Introduction to Automata and Complexity Theory
Winter Quarter 2003

Homework 1 (01/08/03)
Due: Wednesday 01/22 in class

- Solving all of the easy problems can substitute for solving a single moderate problem.
- Each problem must be submitted on a separate sheet.
- Please write your name and SUnet ID at the beginning of each problem. We will use your SUnet ID to communicate grades to you.

Easy Problems

Problem 1 Prove by induction that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

State clearly the basis and inductive hypothesis. Any proof that does not use induction will not be given any credit.

Problem 2 List which of the following sets are closed under the following operations.

(a) The odd integers under multiplication

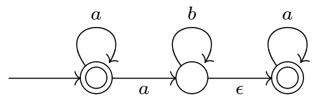
(b) The positive integers under division

(c) The negative integers under addition

(d) The negative integers under multiplication

(e) The odd integers under division

Problem 3



Given $\Sigma = \{a, b\}$, which of the following strings are accepted by the NFA above?

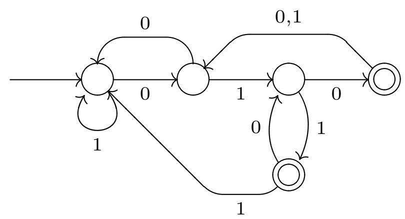
(a) $aaaaaa$

(b) $baaa$

(c) $aabaa$

(d) $aabbb$

Problem 4



Given $\Sigma = \{0, 1\}$, which of the following strings are accepted by the DFA above?

(a) 100010010

(b) 001010

(c) 011011010

(d) 00000010

Problem 5

(a) Construct a DFA recognizing

$$L = \{\, w \mid w \text{ contains “1001”, i.e. } w = x1001y \text{ for some } x,\, y \,\}$$

Just give a state diagram.

(b) Construct an NFA to do the same.

Moderate Problems

Problem 1

(a) Let M be a DFA such that $M = (Q, \Sigma, \delta, q_0, F)$, with $F = \{q_f\}$ and $\forall a \in \Sigma$, $\delta(q_f, a) = \delta(q_0, a)$. Let $w \in L(M)$. Prove that for all $k > 0$, $w^k \in L(M)$.

(b) Let $M = (Q, \Sigma, \delta, q_0, F)$ with $F = \{q_f\}$. Let $\forall w \in L(M)$, $w^k \in L(M)$ for all $k > 0$. Prove, or disprove, that $\forall a \in \Sigma$, $\delta(q_f, a) = \delta(q_0, a)$.

Problem 2 Let $C_n = \{\, x \mid x \text{ is a binary number such that } x \bmod n = 1 \,\}$.

(a) Build a DFA recognizing C_3.

(b) Show that $\forall n$, C_n is regular.

Problem 3 Let

$$\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \cdots \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Σ_3 contains all size-3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row as the *reverse* of a binary number.

Example:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

corresponds to 011 (top row), 110 (middle row), and 010 (bottom row).

Let $B = \{\, w \mid \text{bottom number of } w \text{ is sum of the top and middle numbers} \,\}$. Show that B is regular.

Hint: To verify a binary sum is correct, you just need to remember one carry bit. Can you build a DFA that will "remember" this carry bit and verify the sum?

Problem 4 Show that the regular languages are closed under the following operations:

(a) $\min(L) = \{\, w \mid w \text{ is in } L, \text{ but no proper prefix of } w \text{ is in } L \,\}$

(A *prefix* of w is x such that there is y where $w = xy$. A *proper prefix* of w is a prefix which is not w, i.e. $y \neq \epsilon$. E.g. 110 and 11 are proper prefixes of 1101, while 101 and 1101 are not.)

(b) $\max(L) = \{\, w \mid w \text{ is in } L \text{ and for no } x, \text{ other than } \epsilon, \text{ is } wx \text{ in } L \,\}$

Problem 5 Prove that the following languages are not regular. ($\Sigma = \{0, 1\}$)

(a) $\{ w\overline{w} | \, w \in \Sigma^* \}$

where $\overline{w}$ means w with each occurrence of 0 replaced by 1, and vice versa

(b) $\{ 0^i 1^j 0^k 1^k \mid i, j, k \geq 0 \}$

Extra Credit Problem

(a) Let x and y be strings and let L be any language. We say that x and y are *distinguishable by* L if some string z exists whereby exactly one of the strings xz and yz are members of L; otherwise, for every string z, $xz \in L$ whenever $yz \in L$ and we say that x and y are *indistinguishable by* L. If x and y are indistinguishable by L we write $x \equiv_L y$. Show that $\equiv_L$ is an equivalence relation.

An *equivalence relation* is a relation which is reflexive, symmetric, and transitive. A relation R is *reflexive* if for every element x, $R(x, x)$ is true (e.g. $x = x$). A relation R is *symmetric* if whenever $R(x, y)$ is true, then $R(y, x)$ is also true (e.g. if $x = y$, then $y = x$). A relation R is *transitive* if whenever $R(x, y)$ and $R(y, z)$ are true, then $R(x, z)$ is true (e.g. if $x = y$ and $y = z$, then $x = z$).

(b) Let L be a language and let X be a set of strings. Say that X is *pairwise distinguishable by* L if every two distinct strings in X are distinguishable by L. Define the *index of* L to be the maximum number of elements in any set that is pairwise distinguishable by L. The index of L may be finite or infinite. Show that if L is recognized by a DFA with k states, L has index at most k.

(c) Let L be a language. Show that if the index of L is a finite number k, it is recognized by a DFA with k states.

(d) From (b) and (c), conclude that L is regular iff it has finite index. Moreover, its index is the size of the smallest DFA recognizing it.