Generalized Singular Value Decompositions:
A Proposal for a Standardized Nomenclature

by

Bart L.R. De Moor
Gene H. Golub
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Bart L.R. De Moor †
Gene H. Golub
Department of Computer Sciences
Stanford University
CA 94305 Stanford
tel: 415-723-1923
e-mail: golub@patience.stanford.edu
demoor@patience.stanford.edu

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Abstract

An alphabetic and mnemonic system of names for several matrix decompositions related to the singular value decomposition is proposed: the OSVD, PSVD, QSVD, RSVD, SSVD, TSDV. The main purpose of this note is to propose a standardization of the nomenclature and the structure of these matrix decompositions.

1 Introduction

The ordinary singular value decomposition (OSVD) has become an important tool in the analysis and numerical solution of numerous problems. Not only does it allow for an elegant problem formulation, but at the same time it provides geometrical and algebraic insight together with an immediate numerically robust implementation [10]. It plays a prominent role in numerous

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†Dr. De Moor is on leave from the Katholieke Universiteit Leuven, Belgium and is also with the Information Systems Lab, Department of Electrical Engineering, Stanford University. He is supported by an Advanced Research Fellowship in Science and Technology of the NATO Science Fellowships Programme and a grant from IBM.
applications in linear algebra, systems theory and signal processing (e.g. \[5\]
\[lo\]). Recently, several generalizations of the OSVD have been proposed
and their properties analysed.

This note proposes a mnemonic system of names and abbreviations for sev-
eral matrix decompositions that are related to the OSVD of a (complex)
matrix. At the same time, for each of the factorizations, the specific
structure is emphasised.

A survey, discussing more in detail the properties and connections between
these generalizations such as the relation to (generalized) eigenvalue
problems, variational characterizations, uniqueness issues and typical applica-
tions, including linear and total linear least squares, rank minimization,
generalized inverses, etc . . . , is in preparation \[4\].

Besides the Ordinary SVD, we briefly discuss the Product, Quotient and
Restricted SVD, all of which are referred to as generalized SVDs (GSVD).
We also briefly consider the Structured Singular Value (SSV) arising in sys-
tem theory and the Takagi SVD (TSVD) for a complex symmetric matrix.

Throughout this note, matrices are denoted by capitals, vectors by lower case
letters other than \(i, j, k, l, m, n, p, q, r\), which are positive integers. Scalars
(complex) are denoted by greek letters. \(A (m \times n), B (m \times p), C (q \times n)\) axe
given complex matrices. Their rank will be denoted by \(r_a, r_b, r_c\). We also
define:

\[
\begin{align*}
r_{ac} &= \text{rank} \left( \begin{array}{c} A \\ C \end{array} \right) & r_{abc} &= \text{rank} \left( \begin{array}{cc} A & B \\ C & 0 \end{array} \right) \\
r_{ab} &= \text{rank} \left( \begin{array}{c} A \\ B \end{array} \right) & r_1 &= \text{rank}(A^*B)
\end{align*}
\]

\(A^t\) is the transpose of a (possibly complex) matrix while \(\overline{A}\) is the conjugate of
\(A\) and \(A^*\) the complex conjugate transpose of a (complex) matrix: \(A^* = \overline{A}^t\).

\(A^{-*}\) is the inverse of \(A^*\), \(I_k\) is the \(k \times k\) identity matrix. \(U_a (m \times m), V_a
(n \times n), V_b (p \times p), U_c (q \times q)\) are unitary matrices:

\[
\begin{align*}
U_aU_a^* &= I_m = U_a^*U_a & V_aV_a^* &= I_n = V_a^*V_a \\
V_bV_b^* &= I_p = V_b^*V_b & U_cU_c^* &= I_q = U_c^*U_c
\end{align*}
\]

\(P (m \times m), Q (n \times n)\) are square non-singular matrices. \(S_a (m \times n), S_b
(m \times p), S_c (q \times n)\) are sparse matrices, with real, nonnegative elements,
the structure of which will be explored in detail in the main theorems. The
non-zero elements are denoted by $\alpha_i, \beta_i$ and $\gamma_i$. Moreover, we will adopt the following convention for block matrices: Any (possibly rectangular) block of zeros is denoted by $0$, the precise dimensions being obvious from the block dimensions. The symbol $I$ represents a matrix block corresponding to the square identity matrix of appropriate dimensions. Whenever a dimension indicating integer in a block matrix is zero, the corresponding block row or block column should be omitted. An equivalent formulation would be that we allow $0 \times n$ or $n \times 0$ ($n \neq 0$) blocks to appear in matrices. This allows an elegant treatment of several cases at once.

