0$, where $\bar{u} = P_I Q_I \bar{r}$, $\bar{r} = f(\bar{x})$. This implies $\bar{x} \in \Omega$. If there is a $j \in \bar{I} - I$ then $r_{kj} \to 0$ for $k \to \infty$, $k \in J_o$ and hence $\bar{r}_j = 0$ while $\bar{r}_j$ must not vanish under the assumption $G^{*T} u^* < 0$ for $x^* \in \Omega$. We conclude $I = \bar{I}$.

From (4.6) we have $\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0$ and $f(x_{k+1}) \geq f(x_k)$. Since f has only finitely many local maxima on $K \cap \partial S_\rho$, $\rho > 0$, which are strict maxima according to (3.4c) and because $\tilde{I}_k \equiv I$ for $k \in J_o$, $k \geq k_1$ we finally obtain that the whole sequence $\{x_k\}$, $k = 1, 2, \ldots$ must converge to $\bar{x}$ and that $I_k \equiv \bar{I}$ for $k \geq k_o$.

In the case that for all $k \geq k_o$ there is $\mu_k = 1$ or $\mu_{k+1} = 1$ the proof may be completed combining the above arguments and those of the corresponding part of the proof of Theorem 2 in [11], to which we refer.

The ascent property (4.6) allows to state the following simple consequence of Theorem 3.1.

Corollary *4.2* In addition to the assumptions of Theorem 3.1 we assume that λ_o is the largest critical value of f with respect to $K\cap\partial S_\rho$ and that there is no other critical value in $(\lambda_o-\varepsilon, \lambda_o)$, $\varepsilon > 0$. If $f(x_1) > \lambda_0 - \varepsilon$ then the sequence $\{x_k\}$, k = 1,2,... generated by the above algorithm converges to $x_o \in K\cap\partial S_\rho$ with $f(x_o) = \lambda_o$.

5. The beam problem

In this section we return to the problem of the compressed clamped beam and we first consider to some extent the linear eigenvalue problem, i. e. we search for λ, $u\in K$, K as in (1.3a), such that

$$(5.1) \qquad \lambda\int_0^1 u''(v-u)''dx \geq \int_0^1 u'(v-u)'\,dx$$

for all $v\in K$. We restrict ourselves here to the case that the sets C, D are finite

$$C \cup D = \{x_1, \dots, x_N\}$$

where $0 = x_0 < x_1 \dots < x_N < x_{N+1} = 1$. For the variational inequality of second order

$$(5.2) \qquad \lambda\int_0^1 u'(v-u)'\,dx \geq \int_0^1 u(v-u)\,dx,$$

$u,v\in K = \{u\in H^1[o,1],\ u(x_i) \geq 0,\ i = 1,\dots,N\}$ a description of all the eigenvalues and c&responding eigenfunctions was given in [4]. For the problem (5.1), where the situation is different, it is not our aim here to do the same. Instead we consider only problems with a few obstacles.

In the usual way it can be shown that (5.1) is equivalent to the following set of conditions

(5.3a) $\lambda u^{(4)} + u'' = 0$ on (x_i, x_{i+1}), $i = 0, \ldots, N$,

(5.3b) $u(0) = u'(0) = u(1) = u'(1) = 0$,

(5.3c) u, u' und u'' continuous in x_i, $i = 1, \ldots, N$,

(5.3d) $u'''(x_i + 0) - u'''(x_i - 0) \begin{cases} \geq 0, & \text{if } x_i \in C, \\ \leq 0, & \text{if } x_i \in D, \end{cases}$

(5.3e) $(u'''(x_i + 0) - u'''(x_i - 0))u(x_i) = 0$, $i = 1, \ldots, N$,

(5.3f) $u(x_i) \geq 0$, if $x_i \in C$, $u(x_i) \leq 0$ if $x_i \in D$.

