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A NEW APPROACH FOR SOLVING SYMMETRIC EIGENVALUE PROBLEMS

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Abstract. In this paper, we present a new approach for the solution to a series of slightly perturbed symmetric eigenvalue problems $(A + BS_i B^T)\, x = \lambda\, x$, $0 \leq i \leq m$, where $A = A^T \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, and $S_i = S_i^T \in \mathcal{R}^{p\times p}$, $p \ll n$. The matrix B is assumed to have full column rank, The main idea of our approach lies in a specific choice of starting vectors used in the block Lanczos algorithm so that the effect of the perturbations is confined to lie in the first diagonal block of the block tridiagonal matrix that is produced by the block Lanczos algorithm. Subsequently, for the perturbed eigenvalue problems under our consideration, the block Lanczos scheme needs be applied to the original (unperturbed) matrix only once and then the first diagonal block updated for each perturbation so that for low-rank perturbations, the algorithm presented in this paper results in significant savings. Numerical examples based on finite element vibration analysis iilustrate the advantages of this approach.

Key words. perturbed eigenvalue problems, generalized eigenvalue problems, block Lanczos algorithm, block tridiagonal matrices, low-rank perturbations.

AMS(MOS) subject classifications. 65F15, 65F30, 65L60.

1. **Introduction.** Efficient numerical algorithms for finding eigenvalues and/or eigenvectors of real symmetric matrices are essential in many scientific and engineering applications. In this respect, however, given that one has an efficient algorithm for obtaining selected eigenpairs to the problem Ax = Xx, there is still a real need to search for algorithms of equal efficiency for handling the associated perturbed eigenvalue problem $(A + E)x$ = Ax, where E is of low-rank.

To date, particular cases of this type of problem have been observed; for example, the behavior and error bounds of eigenvalues for perturbed Hermitian matrices have been studied in [Paig74, Thom76, RaBB90] and efficient algorithms for finding the eigenvalues and eigenvectors of Hermitian matrices modified by low-rank pertur-

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bations (given the spectral decomposition of the unperturbed matrices) have been proposed in [ArGo88]. Similarly, in [Golu73, BjGo77] techniques for obtaining the eigenvalues and eigenvectors of perturbed symmetric tridiagonal matrices with rank-one perturbations are studied.

In this paper, we present a new approach for reducing a series of slightly perturbed symmetric matrices

$$A(S_i) = A + BS_iB^T,\ 0 \le i \le m, \tag{1}$$

to block tridiagonal form, where $A = A^T \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $S_i = S_i^T \in \mathcal{R}^{p\times p}$, $p << n$, and throughout, unless otherwise stated, the matrix B is assumed to have full column rank and m is a small positive integer.

The well-known block Lanczos method is employed to carry out the reduction, whence the main idea of the approach is to choose the starting vectors so that they span the column space of the matrix B. Advantage can then be taken of the perturbed nature of the matrices so that firstly, it is necessary to apply the block Lanczos method only once to the matrix A and secondly, the perturbations are confined to lie in the first diagonal block of the block tridiagonal matrix which results. Subsequently, once A has been reduced to block tridiagonal form, only the first diagonal block need be iteratively updated for each perturbation.

In section 2, the basic block Lanczos algorithm is briefly reviewed together with the convergence properties for obtaining the associated extremal eigenvalues. In section 3, the starting vectors for the perturbed symmetric eigevalue problem are derived and the method of solution is presented. In section 4, the inverted eigenvalue problem is considered (as is usual when finding the smallest eigenvalues), and in section 5, the technique is extended to consider perturbed generalized eigensystems. Numerical examples to illustrate the efficiency of the procedure are taken from finite element vibration analysis and are presented in section 6.

2. Block Lanczos Algorithm. Detailed descriptions and analyses of the block Lanczos algorithm can be found in [Unde75, GoUn77, Saad80]. For our purposes, we take the block size to be p, the number of columns in the matrix B. The block Lanczos algorithm then commences with an orthonormal matrix $V_0 \in \mathcal{R}^{n\times p}$ and computes a sequence of mutually orthonormal matrices $V_1, V_2, \ldots, V_k \in \mathcal{R}^{n\times p}$ such that the space of the vectors spanned by the columns of V_i, $1 \le i \le k$, contains the columns of the matrix $A^{k-1}V_0$ where $1 \le k \le n/p$. Also generated in this process will be a

sequence of dense matrices $D_1, D_2, \ldots . D_k \in \mathcal{R}^{p \times p}$ and upper triangular matrices R_2, $R_3, \ldots, R_k \in \mathcal{R}^{p \times p}$ as shown below:

1. Given two integers p and k, $p \ll n$, $1 \leq k \leq n/p$, and a starting orthonormal matrix $V_0 \in \mathcal{R}^{n \times p}$.
2. Let $V_1 = V_0$. Compute AV_1 and $D_1 = V_1^T AV_1$.
   Begin loop
3. For i = 1, 2, ... k- 1, do the following
   - Compute $Z_{i+1} = AV_i - V_i D_i$,
   - Compute $Z_{i+1} = Z_{i+1} - V_{i-1} R_i^T$ if i > I,
   - Compute the orthogonal factorization on Z_{i+1}, $Z_{i+1} = V_{i+1} R_{i+1}$, where R_{i+1} is upper triangular ,
   - Compute AV_{i+1} and $D_{i+1} = V_{i+1}^T AV_{i+1}$.

