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Generalized Singular Value Decompositions:

A Proposal for a Standardized Nomenclature

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Generalized Singular Value Decompositions: A proposal for a standardized nomenclature *

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Abstract

An alphabetic and mnemonic system of names for several matrix decompositions related to the singular value decomposition is proposed: the OSVD, PSVD, QSVD, RSVD, SSVD, TSVD. The main purpose of this note is to propose a standardization of the nomenclature and the structure of these matrix decompositions.

1 Introduction

The *ordinary singular value decomposition (OSVD)* has become an important tool in the analysis and numerical solution of numerous problems. Not only does it allow for an elegant problem formulation, but at the same time it provides geometrical and algebraic insight together with an immediate numerically robust implementation [10]. It plays a prominent role in numerous

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applications in linear algebra, systems theory and signal processing (e.g. [5] [10]). Recently, several generalizations of the OSVD have been proposed and their properties analysed.
This note proposes a mnemonic system of names and abbreviations for several matrix decompositions that are related to the OSVD of a (complex) matrix. At the same time, for each of the **factorizations,** the **specific** structure is emphasised.
A survey, discussing more in detail the properties and connections between these generalizations such as the relation to (generalized) **eigenvalue** problems, variational characterizations, uniqueness issues and typical applications, including linear and total linear least squares, rank minimization, generalized inverses, etc ..., is in preparation **[4].**

Besides the Ordinary **SVD,** we **briefly** discuss the Product, Quotient and Restricted **SVD,** all of which are **referred** to **as *generalized* SVDs** (GSVD). We also briefly consider the Structured Singular Value (SSV) arising in system theory and the **Takagi SVD** (TSVD) for a complex symmetric matrix.

Throughout this note, matrices are denoted by capitals, vectors by lower case letters other than i, j, k, l, m, n, p, q, r, which are positive integers. Scalars (complex) are denoted by greek letters. A $(m \times n)$, B $(m \times p)$, C $(q \times n)$ are given complex matrices. Their rank **will** be denoted by r_a, r_b, r_c. We also **define:**

$$r_{ac} = rank \begin{pmatrix} A \\ C \end{pmatrix} \qquad r_{abc} = rank \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$
$$r_{ab} = rank\begin{pmatrix} A & B \end{pmatrix} \qquad r_1 = rank(A^*B)$$

A^t is the transpose of a (possibly complex) matrix while $\overline{A}$ **is** the conjugate of A and A^* the complex conjugate transpose of a (complex) **matrix:** $A^* = \overline{A}^t$. A^{-*} is the inverse of A^*. I_k is the $k \times k$ identity matrix. U_a $(m \times m)$, V_a $(n \times n)$, V_b $(p \times p)$, U_c $(q \times q)$ *are unitary* **matrices:**

$$U_aU_a^* = I_m = U_a^*U_a \qquad V_aV_a^* = I_n = V_a^*V_a$$
$$V_bV_b^* = I_p = V_b^*V_b \qquad U_cU_c^* = I_q = U_c^*U_c$$

P $(m \times m)$, Q $(n \times n)$ **are** square **non-singular** matrices. S_a $(m \times n)$, S_b $(m \times p)$, S_c $(q \times n)$ are sparse matrices, with real, nonnegative elements, the structure of which will be explored in detail in the main theorems. The non-zero elements are denoted by α_i, β_i and γ_i. Moreover, we will adopt the following convention for block matrices: Any (possibly rectangular) block of zeros is denoted by **0**, the precise dimensions being obvious from the block dimensions. The symbol I represents a matrix block corresponding to the square identity matrix of appropriate dimensions. Whenever a dimension indicating integer in a block matrix is zero, the corresponding block row or block column should be omitted. An equivalent formulation would be that we allow $0 \times n$ or $n \times 0$ $(n \neq 0)$ blocks to appear in matrices. This allows an elegant treatment of several cases at once.

2 The Ordinary Singular Value Decomposition (OSVD)

The *singular-value decomposition* was introduced in its general form by Autonne [1] in 1902 and an important characterization was described by Eckart and Young in 1936 [7].