For the sake of completeness we sketch the proof. We integrate (5.1) by parts

$$\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (\lambda u^{(4)} + u'')(v-u)\,dx$$

$$- \lambda \sum_{i=1}^{N} (u''(x_i+0) - u''(x_i-0))(v'(x_i) - u'(x_i))$$

$$+ \lambda \sum_{i=1}^{N} (u'''(x_i+0) - u'''(x_i-0))(v(x_i) - u(x_i)) \geq 0.$$

Choosing $v \in K$ such that the last terms vanish we see that $\lambda u^{(4)} + u''$ is orthogonal to $w = v - u \in H_0^2[x_i, x_{i+1}]$, $i = 0, \ldots, N$ which yields (5.3a) and the first term vanishes. We have no restrictions on $v'(x_i)$, $i = 1, \ldots, N$ for $v \in K$. If the last term and all but the i-th in the second term is made *zero* then we conclude that this must vanish, too, i. e. u'' has to be continuous. Finally, if $u(x_i) \neq 0$ for an $i = 1, \ldots, N$ then we choose v such that $v(x_i) = 0$ respectively $v(x_i) = 2u(x_i)$ and $v(x_j) = 0$, $j \neq i$, yielding (5.3e) while in the case $u(x_i) = 0$ the condition on $v(x_i)$ gives (5.3d).

A simple computation shows that the sets

$$u_k^{(0)}(x) = c_k(1-\cos(z_k x)), \quad c_k \in \mathbb{R}' = \mathbb{R} - \{0\},$$

(5.4a)

$$\lambda_k^{(0)} = z_k^{-2}, \quad z_k = z_k^{(0)} = 2k\pi, \quad k = 1,2,\ldots,$$

$$u_k^{(1)}(x) = \begin{cases} c_k(\sin(z_k x) - \frac{z_k}{2}\cos(z_k x) + (\frac{1}{2} - x)z_k) \text{ in } [0, \frac{1}{2}], \\ -u_k^{(1)}(1-x) \quad \text{in} \quad [\frac{1}{2}, 1], \ c_k \in \mathbb{R}' \end{cases}$$

(5.4b)

$$\lambda_k^{(1)} = (\frac{1}{z_k})^2, \ k = 1,2,\ldots, \ z_k = z_k^{(1)},$$

$$z_1^{(1)} < z_2^{(1)} < \ldots \text{ the solutions of } \frac{z}{2} = \tan(\frac{z}{2})$$

are eigenfunctions and corresponding efgenvalues of the unrestricted problem ($K = H$) and hence they are also solutions of (5.1) if they fit the condition (5.3f). In Table 1 we have listed the first $\lambda_k^{(i)}$, $i = 1,2$.

k	$\lambda_k^{(o)}$	$\lambda_k^{(1)}$
1	.o2533o2959	.o1238192o7
2	.oo6332574o	.oo4189o42o
3	.oo28144773	.oo21o26o96
4	.oo15831435	.oo12635336

Table 1 The first four $\lambda_k^{(i)}$, $i = o,1$ according to (5.4)

In order to find eigenfunctions of (5.1) which are not solutions of the unrestricted problem we consider the simplest case $N = 1$. Combining the solutions on $[o, x_1]$ and $[x_1, 1]$ such that they satisfy (5.3a) - (5.3c) and vanish in x_1 yields

$$u_k(x) = c_k[(1-\cos(z_k x_1))(\sin(z_k x) - z_k x) - (\sin(z_k x_1) - z_k x_1)(1-\cos(z_k x))] \cdot$$
$$\cdot (z_k x_1 \cos(z_k \bar{x}_1) - \sin(z_k \bar{x}_1)) \text{ on } [0, x_1],$$

.(5.5a)

$$u_k(x) = c_k[(1-\cos(z_k \bar{x}_1))(\sin(z_k(1-x)) - z_k(1-x))$$
$$- (1-\cos(z_k(1-x)))(\sin(z_k \bar{x}_1) - z_k \bar{x}_1) \,]\, (z_k x_1 \cos(z_k x_1) - \sin(z_k x_1))$$

$$\text{on } [x_1, 1],$$

where $\bar{x}_1 = 1 - x_1$, $x_1 \neq \frac{1}{2}$, $\lambda_k = z_k^{-2}$, z_k, $k = 1,2,\ldots,$ the solutions of

$$(zx_1 \sin(zx_1) - 2(1-\cos(zx_1))) \cdot (z\bar{x}_1 \cos(z\bar{x}_1) -$$

(5.5b) $$\sin(z\bar{x}_1)) = (2(1-\cos(z\bar{x}_1)) - z\bar{x}_1 \sin(z\bar{x}_1)) \cdot$$

$$(zx_1 \cos(zx_1) - \sin(zx_1)).$$

If $x_1 = \frac{1}{2}$ then *there* are the eigenfunctions and corresponding eigenvalues $u_k^{(1)}$, $\lambda_k^{(1)}$, $u_{2k}^{(o)}$, $\lambda_{2k}^{(o)}$, $k = 1,2,\ldots$. Additionally we have