   End of the loop.

Now, let V be given by,

$$V = [\, V_1, V_2, \cdots, V_k \,], \tag{2}$$

then, from the orthogonality property of the block Lanczos vectors, we have that

$$V^T AV = T = \begin{bmatrix} D_1 & R_2^T & & & & \\ R_2 & D_2 & R_3^T & & & \\ & R_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & D_{k-1} & R_k^T \\ & & & & R_k & D_k \end{bmatrix}. \tag{3}$$

Once the matrix T is obtained, any standard algorithm such as Francis' QR scheme can be employed to find the eigenpairs, $(\tilde{\lambda}, \tilde{x})$ say, of the symmetric matrix T where $\tilde{\lambda}$ is called the Ritz value. The Ritz pairs, (X, $V\tilde{x}$), are then usually good approximations to the extremal eigenpairs of the original matrix A. Indeed, in order to observe the convergence properties of the block Lanczos algorithm for the smallest eigenvalues of A, we restate now, without proof, the theorem presented and proved in [GoUn77] - for further details, consult [Unde75]:

THEOREM 2.1. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ ***be the eigenvalues of A with orthonormalized eigenvectors*** $q_1, q_2, \cdots, q_n$ *and* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{pk}$ ***be the eigenvalues of*** T. *Let* $W_1 = Q_1^T V_0$ *where* $Q_1 = [q_1, q_2, \cdots q_p]$. ***Suppose that*** W_1 ***is nonsingular so that its smallest singular value,*** $\sigma_{\min}$, is ***great than zero. If*** $\lambda_p < \lambda_{p+1}$, ***then*** *for* i=l, 2, ... p, ***we have*** $\lambda_i \leq \mu_i \leq \lambda_i + \epsilon_i^2$ ***where***

$$\epsilon_i^2 = \frac{(\lambda_n - \lambda_i)\tan^2\theta}{\mathcal{T}_{k-1}^2(\frac{1+\gamma_i}{1-\gamma_i})}, \quad \theta = \arccos \sigma_{min}, \quad \gamma_i = \frac{(\lambda_{p+1} - \lambda_i)}{(\lambda_n - \lambda_i)}, \tag{4}$$

and $\mathcal{T}_{k-1}$ ***is the*** $(k-1)^{st}$ ***Chebyshev polynomial of the first kind,***

An analogous result pertaining to the largest eigenvalues follows immediately from this theorem.

To close this section, it is noted finally that, in the algorithm described **above,** we have assumed that Z_{i+1} is of full column rank. In the event that it is rank deficient, orthogonality amongst the block Lanczos vectors may be lost. Indeed, in practice, the block Lanczos vectors can rapidly lose orthogonality even if Z_{i+1} is not rank deficient, muddying the issue of terminating the algorithm and complicating the relationship between ***A's*** eigenvalues and those of the block tridiagonal matrix ***T.*** It is essential, therefore, that some sort of a reorthogonalization process be employed to maintain the orthogonality. In this respect, one can either orthogonalise each newly computed block Lanczos vector against all its predecessors - a so-called ***complete reorthogonalization*** block Lanczos scheme - or, equivalently, orthogonalise the newly computed block Lanczos vectors against the (typically few) converged Ritz vectors. This scheme is known as block Lanczos with ***selective orthogonalization*** and comes about because loss of orthogonality goes hand in hand with convergence of a Ritz pair. The practical aspects of enforcing orthogonality in either of these ways are discussed in [Lwis77, Sctt79, Ruhe79].

3. Starting Vectors for $A(S_i)x = \lambda x$. We are now in a position to discuss how to choose the starting vectors in order to take advantage of the fact that ***A*** has already been reduced to block tridiagonal form via the algorithm described in section 2. Recalling that

$$A(S_i) = A + BS_iB^T, \; B \in \mathcal{R}^{n\times p}, \; S_i \in \mathcal{R}^{p\times p} \tag{5}$$

where $0 \leq i \leq m$, $p \ll n$, and the matrix ***B*** has full column rank, the idea of our approach is rather simple. We take an orthonormal set of vectors that spans the

column space of the matrix **B.** This can be achieved by performing an orthogonal factorization of **B, B** = $V_0 R_0$, where $V_0 \in \mathcal{R}^{n \times p}$ and $R_0 \in \mathcal{R}^{p \times p}$. We then take V_0 **to** be the starting matrix V_1 for reducing the original matrix **A** via the block Lanczos algorithm to the block tridiagonal matrix **T as** shown in (3).

In the following, we show that this choice of the starting matrix leads to a reduced block tridiagonal form of the perturbed matrices $A(S_i)$ which is only a rank **p** perturbation of the block tridiagonal matrix of **A.** Let **V,** as shown in (2), contain the Lanczos matrices $V_1, V_2, \ldots . V_k$ generated by the algorithm. From the orthogonality property of these matrices, we have

$$V_i^T V_j = \begin{cases} I_p & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{6}$$

where I_p is the identity matrix of order **p. Now** define E_i = $BS_i B^T$. Pre and post multiplying E_i by V^T and **V,** respectively we have

$$V^T E_i V = V^T B S_i B^T V = V^T V_0 R_0 S_i R_0^T V_0^T V, \; i = 1, 2, \bullet \; ^{**}, m. \tag{7}$$

