Background: Functional Dependencies

- We are always talking about a relation R, with a fixed schema (set of attributes) and a varying *instance* (set of tuples).
- Conventions: $A, B, \ldots$ are attributes; $\ldots, Y, Z$ are sets of attributes. Concatenation means union.
- FD is $X \rightarrow Y$, where X and Y are sets of attributes. Two tuples that agree in all attributes of X must agree in all attributes of Y.
- Implication: FD $X \rightarrow Y$ *follows from* $\mathcal{F}$ iff all relation instances that satisfy $\mathcal{F}$ also satisfy $X \rightarrow Y$.

Three Ways to Reason About FD's

1. *Semantic*: FD $X \rightarrow Y$ is the set of relation instances that satisfy it.
   - Say $\mathcal{F} \models X \rightarrow Y$ if every instance that satisfies all FD's in $\mathcal{F}$ also satisfy $X \rightarrow Y$.
   - All approaches assume there is a fixed relation scheme R to which the FD's pertain.
2. *Algorithmic*: Give an algorithm that tells us, given $\mathcal{F}$ and $X \rightarrow Y$, whether $\mathcal{F} \models X \rightarrow Y$.
3. *Logical*: Give a reasoning system that lets us deduce an FD like $X \rightarrow Y$ exactly when $\mathcal{F} \models X \rightarrow Y$.
   - Deduction indicated by $\mathcal{F} \vdash X \rightarrow Y$.

Closure Test for Implication (Algorithmic Approach)

Start with $X^+ = X$. Adjoin V to X^+ if $U \rightarrow V$ is in $\mathcal{F}$, and $U \subseteq X^+$. At end, test if $Y \subseteq X^+$.

- Proof that if $Y \subseteq X^+$, then $\mathcal{F} \models X \rightarrow Y$: Easy induction on the number of additions to X^+ that $X \rightarrow A$ for all A in X^+.
- Proof that if $\mathcal{F}$ implies $X \rightarrow Y$, then $Y \subseteq X^+$: Prove the contrapositive; assume Y is not a subset of X^+ and prove $\mathcal{F}$ does not imply $X \rightarrow Y$. Construct relation R with two tuples that agree on X^+ and disagree elsewhere.

Armstrong's Axioms (Logical Approach)

A *sound* (what may be deduced is correct in the $\models$ sense) and *complete* (what is true in the $\models$ sense can be deduced) axiomatization of FD's.

A1: *Trivial FD's* or *Reflexivity*. $X \rightarrow Y$ always holds if $Y \subseteq X$.

A2: *Augmentation*. If $X \rightarrow Y$, then $XZ \rightarrow YZ$ for any set of attributes Z.

A3: *Transitivity*. If $X \rightarrow Y$ and $Y \rightarrow Z$, then $X \rightarrow Z$.

Deductive Proofs

A series of "lines." Each line is either:

1. A given statement (FD in the given set $\mathcal{F}$ for deductions about FD's), or
2. A statement that follows from previous lines by applying an axiom.

Example

Given $\{AB \rightarrow C,\ CD \rightarrow E\}$, deduce $ABD \rightarrow E$.

$AB \rightarrow C$ (Given)
$ABD \rightarrow CD$ (A2)
$CD \rightarrow E$ (Given)
$ABD \rightarrow E$ (A3)

Proof of Soundness

Easy observations about relations.

Proof of Completeness

1. Given $\mathcal{F}$, show that if A is in X^+, then $X \rightarrow A$ follows from $\mathcal{F}$ by AA's.
2. Show that if $X \rightarrow A_1, \ldots, X \rightarrow A_n$ follow from AA's, then so does $X \rightarrow A_1 \cdots A_n$.
3. Complete the proof by observing that $X \rightarrow A_1 \cdots A_n$ follows from $\mathcal{F}$ iff all of $A_1, \ldots, A_n$ are in X^+, and therefore iff $X \rightarrow A_1 \cdots A_n$ follows by AA's.

Proof (2)

Induction on i that $X \rightarrow A_1 \cdots A_i$ follows.

- Basis: $i = 1$, given.

- Induction: Assume $X \rightarrow A_1 \cdots A_{i-1}$.
  - $X \rightarrow XA_1 \cdots A_{i-1}$ (A2 by X applied to $X \rightarrow A_1 \cdots A_{i-1}$).
  - $XA_1 \cdots A_{i-1} \rightarrow A_i$ (A2 by $A_1 \cdots A_{i-1}$ applied to $X \rightarrow A_i$).
  - $X \rightarrow A_1 \cdots A_i$ (A3 applied to previous two FD's).

Proof (1)

Induction on the number of steps used to add A to X^+.

- Basis: 0 steps. Then A is in X, and $X \rightarrow A$ by A1.
- Induction: Assume $B_1 \cdots B_k \rightarrow A$ in $\mathcal{F}$ used to add A to X^+.
  - By the inductive hypothesis, $X \rightarrow B_j$ follows from $\mathcal{F}$ for all $1 \leq j \leq k$.
  - By part (2) of the theorem, $X \rightarrow B_1 \cdots B_k$ follows from $\mathcal{F}$.
  - By A3, $X \rightarrow A$ follows.

Background: Normal Forms

- A *key* for R is a minimal set of attributes such that $X \rightarrow R$ (note: I use R as both the instance and schema of a relation — shame on me). A *superkey* is any superset of a key.
- BCNF: If $X \rightarrow Y$ holds for R and is nontrivial, then X is a superkey.
- 3NF: If $X \rightarrow Y$ holds for R and is nontrivial, then either X is a superkey or Y contains a *prime* attribute (member of some key).
- *Decomposition*: We can decompose R into schemas $S_1, \ldots, S_n$ if $S_1 \cup \cdots \cup S_n = R$. The instance for S_i is $\pi_{S_i}(R)$. The FD's that hold for S_i are those $X \rightarrow Y$ such that $XY \subseteq S_i$ and $X \rightarrow Y$ follows from the given FD's for R.