(5.6)
$$u_k(x) = c_k[(1-\cos\frac{z_k}{2})(\sin(z_k x) - z_k x) - (\sin\frac{z_k}{2} - \frac{z_k}{2}(1-\cos(z_k x))]$$
on $[0, \frac{1}{2}]$,
$$u_k(x) = u_k(1-x) \text{ on } [\frac{1}{2}, 1],$$

where $z_k = 2 z_k^{(1)}$, $k = 1,\ldots$ and $\lambda_k = z_k^{-2}$. All these eigenvalues are arranged in decreasing order as $\lambda_k^{(2)}$, $k = 1,2,\ldots$ with the eigenfunctions $u_k^{(2)}$. If $x_1 \neq \frac{1}{2}$ *there* again remain certain of the eigenfunctions $u_k^{(o)}$, $u_k^{(1)}$.

The sign of the factor c_k may be chosen such that $u_k^{(2)}$ satisfies (5.3d). In Table 2 we have listed the first ten of the resulting eigenvalues and the corresponding eigenfunctions for $x_1 \in C$, i. e. we have given the range in which c_k in $u_k^{(2)}$ may vary, if $x_1 \in C$.

k	$\lambda_k^{(2)}(x_1 = \frac{1}{2})$	c_k	$\lambda_k^{(2)}(x_1 = \frac{1}{3})$	c_k
1	.02533o2959	$\mathbb{R}'_+$	.02533o2959	$\mathbb{R}'_+$
2	.o1238192o7	R'	.o14662o91o	$\mathbb{R}'_+$
3	.oo6332574o	$\mathbb{R}'$	.o1238192o7	$\mathbb{R}'_+$
4	.oo4189o42o	R'	.oo7o3189o8	R'
5	.oo3o9548o2	$\mathbb{R}'_+$	.oo6332574o	$\mathbb{R}'_+$
6	.oo28144773	$\mathbb{R}'_+$	.oo41944379	$\mathbb{R}'_+$
7	.oo21o26o96	$\mathbb{R}'$	.oo4189o42o	$\mathbb{R}'$
8	.oo15831435	R'	.oo28144773	$\mathbb{R}'$
9	.oo12635336	R'	.oo21o356o7	$\mathbb{R}'_+$
1o	.oo1o4726o5	$\mathbb{R}'_+$	.oo21o26o96	$\mathbb{R}'$

Table 2 The first $\lambda_k^{(2)}$, c_k for $C = \{x_1\}$, $D = \emptyset$.

In the case $x_1 = \frac{1}{2}$ the largest eigenvalues correspond to eigenfunctions of the unrestricted problem. The eigenfunction $u_2^{(1)}$, $c_2 \in \mathbb{R}_+^1$, for $x_1 = \frac{1}{2}$ is also a solution in the case $C = \{\frac{1}{3}\}$, $D = \emptyset$ and it still is if we e. g. .add the condition $D = \{\frac{2}{3}\}$ to exclude the solution $u_1^{(o)}$, $c_1 \in \mathbb{R}_+^1$. It is then, however, not the solution to the largest eigenvalue, which is $u_2^{(2)}$ to the eigenvalue $\lambda_2^{(2)} = 0.146620910$. This solution satisfies (5.3) but it is not a solution of the unrestricted problem. In Figure 1 we have plotted $u_2^{(2)}$, $u_3^{(2)}$.

Figure1

Formulae (5.5), (5.6) are valid for $0 < x_1 < 1$ and in Table 3 we have listed the largest eigenvalue for varying x_1.

x_1	$\lambda_1^{(2)}$
$\frac{1}{4}$	.0168642027
$\frac{1}{5}$	.0183671473
$\frac{1}{6}$	.0194295691
$\frac{1}{7}$	.0202150233
$\frac{1}{8}$	.0208177576

Table 3 Largest eigenvalue for varying $x_1 \in C$, $D = \{\frac{2}{3}\}$.