Since $V_0 = V_1$, by virtue of (6)

$$V_0^T V \; . \; I_p, \; \ldots \ldots \;], \tag{8}$$

so that

$$V^T E_i V = \begin{bmatrix} R_0 S_i R_0^T & & & \\ & 0 & & \\ & & & \\ & & & 0 \end{bmatrix} \tag{9}$$

and, hence,

$$V^T A(S_i) V = V^T (A + B S_i B^T) V = V^T A V + V^T E_i V = T(S_i) \tag{10}$$

where

$$T(S_i) = \begin{vmatrix} \tilde{D}_1 & R_2^T & & & & & \\ R_2 & D_2 & R_3^T & & & & \\ & R_3 & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & D_{k-1} & R_k^T \\ & & & & & R_k & D_k \end{vmatrix} \tag{11}$$

with

$$\tilde{D}_1 = D_1 + R_0 S_i R_0^T, \tag{12}$$

i.e. the matrix $V^T A(S_i)V$ is of the same structure as $V^T AV$. It is then clear that the advantage of choosing such a set of starting vectors is that the block Lanczos algorithm needs be applied only to the matrix A to yield the block tridiagonal matrix T. Once T is formed, all block tridiagonal matrices $T(S_i)$ can be easily obtained by the addition of terms $R_0 S_i R_0^T$ to the first diagonal block D_1 of T. The matrix V is independent of S_i and, hence, remains the same for all $A(S_i)$. Consequently, the computational savings will be significant for large-scale engineering computations which require many small modifications (or reanalyses) of the structure.

4. Starting Vectors for $A(S_i)^{-1}x = \mu x$. The starting set of vectors discussed in the previous section are useful for handling $A(S_i)x = \lambda x$ when the extremal eigenvalues of $A(S_i)$ are required. For many engineering or scientific applications, however, only a few of the smallest (in magnitude) eigenvalues and their corresponding eigenvectors are desired. In such a case, one usually considers the shift and invert procedure instead of working with $A(S_i)x = \lambda x$ directly. In other words, to seek the eigenvalue near zero, one considers instead the problem $A(S_i)^{-1}x = \frac{1}{\lambda}x$. Accordingly, to be able to take advantage of the nature of the perturbations, we must choose an appropriate set of starting vectors.

It is not clear at this point how the starting vectors for $(A + BS_iB^T)^{-1}x = \mu x$, $\mu = \frac{1}{\lambda}$, can be properly chosen so as to yield a block tridiagonal structure analogous to that shown in (11). However, if we assume that both A and S_i are nonsingular, the Woodbury formula yields,

$$(A + BS_iB^T)^{-1} = A^{-1} - A^{-1}B(S_i^{-1} + B^T A^{-1} B)^{-1} B^T A^{-1}. \tag{13}$$

Thus, if in reducing A" to the block tridiagonal form

$$\tilde{V}^T A^{-1} \tilde{V} = \tilde{T} \tag{14}$$

via the block Lanczos scheme, we choose the first block, $\tilde{V}_1$, of $\tilde{V}$ as that orthonormal matrix resulting from the orthogonal factorization

$$A^{-1}B = \tilde{V}_0 \tilde{R}_0 \tag{15}$$

where $\tilde{R}_0$ is upper triangular of order p, then

$$\tilde{V}^T (A + BS_i B^T)^{-1} \tilde{V} = \tilde{T} + \begin{pmatrix} I_p \\ 0 \end{pmatrix} \tilde{E}(S_i) \begin{pmatrix} I_p & 0 \end{pmatrix} = \tilde{T}(S_i) \tag{16}$$

in which $\tilde{E}(S_i)$ is a pxp matrix given by,

$$\tilde{E}(S_i) = \tilde{R}_0 (S_i^{-1} + B^T A^{-1} B)^{-1} \tilde{R}_0^T. \tag{17}$$

Note that $\tilde{T}(S_i)$ is a block tridiagonal matrix identical to $\tilde{T}$ in (14) except for the first diagonal block. Furthermore, note that, as before, $\tilde{V}$ is independent of S_i and, hence, remains constant for $0 \leq i \leq m$.

5. Extensions to Perturbed Generalized Eigenvalue Problems. In this section, we address the extension of this approach to the perturbed generalized eigenvalue problems of types

$$K(S_i)x = \lambda M x, \quad K(S_i) = (K + BS_i B^T) \tag{18}$$

and

$$K x = \lambda M(S_i)x, \quad M(S_i) = (M + BS_i B^T) \tag{19}$$

where $K(S_i)$ and $M(S_i)$ are assumed to be symmetric positive definite for all i, $0 \leq i \leq m$. In structural mechanics, $K(S_i)$ and $M(S_i)$ are referred to as stiffness and mass matrices, respectively. The key point is to convert these perturbed generalized eigenvalue problems into the standard form as follows.