Covers of Sets of FD's

- Sets $\mathcal{F}_1$ and $\mathcal{F}_2$ of FD's are *equivalent* if each implies the other.
  - I.e., exactly the same relation instances satisfy each.

- Any set of FD's equivalent to $\mathcal{F}$ is a *cover* for $\mathcal{F}$.
- A cover is *minimal* if:

1. No right side has more than one attribute.
2. We cannot delete any FD from the cover and have an equivalent set of FD's.
3. We cannot delete any attribute from any left side and have an equivalent set of FD's.

Example

Relation $CTHRSG$ represents courses, teachers, hours, rooms, students, and grades. The FD's: $C \to T$; $HR \to C$; $HT \to R$; $HS \to R$; $CS \to G$; $CH \to R$.

- We can eliminate $CH \to R$.
    - Proof: Using the other 5 FD's, $CH^+ = CHTR$.
- Having done so, we cannot eliminate any attribute from any left side.
    - Sample proof: Suppose we tried to eliminate T from $HT \to R$. We would need that $C \to T$, $HT \to R$, $HR \to C$, $HS \to R$, and $CS \to G$ imply $H \to R$. But $H^+ = H$ with respect to these 5 FD's.
- Thus the remaining five are a minimal cover of the original six.
- Note: minimal cover need not be unique, or even have the same number of FD's.

Lossless Join

The decomposition of R into $S_1, \ldots, S_n$ has a *lossless join* (with respect to some constraints on R) if for any instance r of R that satisfies the constraints:

$$\pi_{S_1}(r) \bowtie \cdots \bowtie \pi_{S_n}(r) = r$$

- Motivation: We can replace R by S and T, knowing that the instance of R can be recovered from the instances of S and T.

Theorem

A decomposition of R into S and T has a lossless join wrt FD's $\mathcal{F}$ if and only if $S \cap T \to S$ or $S \cap T \to T$.

Proof

- First note that $r \subseteq \pi_S(r) \bowtie \pi_T(r)$ always.
- Assume $S \cap T \to S$ and show that

$$\pi_S(r) \bowtie \pi_T(r) \subseteq r$$

  - Suppose s is in $\pi_S(r)$ and t is in $\pi_T(r)$, and s and t join (i.e., they agree in $S \cap T$).
  - Then t is the projection of a tuple t' of r that agrees with s on S.
  - So t' agrees with t on T and with s on S, so t' is $s \bowtie t$; i.e., $s \bowtie t$ is in r.
- Similarly if $S \cap T \to T$.

- Now assume neither FD follows from $\mathcal{F}$.
  - Then there is an instance r consisting of two tuples that agree on $(S \cap T)^+$ and disagree elsewhere. This instance satisfies $\mathcal{F}$.
  - Neither S nor T is contained in $(S \cap T)^+$ (or else one of the FD's in question would follow).
  - Thus, r projected and rejoined yields four distinct tuples, and cannot be r.

Theorem

We can always decompose a relation R with FD's $\mathcal{F}$ into BCNF relations with a lossless join.

Proof

- We decompose when we find a BCNF violation $X \to Y$, into $X \cup Y$ and $(R - Y) \cup X$.
- But $\big((R - Y) \cup X\big) \cap (X \cup Y) = X$. Thus the intersection of the schemas functionally determines one of them, $X \cup Y$.
- To complete the proof, we need to show that when we decompose further, the resulting n relations have a lossless join, but that is an easy induction on n.

Dependency Preservation

When we decompose R with FD's $\mathcal{F}$, will $\mathcal{F}$ be equivalent to the union of its projections onto the decomposed relations?

- One way to guarantee dependency preservation is to use a minimal cover, and convert each FD in the cover $X \to A$ into the schema XA.
  - But if there are some attributes not mentioned in any FD, make them a schema by themselves.

Theorem

A minimal cover $\mathcal{F}$ yields 3NF relations.

Proof

Suppose XA (the relation from $X \to A$) is not in 3NF, because $Y \to B$ is a 3NF violation.

- We know Y is not a superkey, and B is not prime.
- Case 1: $A = B$. Then $Y \subset X$. Since $Y \to B$ follows from $\mathcal{F}$, and $X \to A$ surely follows from $Y \to B$, we know $\mathcal{F}$ is equivalent to $\mathcal{F} - \{X \to A\} \cup \{Y \to B\}$.
  - Then $\mathcal{F}$ was not a minimal cover.
- Case 2: $A \neq B$. Then B is in X.
  - X is surely a superkey for XA, and since B is not prime, B must not be in any key $Z \subseteq X$.
  - Then $(X - B) \to A$ can replace $X \to A$ in $\mathcal{F}$, showing $\mathcal{F}$ is again not minimal.

Decomposition With 3NF, Dependency Preservation and Lossless Join

To the schema from a minimal cover, add a key for the original relation if there is not already some relation schema that is a superkey.

Example

$R = ABCD$; $\mathcal{F} = \{A \to B, C \to D\}$.

- Start with AB and CD from the FD's.
- Only key for R is AC.
- Thus, DP, LJ, 3NF decomposition is $\{AB, AC, CD\}$.

- Proof that the decomposition is always LJ will have to wait for the theory of "generalized dependencies."

  ❖ This decomposition is obviously LJ, since $AB \bowtie AC$ is lossless because $A \rightarrow B$, and then $ABC \bowtie CD$ is lossless because of $C \rightarrow D$.