For $0 < x_1 < 1$ and $\lambda > 0$ we define

$$w(\lambda) = w(\lambda; x_1) = u^{(2)'''}(x_1 + 0) - u^{(2)'''}(x_1 - 0)$$

where $u^{(2)}$ is the function (5.5), (5.6) to the given x_1 and $z_k = \lambda^{-1/2}$, $c_k = 1$. $w(\lambda)$ has simple zeroes at the λ_k which coincide with $\lambda_i^{(0)}$, $\lambda_j^{(1)}$. For $0 < x_1 \leq \frac{1}{2}$ the value of $\lambda_2^{(2)}(x_1)$ varies in the range $\lambda_1^{(1)} \leq \lambda_2^{(2)} < \lambda_1^{(0)}$ and e. g. $w(\lambda_2^{(2)}; \frac{1}{3})$ is positive as a computation shows. We can thus state

<u>Lemma 5.1</u> For $0 < x_1 \leq \frac{1}{2}$ ($\frac{1}{2} \leq x < 1$) the functions u_1 according to (5.5), (5.6) with $c_1 \in \mathbb{R}'$ ($c_1 \in \mathbb{R}'_-$) are eigenfunctions of (5.1) where $C = \{x_1\}$.

Continuing this argument we can explain the choice of c_k in Table 2. If e. g. $x_1 = \frac{1}{2}$ and the eigenvalue is derived from (5.6) as e. g. $\lambda_5^{(2)}$, $\lambda_{10}^{(2)}$ in Table 2, then $w(\lambda_k, \frac{1}{2}) = -2u_k'''(\frac{1}{2} - 0) = 4c_k z_k^3 \sin^2(\frac{z_k}{4})$. Hence $c_k \in \mathbb{R}'_+$ has to be chosen in this case.

We did not exclude the case $C \cap D \neq \emptyset$. If e. g. $C = D = \{x_1\}$ then (5.3) shows that also the third derivative of an eigenfunction must be continuous. The form of the eigenfunction shows that the fourth derivative is continuous together with the second and only the eigenfunctions of the unrestricted problem remain. *If* $x_1 = \frac{p}{q}$, $p < q$, $p, q \in \mathbb{N}$, then we have the'eigenfunctions and -values $u_k^{(0)}$, $\lambda_k^{(0)}$, $k = j(p + q)$, $j = 1,2,\ldots$ For certain x_1 also some of the $u_k^{(1)}$ are eigenfunctions.

The branch bifurcating from the solutions of (5.1) may be computed from

$$(5.7) \qquad f(u_\rho) = \max_{K \cap \partial S_\rho} f(u)$$

K, f and g as in (1.3). We refer to theoretical results of [6]. It may, however, not be expected that analytic expressions for u_ρ, $\lambda(u_\rho)$ could be derived. For the largest eigenvalues the existence of a continuous branch extending to infinity is assured by the results in [6, 18]. Hence in the following we concentrate on the maximal eigenvalue for the restricted case and compute the branches.

6. Discretization and results

As in 3. it suffices to consider the problem (5.7). We use a finite element method to obtain a finite-dimensional problem. Instead of doing this in an abstract setting we again give a concrete application. In order to discretize (1.3), (5.7) we use a subdivision of $[0,1]$: $0 = x_0 < x_1 < \ldots < x_N = 1$ which for the sake of simplicity we assume to be equidistant with distance h. We use the Hermite cubic finite element functions u_h. In (1.3b) it is also necessary to integrate numerically. On each subinterval $[x_i, x_{i+1}]$, $i = 0,\ldots,N-1$ we use the Q-point Gauss-Lobatto formula with x_i, x_{i+1} as two of the nodes. We obtain a problem-for the vector $y = y_h \in \mathbb{R}^{2N-2}$ of unknowns $u_h(x_1), u_h'(x_1), \ldots, u_h'(x_{N-1})$. For the second integral in (1.3b) we have

$$\text{(6.1a)} \qquad g_h(y) = \frac{1}{h^3} y^T \begin{bmatrix} 12 & 0 & -6 & 3h & & & & & \\ 0 & 4h^2 & -3h & h^2 & & & & & \\ -6 & -3h & 12 & 0 & -6 & 3h & & & \\ 3h & h^2 & 0 & 4h^2 & -3h & h^2 & & & \\ & & & & \ddots & & & & \\ & & & & & & -6 & -3h & 12 \end{bmatrix} Y$$