Let $K = L_k L_k^T$ and $M = L_m L_m^T$ be the Cholesky factorization of K and M, respectively. Premultiplying both sides of (18) and (19) by L_m^{-1} and L_k^{-1}, respectively, yields the standard counterparts

$$(\tilde{K} + \tilde{B} S_i \tilde{B}^T) y = \lambda y \tag{20}$$

and

$$(\hat{M} + \hat{B} S_i \hat{B}^T) z = \lambda^{-1} z \tag{21}$$

where

$$\tilde{K} = L_m^{-1} K L_m^{-T},\ \tilde{B} = L_m^{-1} B, \text{ and } y = L_m^T x \tag{22}$$

and

$$\hat{M} = L_k^{-1} M L_k^{-T},\ \hat{B} = L_k^{-1} B, \text{ and } z = L_k^T x. \tag{23}$$

Now, both problems can be treated as discussed above. If one seeks those eigenpairs closest to zero in (18), then we need only obtain the block tridiagonal form associated with $\tilde{K}^{-1}$ once the relevant information about the starting orthonormal block is obtained from the orthogonal factorization of $\tilde{B}$. Similarly, in (19) one needs the block tridiagonal form associated with $\hat{M}$ based on the orthogonal factorization of $\hat{B}$.

6. Numerical Experiments. Typical examples of the class of problem described in the preceeding sections arise in the dynamic analysis of modified structures, where the natural frequencies and corresponding modes of a free vibrating system are intimately related to the eigenvalues and eigenvectors respectively of the generalized system of characteristic equations corresponding to the discrete model approximating that vibrating system.

A frequently encountered problem is how to take into account, in analysis and design, changes introduced after the initial structural dynamic analysis has been completed. Typically, the solution process is of an iterative nature and consists of repeated modification to either the stiffness or the mass of the structure in order to fine tune or improve constraint conditions/violations. The number of iterations depends on the complexity of the problem together with the nature and number of constraints. Even though, these modifications may be only slight, a complete reanalysis for the new eigenvalues and eigenvectors of the modified eigensystem is often necessary, and for large-scale problems, this can drive the computational cost of the entire process up dramatically.

The question then is how information from the initial/previous analysis can be readily exploited to derive the response of the new modified structure without extensive additional computations. To illustrate the usefulness of our approach in this respect, we now present some numerical applications from the free vibration analysis

of an undamped cantilever beam using finite elements. Without loss of generality, we consider only modifications to the system stiffness matrix.

We assume that the beam is uniform along its span and that it is composed of a linear, homogeneous, isotropic, elastic material. Further, the beam is assumed to be slender, i.e. deformation perpendicular to the beam axis is due primarily to bending (flexing) and shear deformation perpendicular to the beam axis can be neglected; shear deformation and rotary inertia effects only become important when analyzing deep beams at low frequencies or slender beams at high frequencies. Subsequently, we consider only deformations normal to the undeformed beam axis and are only interested in the fewest lowest natural frequencies, since they usually have the simplest mode shapes, the largest amplitudes of structural distortion and stress, and are the most easily excited. All corresponding natural modes of vibration have the property of ort hogonali ty.

The beam possesses an additional support at its free end by a spring (assumed to be massless) with various stiffness coefficients α_i, i = 0, 1, ... m, as shown in Figure 1. The beam is assumed to have length L = 3.0, a distributed mass $\bar{m}$ = 390.0 per unit length, and a flexural rigidity $EI = 10^7$.

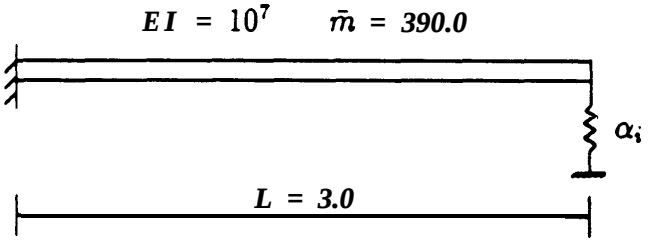

FIG. 1. *A cantilever beam with* **additional** ***spring*** *support.*

We first discretize the beam using one-dimensional solid beam finite elements, each of length l = 0.3, as shown in Figure 2, where the downward and arc arrows represent translational and rotational displacements, respectively. Although, in general, the consistent mass approach is more accurate, for the sake of simplicity, we employ the diagonal mass approach [Cook81, pp.306–308]. Following the standard finite element process [Paz80, pp.237–240], the system mass matrix M and the system stiffness matrix K, both of order 20. are obtained as shown below:

$$(24) \qquad M = \left(\frac{\bar{m}l}{78}\right) \text{diag } (78.0,\ 2l^2,\ 78.0,\ 2l^2, \cdots,\ 78.0,\ 2l^2,\ 39.0,\ l^2)$$

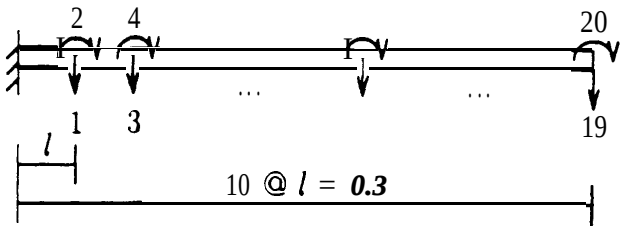

FIG. 2. *The ordenng* **of** *unknowns* **of** *the* ***uniformly*** **discntized** *beam.*

where the term ***diag*** denotes a diagonal matrix and

$$K = \frac{EI}{l^3}\begin{bmatrix} A, & C^T & & & & \\ C & A, & C^T & & & \\ & C & \cdot & \cdot & & \\ & & \cdot & & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & A, \quad C^T \\ & & & & C & A_r \end{bmatrix} \tag{25}$$

where

$$A_s = \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}, \quad A_r = \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}, \text{ and } \quad C = \begin{bmatrix} -12 & -61 \\ 61 & 2l^2 \end{bmatrix}. \tag{26}$$