2 The Ordinary Singular Value Decomposition (OSVD)

The singular-value decomposition was introduced in its general form by Autonne [1] in 1902 and an important characterization was described by Eckart and Young in 1936 [7].

With the notations and conventions of section 1, we have the following:

**Theorem 1** The Ordinary Singular Value Decomposition: The Autonne-Eckart-Young theorem

Every $m \times n$ matrix $A$ can be factorized as:

$$A = U_a S_a V_a^*$$

where $U_a$ and $V_a$ are unitary matrices and $S_a$ is a real $m \times n$ diagonal matrix with $r_a = \text{rank}(A)$ positive diagonal entries:

$$S_a = \begin{pmatrix}
 r_a & n - r_a \\
 0 & 0 \\
 0 & 0
\end{pmatrix}$$

where $D_a = \text{diag}(\sigma_i)$, $\sigma_i > 0$, $i = 1, \ldots, r_a$.

The columns of $U_a$ are the left singular vectors while the columns of $V_a$ are the right singular vectors. The diagonal elements of $S_a$ are the so-called singular values and by convention they are ordered in non-increasing order. A proof of the OSVD and numerous properties can be found in e.g. [5] [10]. Applications include rank reduction with unitarily invariant norms, linear and total linear least squares, computation of canonical correlations, pseudo-inverses and canonical forms of matrices.
3 The Product Singular Value Decomposition (PSVD)

The product singular value decomposition (PSVD) was introduced by Fernando and Hammarling [9] in 1987 but it is also implicit in the work of Heath et al. [11] [13].

With the notations and conventions of section 1, we have the following:

**Theorem 2 The Product SVD**

Every pair of matrices $A$, $m \times n$ and $B$, $m \times p$ can be factorized as:

$$
A = P^{-*} S_a V_a^* \\
B = P S_b V_b^*
$$

where $V_a, V_b$ are unitary and $P$ is square nonsingular. $S_a$ and $S_b$ are real and have the following structure:

$$
S_a = \begin{pmatrix}
    r_1 & r_a - r_1 & n - r_a \\
    r_a - r_1 & 0 & 0 \\
    r_b - r_1 & 0 & 0 \\
    m - r_a - r_b + r_1 & 0 & 0
\end{pmatrix}
\quad
D_a = \begin{pmatrix}
    r_1 & r_b - r_1 & p - r_b \\
    0 & I & 0 \\
    0 & 0 & 0
\end{pmatrix}
$$

where $D_a = D_b$ is square diagonal with positive diagonal elements and $r_1 = \text{rank}(A^*B)$.

A constructive proof based on the OSVDs of $A$ and $B$, can be found in [2], where also all possible sources of non-uniqueness are explored.

The name PSVD originates in the fact that the OSVD of the product $A^*B$ is a direct consequence of the PSVD of the pair $A, B$. The matrix $D_a^2 = D_b^2$ contains the nonzero singular values of $A^*B$. The column vectors of $P$ are the eigenvectors of the eigenvalue problem $(BB^*AA^*)P = PA$. The column vectors of $V_a$ are the eigenvectors of $(A^*BB^*)V_a = V_a \Lambda$ while those of $V_b$ are the eigenvectors in $(B^*AA^*)V_b = V_b \Lambda$. The pairs of diagonal elements
of $S_a$ and $S_b$ are called the product singular value pairs while their products are called the product singular values. By convention, the diagonal elements of $S_a$ and $S_b$ are ordered such that the product singular values are non-increasing.

Applications will be surveyed in [2], including the orthogonal Procrustes problem, balancing of state space models and computing the Kalman decomposition.

4 The Quotient Singular Value Decomposition

The quotient singular value decomposition was introduced by Van Loan in [16] (the BSVD) in 1976 although the idea had been around for a number of years, albeit implicitly (disguised as a generalized eigenvalue problem). Paige and Saunders extended Van Loan’s idea in order to handle all possible cases [14] (they called it the generalized SVD).

With the notations and conventions of section 1, we have the following:

**Theorem 3** The Quotient SVD

Every pair of matrices $A$, $m \times n$ and $B$, $m \times p$ can be factorized as:

$$
A = P^{-*}S_aV_a^*
$$
$$
B = P^{-*}S_bV_b^*
$$

where $V_a$ and $V_b$ are unitary and $P$ is square nonsingular. The matrices $S_a$ and $S_b$ are real and have the following structure:

$$
S_a = 
\begin{pmatrix}
    r_{ab} - r_b & r_a + r_b - r_{ab} & n - r_a \\
    r_a + r_b - r_{ab} & I & 0 \\
    r_{ab} - r_a & 0 & Da \\
    m - r_{ab} & 0 & 0
\end{pmatrix}
$$

$$
S_b = 
\begin{pmatrix}
    p - r_b & r_a + r_b - r_{ab} & r_{ab} - r_a \\
    0 & 0 & Da \\
    0 & 0 & D_b \\
    0 & 0 & I
\end{pmatrix}
$$
where $D_a$ and $D_b$ are square diagonal matrices with positive diagonal elements, satisfying:

$$D_a^2 + D_b^2 = I_{r_a+r_b-r_{ab}}$$

There are 4 different kinds of pairs of diagonal elements of $S_a$ and $S_b$:

- $r_{ab} - r_b$ pairs $(\alpha_i, \beta_i) = (1,0)$
- $r_a - r_b - r_{ab}$ pairs $(\alpha_i, \beta_i)$ with $\alpha_i \neq 0$ and $\beta_i \neq 0$.
- $r_{ab} - r_a$ pairs $(\alpha_i, \beta_i) = (0,1)$
- $m - r_{ab}$ pairs $(\alpha_i, \beta_i) = (0,0)$

The first three kinds of pairs are called non-trivial while the zero pairs are called trivial quotient singular value pairs.

The quotient-singular values are defined as the ratios of elements of these pairs. Hence, there are zero, non-zero, infinite and arbitrary (or undefined) quotient singular values. By convention, the non-trivial quotient singular value pairs are ordered such that the quotient singular values are non-increasing.

The name QSVD originates in the fact that under certain conditions, the QSVD provides the OSVD of $A+B$, which could be considered as a matrix quotient. Moreover, in most applications, the quotient singular values are relevant (not the diagonal elements of $S_a$ and $S_b$ as such). A typical example is the prewhitening of data (Mahalanobis transformation) when the (possibly singular) square root of the) noise covariance matrix is known.

The column vectors of $P$ are the eigenvectors of the generalized eigenvalue problem $AA^*P = BB^*PA$.

Applications include rank reductions of the form $A + BD$ with minimization of any unitarily invariant norm of $D$, least squares (with constraints) and total least squares (with exact columns), signal processing and system identification, etc… [4][5][10][14][16].

5 The Restricted Singular Value Decomposition (RSVD)

The idea of a generalization of the OSVD for three matrices is implicit in the S, T-singular value decomposition of Van Loan [16] via its relation to a generalized eigenvalue problem. An explicit formulation and derivation of
the restricted singular value decomposition was introduced by Zha in 1988 [17]. Constructive proofs and a lot of applications are discussed in [3].

With the notations and conventions of section 1, we have the following:

**Theorem 4 The Restricted SVD**

Every triplet of matrices $A (m \times n)$, $B (m \times p)$ and $C (q \times n)$ can be factorized US:

$$
A = P^{-*} S_a Q^{-1} \\
B = P^{-*} S_b V_b^* \\
c = U_c S_c Q^{-1}
$$

where $P (m \times m)$ and $Q (n \times n)$ are square nonsingular, $V_b (p \times p)$ and $U_c (q \times q)$ are unitary, $S_a (m \times n), S_b (m \times p)$ and $S_c (q \times n)$ are real matrices with nonnegative elements and the following structure:

$$
S_a = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & S_1 & 0 & 0 & 0 & 0 \\
2 & 0 & I & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$$
S_b = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & I & 0 & 0 \\
2 & 0 & I & 0 \\
3 & 0 & 0 & I \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
6 & 0 & 0 & 0
\end{array}
$$

$$
S_c = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & I & 000 & 0 & 0 \\
2 & 0 & 100 & 0 & 0 \\
3 & 0 & 000 & 0 & 0 \\
4 & 0 & 0 & 0 & S_3 & 0
\end{array}
$$

The block dimensions of the matrices $S_a, S_b, S_c$ are:
Blockcolumns of $S_a$ and $S_c$:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. $r_{ac} - r_a$
6. $n - r_{ac}$

Blockcolumns of $S_b$:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ac} + r_b - r_{abc}$
3. $p - r_b$
4. $r_{ab} - r_a$

Block rows of $S_a$ and $S_b$:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. $r_{ab} - r_a$
6. $m - r_{ab}$

Block rows of $S_1$:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$
2. $r_{ab} + r_c - r_{abc}$
3. $q - r_b$
4. $r_{ac} - r_a$

*The matrices $S_1, S_2, S_3$ are square nonsingular diagonal.*

The restricted singular value triplets are the following triplets of numbers:
• \( r_{abc} + r_a - r_{ab} - r_{ac} \) triplets of the form \((\alpha_i, 1, 1)\) with \(\alpha_i > 0\).
• \( r_{ab} + r_c - r_{abc} \) triplets of the form \((1, 0, 1)\).
• \( r_{ac} + r_b - r_{abc} \) triplets of the form \((1, 1, 0)\).
• \( r_{abc} - r_b - r_c \) triplets of the form \((1, 0, 0)\).
• \( r_{ab} - r_a \) triplets of the form \((0, \beta_j, 0)\), \(\beta_j > 0\) (elements of \(S_2\)).
• \( r_{ac} - r_a \) triplets of the form \((0, 0, \gamma_i)\), \(\gamma_k > 0\) (elements of \(S_3\)).
• \( \min(m - r_{ab}, n - r_{ac}) \) trivial triplets \((0, 0, 0)\).