and for the general integrand $F(u_h')$ in the first term

$$\text{(6.1b)} \qquad f_h(y) = \frac{h}{12} \sum_{i=0}^{N-1} [2\, F(y_{2i}) + 5(F(z_i^{(1)}) + F(z_i^{(2)}))]$$

where $\quad z_i^{(j)} = y_{2i} + 2\gamma_j \xi_i^{(1)} + 3\gamma_j^2\, \xi_i^{(2)}, \; j = 1,2,$

$$\xi_i^{(1)} = \frac{3}{h}(y_{2i+1} - y_{2i-1}) - 2y_{2i} - y_{2i+2},$$

$$\xi_i^{(2)} = \frac{2}{h}(y_{2i-1} - y_{2i+1}) + y_{2i} + y_{2i+2},$$

$$\gamma_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right) \quad \gamma_2 = 1 - \gamma_1, \quad y_{-1} = y_0 = y_{2N-1} = y_{2N} = 0 .$$

We shall use the notation $f_h^{(1)}(y)$ when $F(u_h') = \frac{1}{2} u_h'^2$ and $f_h^{(2)}(y)$ when $F(u_h') = \sqrt{1+u_h'^2} - 1$. Obviously we have $f_h^{(1)}(y) = \frac{1}{2}(f''(0)u_h, u_h)$. The problem of determining a $y_o \in \mathbb{R}^{2N-2}$ with

(6.2) $$f_h(y_o) = \max_{K_h \cap \partial S_\rho^h} f_h(y)$$

$$K_h = \{y \in \mathbb{R}^{2N-2},\ y_{2i-1} \geq 0 \text{ if } x_i \in C,\ y_{2i-1} \leq 0 \text{ if } x_i \in D,\ i = 1,\dots,N\}$$

$$\partial S_\rho^h = \{y \in \mathbb{R}^{2N-2},\ g_h(y) = \tfrac{1}{2}\rho^2, \rho > 0\}$$

may now be solved by the algorithm of section 3.

We remark that in the finite-dimensional case it is easy to describe the set of all solutions by considering the eigenvalues and -vectors of the general eigenvalue problem $Ay = \lambda By$ together with those for certain submatrices of $\bar{A}$ and B.

In the following we shall restrict ourselves in the choice of numerical examples as we did in the continuous problem in section 5.

The functional $f_h^{(1)}(y)$ has a finite number of critical points with respect to $K \cap \partial S_1$ and in general the algorithm converges only locally. If e. g. $h = \frac{1}{3}$, $C = \{\frac{1}{3}\}$, $D = \{\frac{2}{3}\}$ and we choose the vector $y^{(o)} = (1, 0, -1, 0)^T$ after normalization according to $\|\bar{y}^{(o)}\|_B = 1$ as starting vector then $f_h^{(1)}(\bar{y}^{(o)}) = .01111111 < \lambda_1^{(1)}$ and the sequence $\{y^{(k)}\}$ generated by our algorithm converges to a solution of the form of $u_1^{(1)}$. The starting vector $y^{(o)} = (0,0,-1,0)^T$, however, which has the same function value, lea& to $u_1^{(2)}$. Since we are interested in bifurcation from the largest eigenvalue for restricted problems we look at this case in more detail. Denoting $\lambda_{1h}^{(?)} = f_h^{(1)}(y_h)$ and recalling that $u_h(\frac{1}{3}) = 0$ we list in Table 4 the values of the approximate solution for different h and the exact solution which is normalized by choosing

$$c_1 = \left(\int_0^1 [u_1^{(2)''}]^2 dx \right)^{-1/2} = |u_1^{(2)}|_2^{-1}$$

in (5.5).

h	$\lambda_{1h}^{(2)}$	$u_h'(\frac{1}{3})$	$u_h(\frac{2}{3})$	$u_h'(\frac{2}{3})$
$\frac{1}{3}$	.a1432293	-.133851	-.043313	.050980
$\frac{1}{6}$	.01459822	-.138142	-.042493	.056045
$\frac{1}{12}$	.01465767	-.137874	-.042402	.055918
ex.	.01466209	-.137853	-.042396	.055910