Note here that both matrices are symmetric positive definite. The system stiffness matrix is now perturbed to form $K(\alpha_i)$ which is just K plus the contribution due to the spring stiffness α_i

$$K(\alpha_i) = K + b\,\alpha_i\,b^T \tag{27}$$

where b^T = [o, 0, 0, . . . , 0, 0, 1, O]. Hence, the generalized eigenvalue problem considered here is

$$K(\alpha_i)x = (K + \alpha_i bb^T)x = \lambda Mx, \quad 0 \le i \le m \tag{28}$$

which is transformed to ***a*** standard eigenvalue problem by symmetrically scaling $K(\alpha_i)$ using the matrix M. For our experiments, six values are taken for the spring stiffness

$$\alpha_0 = 0 \text{ and } \alpha_i = \frac{EI}{l^3}(10^{2i-6}), \; 1 \le i \le 5. \tag{29}$$

Three different numerical procedures, to be referred to as PERL, LAN1, and LAN2, are performed to find the smallest eigenvalues and their corresponding eigenvectors for each α_i. In PERL, we separate the perturbations from the original stiffness matrix K using the approach presented in Section 4, which involves the Woodbury updating formula. LAN1 applies the same Lanczos algorithm with the same starting vector as used in PERL to the problem directly without separating the perturbations from the stiffness matrices $K(\alpha_i)$. LAN2 is the same as LAN1 except that a random vector is used as the starting vector.

In all experiments, the number of Lanczos steps is taken to be 5 for all cases, and complete reorthogonalization is performed in the Lanczos process to avoid the loss of orthogonality. All experiments use the EISPACK subroutine TQL2 [GBDM77, pp.7.1-155–7.1-161] to extract the eigenvalues and eigenvectors of T. Indeed, in this particular case, the block size taken in the block Lanczos algorithm is one so that matrix T is tridiagonal.

Table 4 lists all three smallest eigenvalues obtained from PERL for all perturbations and compares the two extreme cases, α_0 and α_5, with the theoretical results of a cantilever beam and a fixed-hinged beam [HuRu64, pp.198-203], respectively. As can be seen from Table 4, the eigenvalues (frequencies) increase as both i and the value of the spring constant, α, increase. This latter effect corresponds to a stiffening of the beam as the end supports become more constraining leading to the higher frequencies produced by increased stiffness.

Note that the difference between the analytical solution and the numerical one for α_0 is mainly due to the finite element discretization and the diagonal mass approach. Better approximations can be achieved either by refining the discretization or by using the consistent mass approach. Further, note that in the lumped mass approach, the natural frequencies obtained may be lower than the true values (see table). This illustrates the point that consistent mass matrices must be used to achieve monotonic convergence of the natural frequencies to the true values as the finite element mesh is refined. However, the lumped mass approach is used in practice when a diagonal mass matrix leads to computational advantages in response calculations.

Table 2 presents the comparison among PERL, LAN1, and LAN2 for the third smallest eigenvalue λ_3. This table shows that the differences among PERL, LAN1, and LAN2 are very small and negligible. For the first two smallest eigenvalues, λ_1 and λ_2, they agree even better. It should also be mentioned that in our experiments, both PERL and LAN1 produce identical results up to at least the 4"' significant

TABLE 1
Comparison of the first 3 smallest eigenvalues obtained from PERL.

Approach	Exact	PERL (Numerical)						Exact
$\alpha * l^3/EI$	0	0	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}	∞
$\lambda_1/10^4$	0.391	0.387	0.400	1.526	7.292	7.501	7.503	7.528
$\lambda_2/10^6$	0.154	0.148	0.148	0.160	0.706	0.781	0.781	0.791
$\lambda_3/10^7$	0.121	0.112	0.112	0.113	0.261	0.335	0.336	0.344

TABLE 2
Companson of λ_3 for PERL, LAN1, and LAN2.

Approach	Perturbation $(\alpha * l^3/EI)$					
	0	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
PERL	0.1123	0.1123	0.1134	0.2605	0.3350	0.3356
LAN1	0.1123	0.1123	0.1134	0.2605	0.3350	0.3356
LAN2	0.1123	0.1124	0.1134	0.2607	0.3348	0.3354

digit for all other cases. For the 4^{th} and 5^{th} eigenvalues, PERL and LAN1 yield more accurate results than LAN2.

Additionally, we wish to stress that the PERL approach requires the Lanczos tridiagonalization only once for all different perturbations α_i. This can be a substantial advantage over other Lanczos algorithms for large-scale problems with low-rank perturbations as the number of perturbations, *m,* becomes large. Let t_l be the time required in the Lanczos process and t_e be the time needed for extracting the eigenvalues/eigenvectors From the reduced (block) tridiagonal matrix. Then, the algorithm without separating the perturbations will take $m(t_l + t_e)$ to complete the computations; while our approach needs only $t_l + m * (t_e + t_u)$, approximately, where t_u is the time for updating a single perturbation. For each α_i in the small example presented above, the Lanczos tridiagonalization with 5 steps took about 0.0118 CPU seconds on the Alliant FX/80 using all 8 processors. The extraction of the eigenpairs of the reduced tridiagonal matrix T using the EISPACK subroutine TQL2 plus the computation of the generalized eigenpairs took only about 0.0052 CPU seconds. Therefore, for *m* equal to 5, PERL should be more than 2 times as fast as LAN1. Note that in our implementation, we performed complete reorthogonalization. Subsequently, for large-scale (generalized) eigenvalue problems with low-rank perturbations, the time

