cs154 Homework #5, Winter 2002-2003

Due date: Monday, Mar 3, 2003, in class.

Please write your **Leland login ID** for each problem you submit.

Easy Problems

(A) (Sipser 5.3) Find a match in the following instance of the PCP: $\{[\frac{ab}{abab}], [\frac{b}{a}], [\frac{aba}{b}], [\frac{aa}{a}]\}$

(B) (Sipser 7.1) Answer each part TRUE or FALSE:

(a) $2n = 0(n)$ (b) $n^2 = O(n)$ (c) $n^2 = O(n \log^2 n)$

(d) $n \log n = O(n^2)$ (e) $3^n = 2^{O(n)}$ (f) $2^{2^n} = O(2^{2^n})$

(C) (Sipser 7.5) Is the following formula satisfiable? $(x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{x} \vee y) \wedge (\overline{x} \vee \overline{y})$

(D) (Sipser 7.6) Show that P is closed under union, concatenation and complement.

(E) (Sipser 7.10) A **triangle** in an undirected graph is a 3-clique. Show that $TRIANGLE \in P$, where $TRIANGLE = \{\langle G \rangle \mid G \text{ contains a triangle } \}$.

Problem 1 (Sipser 5.17) Show that PCP is decidable over a unary alphabet, i.e., over $\Sigma = \{1\}$.

Problem 2 (Sipser 5.21) **A two dimensional finite automaton** (2DIM-DFA) is defined as follows. The input is an $m \times n$ rectangle, for any $m, n \geq 2$. The squares along the boundary of the rectangle contain the symbol # and the internal squares contain symbols over the input alphabet Σ. The transition function is a mapping $Q \times \Sigma \rightarrow Q \times \{L, R, U, D\}$ to indicate the next state and the new head position (Left, Right, Up, Down). The machine accepts when it enters one of the designated accept states. It rejects if it tries to move off the input rectangle or if it never halts. Two such machines are equivalent if they accept the same rectangles. Consider the problem of testing whether two of these machines are equivalent. Formulate this problem as a language, and show that it is undecidable.

Problem 3 (Sipser 7.12) Let $MODEXP = \{\langle a, b, c, p \rangle \mid a, b, c$ and p are binary integers such that $a^b \equiv c \pmod{p}\}$. Show that $MODEXP \in P$. Note that the most obvious algorithm does not run in polynomial time. *Hint: Try it first where b is a power of 2.*

Problem 4 (Sipser 7.13) Show that P is closed under the star operation. *Hint: On input $y = y_1 \cdots y_n$ for $y_i \in \Sigma$, build a table indicating for each $i \leq j$ whether the substring $y_i \cdots y_j \in A^*$ for any $A \in P$.*

Problem 5 (Modified Sipser 7.15) Let $UNARY - SSUM$ be the following problem: The input consists of a set of k numbers $x_1, x_2, \ldots, x_k$, alongwith with a target number t. All numbers are represented in unary. The input is accepted iff there exists a subset of the k numbers whose members add up to exactly t. Show that $UNARY - SSUM \in P$.

Extra Credit (Cat and Mouse Game) A two-player game is played on a grid of $N \times N$ squares that resembles a maze. Each square has at most four neighbours: in the North, South, East and West directions. Two neighbours are either *connected* or there is a *wall* separating them. At the beginning of the game, a *cat*, a *mouse* and a *piece of cheese* are placed in three different squares. Further, one of the corner squares is a *hole* where the mouse can hide.

The game is played as follows: Mouse moves first, then cat, then mouse, and so on. The cat wins if it catches the mouse, that is, if it lands on the same square as the mouse. The mouse wins if it (a) takes the cheese, and then (b) reaches its hiding place. If at any point the mouse reaches the hole without first getting the cheese, or if the position of cat and mouse repeats, the game is a draw.

An instance of the cat and mouse game G consists of the following: (a) description of the maze (whether two neighbours are connected or separated by a wall), (b) the position of the hole and (c) the initial positions of cat, mouse and cheese. If each player has unlimited time to think, then we can predict the outcome of the game from the beginning. Consider the following languages:

$L_{win} = \{G \mid G$ is a $N \times N$ cat and mouse game and the cat wins $\}$

$L_{draw} = \{G \mid G$ is a $N \times N$ cat and mouse game and the result is a draw $\}$

$L_{lose} = \{G \mid G$ is a $N \times N$ cat and mouse game and the mouse wins $\}$

Prove that all three languages are in P.