Table 4 Approximate and exact values for $C = \{\frac{1}{3}\}$, $D = \{\frac{2}{3}\}$

We have then computed the solutions $y_{h\rho}$ of (6.2) with $f_h = f_h^{(2)}$, i. e. points on the branch bifurcating from $\lambda_{1h}^{(2)}$ by starting with the approximate solution of the eigenvalue problem and using an increasing sequence $\{\rho_k\}$, $k = 1,2,\ldots$. In Table 5 we have listed the values of $P_{h\rho} = (\lambda_h(y_{h\rho}))^{-1}$, i. e. of the axial force applied to the beam, and $u_{h\rho}$ at the same points as in the last table ($h = \frac{1}{6}$). We have $P_{ho} = 68.5015$.

ρ	$P_{h\rho}$	$u_{h\rho}'(\frac{1}{3})$	$u_{h\rho}(\frac{2}{3})$	$u_{h\rho}'(\frac{2}{3})$
1	68.9679	-.138252	-.042496	.055980
2	70.3172	-.277074	-.085001	.111859
5	78.3242	-.702718	-.212236	.277923
10	98.0209	-1.43655	-.421930	.552830

Table 5 Approximate values for the buckled beam

In order to show the change in the solution and to check (2.5) we list the values of $\bar{u}_{h\rho} = \frac{1}{\rho} u_{h\rho}$ for $h = \frac{1}{6}$ and for $\rho = 0$ that of the eigenvalue problem in Table 6.

ρ	$\bar{u}'_{h\rho}(\frac{1}{3})$	$\bar{u}_{h\rho}(\frac{2}{3})$	$\bar{u}'_{h\rho}(\frac{2}{3})$
0	-.138142	-.042493	.056045
1	-.138252	-.042496	.055980
2	-.138537	-.042501	.055930
5	9.140544	-.042447	.055585
10	-.143655	-.042193	.055283

Table 6 Normalized values on the bifurcating branch

It presented no difficulties to follow the branch up to larger values of ρ but then, of course the variational inequality (5.1) ceases to decribe the actual behavior of the beam.

Finally we have plotted the buckled beam for two different values of the force and the branch for $0 \leq \rho \leq 10$.

Figure 2

Figure 3

We have seen that both problems which have been attacked in this paper, namely the approximate computation of bifurcating branches for nonlinear variational inequalities and the determination of the bifurcation points may be solved satisfactorily. Already for rather crude discretizations, the computations were performed in BASIC on a cbm 3032, reasonable accuracy was obtained in the solution of the linearized problem.

For the numerical treatment of similar problems in higher dimensions, as e. g. the buckling of plates, the efficiency of the algorithm should be increased. For variational inequalities a preconditioned cg-method was considered in [11]. The nonlinear restriction $u \in \partial S_\rho$ in (5.7) should eventually be handled in an indirect way. We would suggest an augmented Lagrangian method. Especially for the following of the branch good starting values for this algorithm will be available after the linear eigenvalue problem has been solved. Finally, another question-which was not considered here is the convergence of the discrete approximations for $h \to 0$.

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Figure-legends

Fig. 1 two eigenfunctions in the case C = $\{\frac{1}{3}\}$, D = $\{\frac{2}{3}\}$

Fig. 2 Approximate deflection of the buckled beam. P = 68.9679 and P = 70.3172

Fig. Bifurcation diagram obtained for h = $\frac{1}{6}$.