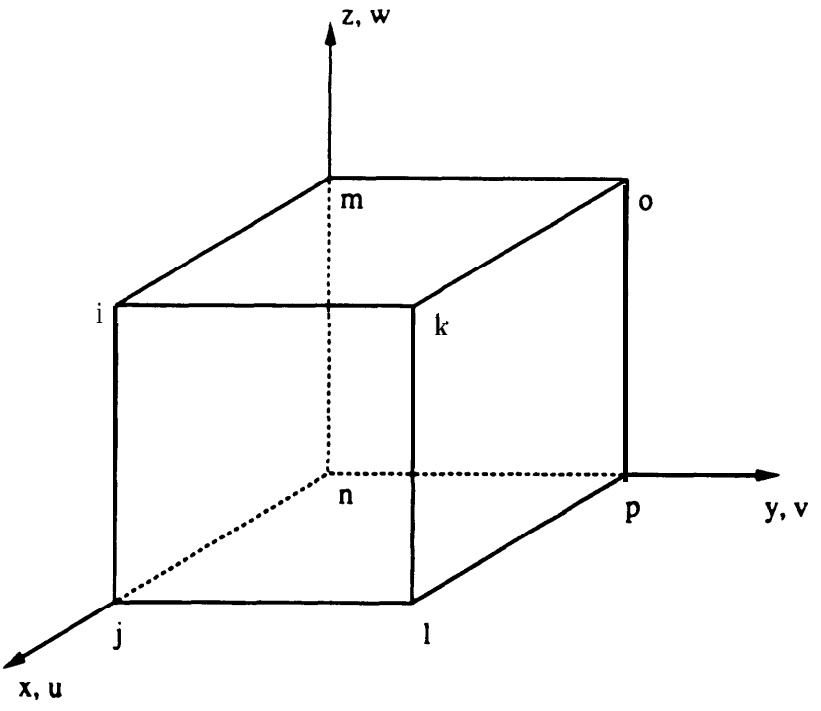

FIG. *3. The regular* ***hexahedral finite*** *element.*

t_l can be much greater than either t_e or t_u. This is especially true if only a few of the eigenpairs are needed, which is usually the case in structural analysis problems. Therefore, in order to assess the efficiency of the reanalysis technique when applied to more complicated large-scale (generalized) eigenvalue problems with low-rank perturbations which are of more practical interest, it is now appropriate to consider modelling of the cantilever beam in three dimensions. We employ the three dimensional regular hexahedral element, see fig 3, for this purpose. This element possesses eight nodes, one at each corner, each supporting three degrees of freedom - namely, the components of displacement u, v and w in the directions of the x, y and z axes, respectively. The element stiffness and mass matrices are given in [Melo63, Prze68, Chen88]

We analyse the same cantilever beam as before, where now the isotropic material constants include the modulus of elasticity $E = 2.068x10^{11}$, Poisson's ratio $\nu = 0.3$ and the mass density $\rho = 8058$; for orthotropic materials, additional elastic constants would be required. Once again, six values for the spring stiffnesses are taken

$$\kappa_1^0 = 0 \text{ and } \kappa_1^i = \frac{EI}{l^3}(10^{2i-6}),\ 1 \leq i \leq 5. \tag{30}$$

TABLE 3
Comparison of the first 3 smallest eigenvalues obtained from PERL.

Grid Size	Approach	Exact	PERL (Numerical)					
	$\kappa_1 * l^3/EI$	0	0	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
	$\kappa_2 * l^3/EI$	0	0	10^{-2}	10^{0}	10^{2}	10^{4}	10^{6}
15 × 5 × 5	λ_1	18.6	21.0	21.3	22.4	25.3	27.1	28.3
	λ_2	37.3	38.2	38.5	39.8	41.4	46.9	47.6
	λ_3	116.8	135.1	135.6	137.2	140.1	141.5	142.3
45 × 5 × 5	λ_1	18.6	18.7	20.0	21.9	23.4	24.6	25.3
	λ_2	37.3	37.7	38.2	39.5	40.2	42.7	44.1
	λ_3	116.8	121.3	121.9	123.2	126.1	128.1	130.1
75 × 5 × 5	λ_1	18.6	18.6	19.0	21.3	22.4	24.1	24.5
	λ_2	37.3	37.3	37.5	38.0	39.9	40.3	40.3
	λ_3	116.8	117.4	117.7	118.2	119.1	119.9	120.2
105 x 5 x 5	λ_1	18.6	18.6	18.9	19.2	21.3	24.0	24.1
	λ_2	37.3	37.3	37.6	38.2	39.1	39.3	39.3
	λ_3	116.8	116.8	116.9	117.0	117.0	117.1	117.3

$$\kappa_2^0 = 0 \text{ and } \kappa_2^i = \frac{EI}{l^3}(10^{2i-6}),\ 2 \le i \le 6. \tag{31}$$

The frequencies obtained are compared in table 3 with the exact values from slender beam theory for four different grid sizes. In each case, comparisons of CPU time for the non-SAS approach (subdomain = 1) are then presented in table 4 for PERL and LAN1, from which it is evident that significant savings arise with a refinement in mesh for our approach.