With the notations and conventions of section 1, we have the following:

Theorem 1 *The Ordinary Singular Value Decomposition: The Autonne-Eckart- Young theorem*
Every $m \times n$ matrix A can be factorized as:

$$A = U_a S_a V_a^*$$

where U_a and V_a are unitary matrices and S_a is a real $m \times n$ diagonal matrix with r_a = rank(A) positive diagonal entries:

$$S_a = \begin{array}{c} \\ r_a \\ m - r_a \end{array} \begin{array}{c} \begin{array}{cc} r_a & n - r_a \end{array} \\ \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

where $D_a = diag(\sigma_i)$, $\sigma_i > 0$, $i = 1, \ldots, r_a$.

The columns of U_a are the left singular vectors while the columns of V_a axe the right singular vectors. The diagonal elements of S_a are the so-called singular values and by convention they are ordered in non-increasing order. A proof of the OSVD and numerous properties can be found in e.g. [5] [10]. Applications include rank reduction with unitarily invariant norms, linear and total linear least squares, computation of canonical correlations, pseudo-inverses and canonical forms of matrices.

3 The Product Singular Value Decomposition (PSVD)

The ***product singular value decomposition*** (PSVD) was introduced by Fernando and Hammarling [9] in 1987 but it is also implicit in the work of Heath et al. [11] [13].

With the notations and conventions of section 1, we have the following:

Theorem 2 The Product SVD
Every pair of matrices A, $m \times n$ and B, $m \times p$ can be factorized as:

$$\begin{aligned} A &= P^{-*} S_a V_a^* \\ B &= P S_b V_b^* \end{aligned}$$

where V_a, V_b are unitary and P is square nonsingular. S_a and S_b are real and have the following structure:

$$S_a = \begin{array}{c} \\ r_1 \\ r_a - r_1 \\ r_b - r_1 \\ m - r_a - r_b + r_1 \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & r_a - r_1 & n - r_a \end{array} \\ \begin{pmatrix} D_a & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_b = \begin{array}{c} \\ r_1 \\ r_a - r_1 \\ r_b - r_1 \\ m - r_a - r_b + r_1 \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & r_b - r_1 & p - r_b \end{array} \\ \begin{pmatrix} D_b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

where $D_a = D_b$ is square diagonal with positive diagonal elements and $r_1 = rank(A^*B)$.

A constructive proof based on the **OSVDs** of *A* and *B*, *can* be found in [2], where also all possible sources of non-uniqueness are explored.
The name **PSVD originates** in the fact that the OSVD **of** the product A^*B is a direct consequence of the **PSVD of** the pair *A, B*. The matrix $D_a^2 = D_b^2$ contains the **nonzero** singular values of A^*B. The column vectors of *P are* the eigenvectors of the eigenvalue problem $(BB^*AA^*)P = P\Lambda$. The. column **vectors** of V_a are the eigenvectors of $(A^*BB^*A)V_a = V_a\Lambda$ *while* those of V_b are the eigenvectors in $(B^*AA^*B)V_b = V_b\Lambda$. The pairs of diagonal elements

of S_a and S_b are called the ***product singular value pairs*** while their products are called the ***product singular values.*** By convention, the diagonal elements of S_a and S_b are ordered such that the product singular values are non-increasing.

Applications will be surveyed in [2], including the orthogonal Procrustes problem, balancing of state space models and computing the Kalman decomposition.

4 The Quotient Singular Value Decomposition

The ***quotient singular value decomposition*** was introduced by Van Loan in [16] ('the BSVD') in 1976 although the idea had been around for a number of years, albeit implicitly (disguised as a generalized eigenvalue problem). Paige and Saunders extended Van Loan's idea in order to handle all possible cases [14] (they called it the generalized SVD).

With the notations and conventions of section 1, we have the following:

Theorem 3 *The Quotient SVD*
Every pair of matrices A, $m \times n$ and B, $m \times p$ can be factorized as:

$$\begin{aligned} A &= P^{-*} S_a V_a^* \\ B &= P^{-*} S_b V_b^* \end{aligned}$$

where V_a and V_b are unitary and P is square nonsingular. The matrices S_a and S_b are real and have the following structure:

$$S_a = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \\ m-r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} r_{ab}-r_b & r_a+r_b-r_{ab} & n-r_a \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & D_a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_b = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \\ m-r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} p-r_b & r_a+r_b-r_{ab} & r_{ab}-r_a \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_b & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

where D_a and D_b are square diagonal matrices with positive diagonal elements, satisfying:

$$D_a^2 + D_b^2 = I_{r_a+r_b-r_{ab}}$$

There are 4 different kinds of pairs of diagonal elements of S_a and S_b:

- $r_{ab} - r_b$ **pairs** $(\alpha_i, \beta_i) = (1, 0)$
- $r_a + r_b - r_{ab}$ pairs (α_i, β_i) **with** $\alpha_i \neq 0$ and $\beta_i \neq 0$.
- $r_{ab} - r_a$ **pairs** $(\alpha_i, \beta_i) = (0, 1)$
- $m - r_{ab}$ pairs $(\alpha_i$, pi$) = (0, 0)$

The first three kinds of pairs are called ***non-trivial while*** the zero pairs are ***called trivial quotient singular value pairs.***
The quotient-singular values are defined as the ratios of elements of these pairs. Hence, there are zero, non-zero, infinite and arbitrary (or undefined) quotient singular values. By convention, the non-trivial quotient singular value pairs are ordered such that the quotient singular values are non-increasing.
The name QSVD originates in the fact that under certain conditions [4], the QSVD provides the OSVD of A^+B, which could be considered as a matrix quotient. Moreover, in most applications, the quotient singular values are relevant (not the diagonal elements of S_a and S_b as such). A typical example is the prewhitening of data (**Mahalanobis** transformation) when the (possibly **singular**) (square root of the) noise covariance matrix is known.
The column vectors of P are the eigenvectors of the generalized eigenvalue problem $AA^*P = BB^*P\Lambda$.
Applications include rank reductions of the form $A + BD$ with minimization of any unitarily **invariant** norm of D, least squares (with constraints) and total least squares (with exact columns), signal processing and system identification, etc ... [4] [5] [10] [14] [16].

5 The Restricted Singular Value Decomposition (RSVD)

The idea of a generalization of the OSVD for three matrices is implicit in the S, T-singular value decomposition of Van Loan [16] via its relation to a generalized eigenvalue problem. An explicit formulation and derivation of the ***restricted singular value decomposition*** was introduced by Zha in **1988** **[17].** Constructive proofs and a **lot** of applications are discussed in **[3].**

With the notations and conventions of section **1, we have the following:**

Theorem *4 The Restricted SVD*
Every triplet of matrices A (m × n), B (m × p) and C (q × n) can be factorized US:

$$
\begin{aligned}
A &= P^{-*} S_a Q^{-1} \\
B &= P^{-*} S_b V_b^{*} \\
c &= U_c S_c Q^{-1}
\end{aligned}
$$

where P (m x m) and Q (n x n) are square nonsingular, V_b (p × p) and U_c (q × q) are unitary. S_a (m × n), S_b (m × p) and S_c (q × n) are real matrices with nonnegative elements and the following structure:

$$
S_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
S_1 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

$$
S_b = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_2 \\
0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

$$
S_c = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & S_3 & 0
\end{pmatrix} \end{array}
$$

***The** block dimensions of the matrices S_a, S_b, S_c are:*

Blockcolumns of S_a and S_c:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. r_{ac} - r_a
6. n - r_{ac}

Blockcolumns of S_b:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ac} + r_b$ - r_{abc}
3. $p - r_b$
4. $r_{ab} - r_a$

Block rows of S_a and S_b:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. r_{ab} - r_a
6. $m - r_{ab}$

Block rows of S,:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$
2. r_{ab} + r_c - r_{abc}
3. $q - r_b$
4. $r_{ac} - r_a$

The matrices S_1, S_2, S_3 are square nonsingular diagonal.

The ***restricted singular value' triplets*** are the following triplets of numbers:

- $r_{abc} + r_a - r_{ab} - r_{ac}$ triplets of the form $(\alpha_i, 1, 1)$ with $\alpha_i > 0$.
- $r_{ab} + r_c - r_{abc}$ triplets of the form $(1, 0, 1)$.
- $r_{ac} + r_b - r_{abc}$ triplets of the form $(1, 1, 0)$.
- $r_{abc} - r_b - r_c$ triplets of the form $(1, 0, 0)$.
- $r_{ab} - r_a$ triplets of the form $(0, \beta_j, 0), \beta_j > 0$ (elements of S_2).
- $r_{ac} - r_a$ triplets of the form $(0, 0, \gamma_i), \gamma_k > 0$ (elements of S_3).
- $min(m - r_{ab}, n - r_{ac})$ trivial triplets $(0, 0, 0)$.