Figure 1

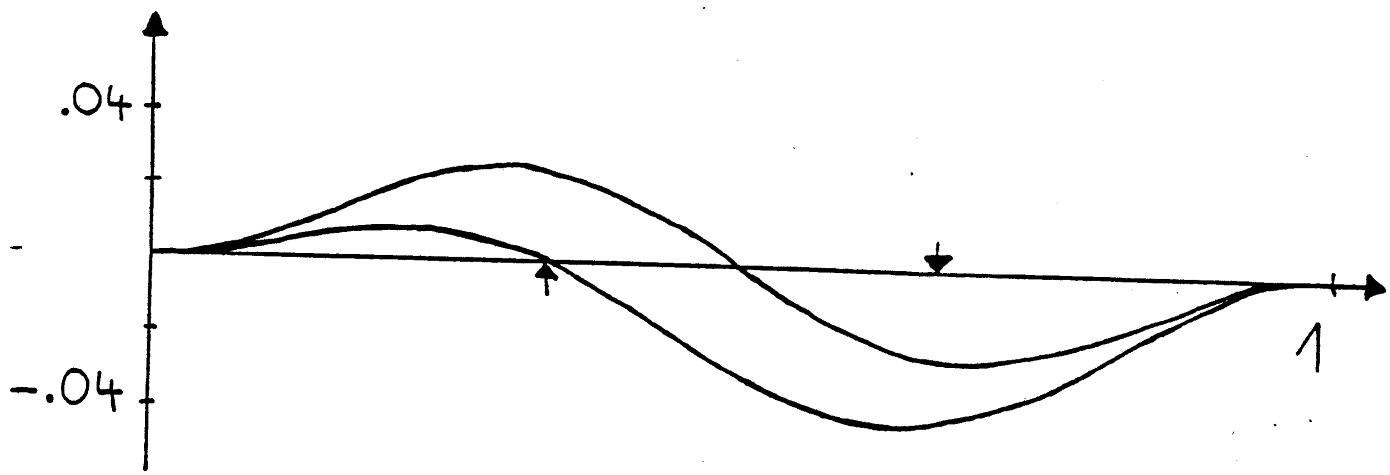

Figure 2

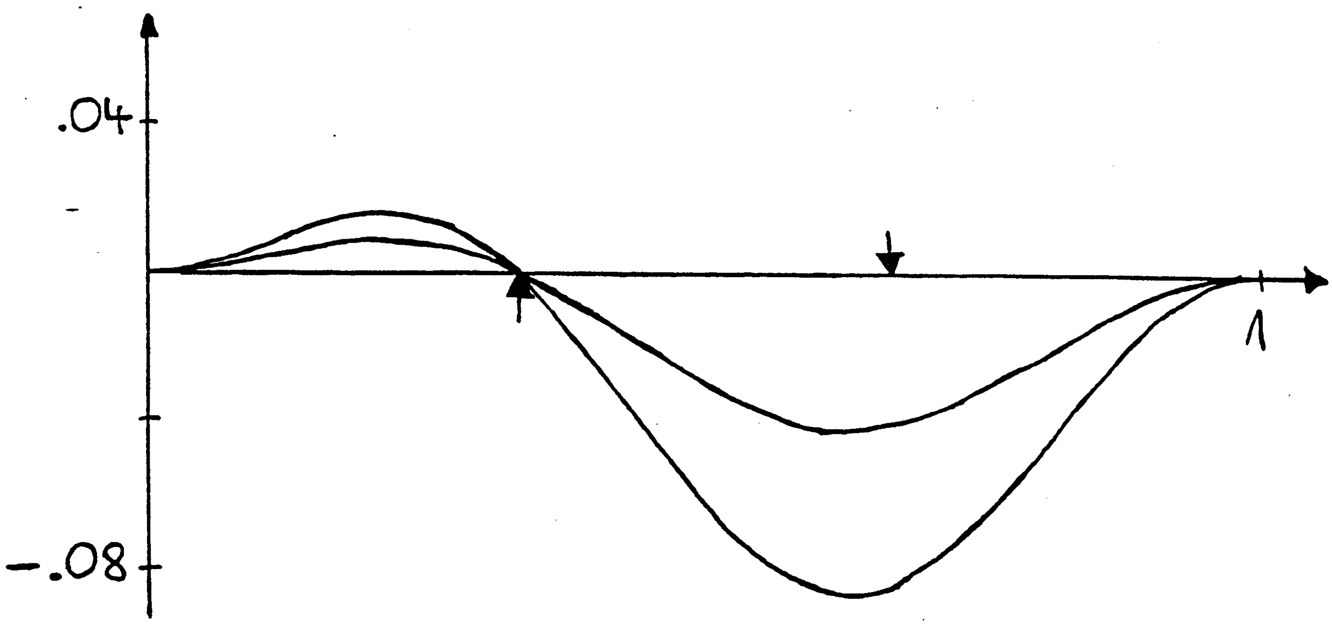

Figure 3