Now, the construction of an efficient numerical scheme for three-dimensional elasticity problems depends not only upon understanding the nature of the physical problem involved, but also upon exploiting special properties associated with its discretized system and incorporating these properties into the numerical algorithm.

In this sense, we can further reduce the time and storage if we incorporate into our procedure some sort of technique which exploits the symmetry of the structure and its boundary conditions. The traditional approach in this sense is usually achieved by observing how many axes or planes of symmetry there are in the physical problem, then reducing the problem accordingly and manually specifying the

proper geometrical and/or natural boundary conditions associated with the resulting subproblem. The approach that we choose to pursue, however, is quite different; it is known as the SAS domain decomposition method, where the term SAS stands for "symmetrical and antisymmet rical", a comprehensive analysis of which is given in [ChSa87, Chen88, ChSa88]. Essentially, it is a special decomposition method which in physical terms takes advantage of the symmetry of a given problem and decomposes the whole domain of the original problem into independent subdomains. Mathematically, this approach exploits the important SAS property possessed by a special class of matrices, A say, which satisfy the relation $A = PAP$, where P is some reflection (symmetrical signed permutation) matrix, and uses numerically stable orthogonal transformations to decompose the matrices defining the discretized system into smaller independent submatrices. The direct application of the SAS approach to physical problems is constrained only by the conditions of symmetry of the domain, the boundary conditions and the material properties. As such, it enables decomposition of the problem in an automatic way and, therefore, eliminates the need for manually specifying the boundary conditions associated with the reduced subproblems as in the traditional approach. Subsequently, it lends itself to parallelism on three levels so that it is, therefore, useful not only for implementation on supercomputers like the Cray X-MP series but also on multiprocessors. Additionally, it has the potential for reducing the bandwidth of the matrix, which reduces the storage requirements when the matrix in question is stored in banded form.

In the case of the hexahedral element, [ChSa87, Chen88, ChSa88] show that the element stiffness and mass matrices possess the SAS property with respect to some reflection matrix and, hence, they can each be recursively decomposed into eight submatrices.

Because of these properties of the hexahedral element (i.e. three levels of symmetric and antisymmetric decomposability), [ChSa87, Chen88, ChSa88] then show that if SAS ordering of the nodes is employed between subdomains then the system stiffness matrix K satisfies the relation

$$PKP = K, \tag{32}$$

where $P = P^T$ is a permutation matrix ($P^2 = I$) and that, therefore, depending on the number of planes of symmetry, the problem can be decomposed into as many as eight subproblems. In the case of the cantilever beam studied here, figure 4, having two axes of symmetry with the springs removed then leads to the following

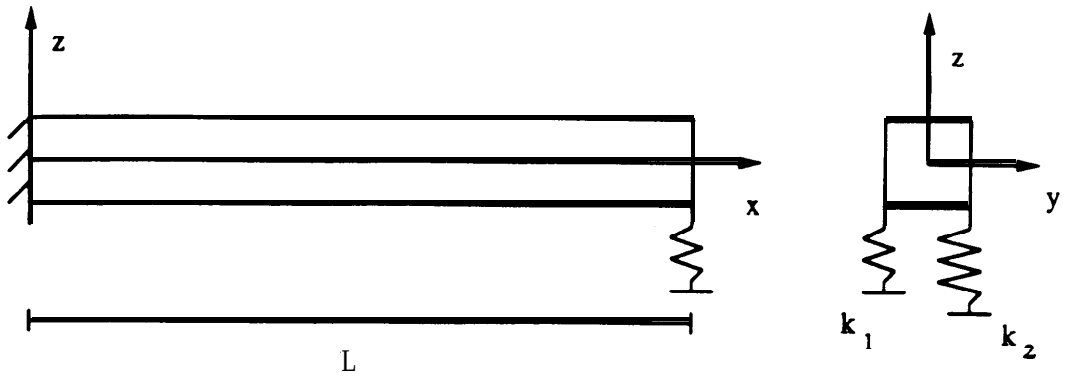

FIG. *4. A cantilever beam with additional spring supports.*

TABLE 4
CPU tames in seconds for the SAS and non-SAS approaches.

Grid Size	15 × 5 × 5		45 × 5 × 5		75 × 5 × 5		105 x 5 x 5	
no. of subdomains	PERL	LAN1	PERL	LAN1	PERL	LAN1	PERL	LAN1
1	24.41	142.05	66.35	364.75	113.91	611.60	207.22	1094.1
2	11.91	69.55	31.40	172.00	54.40	293.00	101.01	530.05
4	5.13	30.62	11.20	62.99	21.24	117.20	50.22	264.10

decomposition of K

$$Q^T K Q = diag(K_1, \ldots . K_4) \tag{33}$$

in which Q is an orthogonal matrix that can be easily constructed from the permutation submatrices that constitute P. The matrices K_i are independent of each other, are each of order $n/4$ and of a much smaller bandwidth.

When we recall that at each step in the block Lanczos algorithm we need to solve systems of the form

$$Kz = g, \tag{34}$$

it is evident that solving such systems is now simplified by solving four independent systems by virtue of (31).

For each discretization grid, identical numerical results were observed when the problem was solved via the SAS approach (subdomain > 1) as were observed when the problem was solved as a single domain without using decompositions - see table

3. The comparisons of CPU time for the SAS approach are given in table 4 for PERL and LAN1, from which it is observed that the SAS approach employed with PERL is much more efficient - this portion of savings in CPU time results mainly from the reduction of the bandwidth of the decomposed submatrices.