Formally, the ***restricted singular values*** **are** the numbers:

$$\sigma_i = \frac{\alpha_i}{\beta_i \gamma_i}$$

Hence, there are zero, **infinite, nonzero** and **undefined** (arbitrary, trivial) restricted singular values.
A constructive proof, based upon the OSVD-PSVD or OSVD-QSVD is derived in **[3]**. It is not too difficult to show that the **OSVD**, PSVD and **QSVD** are special cases of the RSVD (see theorem 5 in **[3]**).
The name RSVD originates in some of its applications. A typical one is **finding** the matrix D of minimal (unitarily invariant) norm that reduces the rank of $A + BDC$ where A, B and C are given. Hence, one attempts reducing the rank of A by ***restricting*** the **modications** to the column space of B and the row space of C. A detailed analysis and many other applications can be found in **[3]**, including the analysis .of the extended shorted operator, **unitar**ily **invariant** norm minimization with rank constraints, rank minimization in matrix balls, the analysis and solution of linear matrix equations, rank minimization of a partitioned matrix and the connection with generalized **Schur** complements, constrained linear and total linear least squares problems with mixed exact and noisy data, including a generalized Gauss-Markov estimation scheme.

6 The Structured Singular Value (SSV)

The concept of ***structured singular value*** was introduced by Doyle in **1982** **[6]** as a tool for analysis and synthesis of feedback systems with structured

uncertainties.
Consider a block partition of a matrix A as:

$$A = \begin{pmatrix} A_{11} & \dots & A_{1q} \\ \dots & \dots & \dots \\ A_{p1} & \dots & A_{pq} \end{pmatrix}$$

and a matrix AA, partitioned in the same way as A, consisting of zero and nonzero blocks ΔA_{ij}, with possibly some constraints $\Delta A_{ij} = \Delta A_{kl}$.

Definition 1 *The structured singular value*
The structured singular value σ_{SSV} is defined as:

$$\sigma_{SSV} = min\|\Delta A\|_\sigma \text{ such that } rank(A + \text{AA}) < rank(A)$$

where $\|.\|_\sigma$ is the largest singular value of a matrix.

Applications are mainly in H_∞ control theory. and some characterizations and algorithms may be found in [8]. For instance, it can be shown that it suffices to investigate matrices AA that are block diagonal. For some structures of the matrix AA, the solution can also be found via the **P-Q-R SVD** [4].

7 The Takagi Singular Value Decomposition (TSVD)

A (possibly complex) matrix A is symmetric whenever $A = A^t$. If $A = A_r + iA_i$, then A is symmetric if and only if both A_r and A_i are real symmetric. Every complex symmetric matrix has the property that all the eigenvalues of $A\overline{A} = AA^*$ are nonnegative. This leads to the so called Takagi factorization, which is a special singular value decomposition for complex symmetric matrices and was derived by Takagi in 1925 [15].

Theorem 5 *Takagi's factorization*
If A is symmetric, there tits a unitary U and a real nonnegative diagonal matrix $\Sigma = diag(\sigma_1, \dots, \sigma_n)$ such that $A = U\Sigma U^t$. The columns of U am an orthonormal set of eigenvectors for $A\overline{A}$ and the corresponding diagonal entries of Σ are the nonnegative square roots of the corresponding eigenvalues of $A\overline{A}$.

The original proof can be found in [15]. Further properties are described in[12].

8 Conclusions and summary

In this note, we have proposed a standardized nomenclature for some generalizations and special cases of the singular value decomposition. Summarizing, we propose the following set of names and abbreviations:

OSVD: Ordinary Singular Value Decomposition (theorem 1)'

PSVD: Product Singular Value Decomposition (theorem 2)

QSVD: Quotient Singular Value Decomposition (theorem *3)*

RSVD: Restricted Singular Value Decomposition (theorem *4)*

The last three cases can be considered as Generalized Singular Value Decomposition-s, **(GSVD).** The RSVD contains the others as special cases and hence is the most general. **Furthermore,** we have also mentioned:

SSV: The Structured Singular Value

TSVD: The Takagi Singular Value Decomposition

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