Formally, the restricted singular values are the numbers:

\[
\sigma_i = \frac{\alpha_i}{\beta_i \gamma_i}
\]

Hence, there are zero, infinite, nonzero and undefined (arbitrary, trivial) restricted singular values.

A constructive proof, based upon the OSVD-PSVD or OSVD-QSVD is derived in [3]. It is not too difficult to show that the \textbf{OSVD, PSVD and QSVD} are special cases of the RSVD (see theorem 5 in [3]).

The name RSVD originates in some of its applications. A typical one is finding the matrix \(D\) of minimal (unitarily invariant) norm that reduces the rank of \(A + BDC\) where \(A\), \(B\) and \(C\) are given. Hence, one attempts reducing the rank of \(A\) by restricting the modifications to the column space of \(B\) and the row space of \(C\). A detailed analysis and many other applications can be found in [3], including the analysis of the extended shorted operator, unitarily invariant norm minimization with rank constraints, rank minimization in matrix balls, the analysis and solution of linear matrix equations, rank minimization of a partitioned matrix and the connection with generalized \textbf{Schur} complements, constrained linear and total linear least squares problems with mixed exact and noisy data, including a generalized Gauss-Markov estimation scheme.

6 The Structured Singular Value (SSV)

The concept of \textit{structured singular value} was introduced by Doyle in 1982 [6] as a tool for analysis and synthesis of feedback systems with structured
uncertainties. Consider a block partition of a matrix $A$ as:

$$
A = \begin{pmatrix}
A_{11} & \cdots & A_{1q} \\
\vdots & \ddots & \vdots \\
A_{p1} & \cdots & A_{pq}
\end{pmatrix}
$$

and a matrix $AA$, partitioned in the same way as $A$, consisting of zero and nonzero blocks $\Delta A_{ij}$, with possibly some constraints $\Delta A_{ij} = \Delta A_{kl}$.

**Definition 1** The structured singular value

The structured singular value $\sigma_{SSV}$ is defined as:

$$
\sigma_{SSV} = \min \|\Delta A\|_\sigma \text{ such that } \text{rank}(A + AA) < \text{rank}(A)
$$

where $\|\cdot\|_\sigma$ is the largest singular value of a matrix.

Applications are mainly in $H_\infty$ control theory, and some characterizations and algorithms may be found in [8]. For instance, it can be shown that it suffices to investigate matrices $AA$ that are block diagonal. For some structures of the matrix $AA$, the solution can also be found via the P-Q-R SVD [4].

### 7 The Takagi Singular Value Decomposition (TSVD)

A (possibly complex) matrix $A$ is symmetric whenever $A = A^t$. If $A = A_1 + iA_i$, then $A$ is symmetric if and only if both $A_1$ and $A_i$ are real symmetric. Every complex symmetric matrix has the property that all the eigenvalues of $AA^* = AA^t$ are nonnegative. This leads to the so called Takagi factorization, which is a special singular value decomposition for complex symmetric matrices and was derived by Takagi in 1925 [15].

**Theorem 5** Takagi’s factorization

If $A$ is symmetric, there sits a unitary $U$ and a real nonnegative diagonal matrix $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$ such that $A = U\Sigma U^t$. The columns of $U$ are an orthonormal set of eigenvectors for $AA^*$ and the corresponding diagonal entries of $\Sigma$ are the nonnegative square roots of the corresponding eigenvalues of $AA^*$.

The original proof can be found in [15]. Further properties are described in[12].
8 Conclusions and summary

In this note, we have proposed a standardized nomenclature for some generalizations and special cases of the singular value decomposition. Summarizing, we propose the following set of names and abbreviations:

OSVD: Ordinary Singular Value Decomposition (theorem 1)
PSVD: Product Singular Value Decomposition (theorem 2)
QSVD: Quotient Singular Value Decomposition (theorem 3)
RSVD: Restricted Singular Value Decomposition (theorem 4)

The last three cases can be considered as Generalized Singular Value Decompositions, (GSVD). The RSVD contains the others as special cases and hence is the most general. Furthermore, we have also mentioned:

SSV: The Structured Singular Value
TSVD: The Takagi Singular Value Decomposition

References