7. Summary. In this paper, we have presented a new approach for handling a series of slightly perturbed eigenvalue problems with symmetric matrices of the form $A(S_i) = A + BS_iB^T$, $0 \leq i \leq m$, by employing the block Lanczos algorithm with a special set of starting vectors. This approach confines the effect of the perturbations to the first diagonal block of the block tridiagonal matrix reduced by the block Lanczos algorithm so that the algorithm needs only be applied to the unperturbed matrix A, once and for all. Numerical experiments using a simple example from the vibration analysis of a cantilever beam with an additional support at its free end by a spring with various stiffness coefficients have also been performed to demonstrate this approach. For low-rank perturbations, this approach results in significant savings given that the unperturbed matrix A has been reduced to a block tridiagonai form using the starting vectors proposed in this paper.

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REFERENCES

[ArGo88] P. Arbenz and G.H. Golub, On *the spectral* decomposition *of* Hermitian *matrices* modified *by* low *rank perturbations* with applications, *SIAM J. Matrix Anal. Appl.*, 9(1), 40-58 (1988).

[BjGo77] Åkeböck and G.H. Golub, *Some general problems* for *matrices associated* with periodic *boundary* conditions, *SIAM Review*, 19(1), 5-16 (1977).

[ChSa87] H.C Chen and A. Sameh, Numerical *linear algebra algorithms on the Cedar system*, Parallel computations and their impact on mechanics (ed. A.K. Noor), The American Society of mechanical engineers, AMD-86, pp. 101-125, 1987.

[Chen88] H.C. Chen, *The SAS Domain Decomposition Method for Structural* Analysis, PhD Thesis, University of Illinois, 1988.

[ChSa88] H.C Chen and A. Sameh, *A matrix* decomposition *method for orthotropic elasticity problems*, CSRD report, no. 798, university of Illinois, 1988.

[Cook81] R.D. Cook, *Concepts* and *Applications of* Finite *Element Analysis*, 2nd Ed., John Wiley & Sons, New York, 1981.

[Dawe84] D.J. Dawe, *Matnx and Finite Element Displacement Analysis of Structures,* Clarendon Press, Oxford, 1984.

[GBDM77] B.S. Garbow, J.M. Boyle, J.J. Dongarra, C.B. Moler, Matrix *Eigensystem Routines - EISPACK Gurde Extension*, Springer-Verlag, New York, 1977.

[GoLo89] G.H. Golub and C.F. Van Loan, *Matnx Computatrons,* 2nd Ed., The Johns Hopkins University Press, Baltimore, Maryland, 1983.

[Golu73] G.H. Golub, *Some modified matnx eigenvalue problems, SIAM Review*, 15(2), 319-334 (1973).

[GoUn77] G.H. Golub and R. Underwood, *The block Lanczos method for Computing eigenvalues*, *Mathematical* Software III (J.R. Rice, Ed.), Academic Press, New York, 1977.

[HuRu64] W.C. Hurty and M.F. Rubinstein, *Dynamics* of *Structures,* Prentice-Hall, New Jersey, 1964.

[Lwis77] J. Lewis, *Algorithms for Sparse* Matnx *Eigenvalue Problems,* Technical Report, STAN-CS-77-595, Computer Science Dept., Stanford CA, 1977.

[Melo63] R.J. Melosh, *Structural Analysts of Solids*, *ASCE J. of Structural* Division, 89, 205-223 (1963).

[Paig74] C.C. Paige, *Eigenvalues of perturbed Hermitian matrices, Linear Algebra and Its Applications*, *8,* l-10 (1974).

[Parl80] B.N. Parlett, *The Symmetnc Eigenvalue Problem,* Prentice-Hall, New Jersey, 1980.

[Paz80] M. Paz, *Structural Dynamics: Theory and Computations,* Van Nostrand Reinhold, New York, 1980.

[Prze68] J.S. Przemieniecki, *Theory of Matrix Structural Analysis,* McGraw-Hill, New York, 1968.

[RaBB90] Y.M. Ram, J.J. Blech, and S.G. Braun, *Eigenproblem error bounds wrth application* to *symmetnc dynamic* system modification, *SIAM J.* Matnx Anal. *Appl.,* 11(4), *553-564* (1990).

[Ruhe79] A. Ruhe, *Implementation Aspects* of *Band Lanczos Algorithms for Computation* of *Eigenvalucs of Large Sparse Symmetric* Matrices, *Math. Comp., 33,* 680-87 (1979).

[Saad80] *Y.* Saad, *On the rates of convergence of the Lanczos and the block Lancros methods, SIAM J. Numer. Anal.,* 17(5), 687-706 (1980).

[Sctt79] D.S. Scott, *Block Lanczos Software* for *Symmetric Eigenvalue Problems, Report ORNL/CSD-48*, Oak Ridge National Laboratory, Oakridge, TN (1979).

[Thom76] R.C. Thompson, *The* behavior *of eigenvalues and singular values under perturbations of restricted rank, Linear Algebra and Its Applications*, 13, 69-78 (1976).

[Unde75] R.R. Underwood, *An iterative block Lancros method for the solution of large sparse symmetric eigen-problems*, Ph.D. dissertation, Stanford University, Stanford, California.

[Wilk65] J.H. Wilkinson, *The Algebmic Eigenvalue Problem,* Oxford University Press, London, 1965.