CS109A Notes for Lecture 2/14–16/96

Orders With Some Equivalent Items

Suppose we have a Scrabble rank of 7 tiles, which are $STAAEEE$. How many different 7-letter words could we make?

- ☐ As before, we're not concerned with whether the word is "legal."

- Similar to permutations of 7 items, but now, some orderings are indistinguishable because the letters are the same.

- Make the letters distinguishable by marking, e.g., subscripts on the A's and E's: $STA_1A_2E_1E_2E_3$.

- We can order these distinguished letters in $7! = 5040$ ways.

- But orders that differ by exchanging A_1 and A_2 are really the same.

  - ☐ e.g.: $E_3TA_1E_1SA_2E_2 = E_3TA_2E_1SA_1E_2$.

- The two A's can be ordered in $2! = 2$ ways; if we pick one of the two orders, say A_2 must follow A_1, then we eliminate $1/2$ of all the orders.

  - ☐ But we keep exactly one of the two equivalent orders that differ only in the order of A_1 and A_2.

- Similarly, the E's can appear in $3! = 6$ ways. If we select only the orders of the 7 letters with this order of the E's, we eliminate 5 of every group of 6 orders that differ only in the order of their E's.

- Conclusion, the number of different words is $7!/(2!3!) = 5040/(2 \times 6) = 420$.

- **General rule**: The orders of n items with groups of $i_1, i_2, \ldots, i_k$ equivalent elements is $n!/(i_1!i_2! \cdots i_k!)$.

Items to Bins

Suppose we throw 7 identical, 6-sided dice. How many outcomes are there?

- Equivalent problem: place each of seven identical tokens into one of 6 bins.
  - □ The tokens are the dice, and the bins are the numbers on the dice; e.g., putting the second token into bin 3 means that the second die showed 3.
- Trick: Imagine 5 markers, denoted `*`, that represent the separations between bins and 7 tokens `T` that represent the dice.
  - □ The outcomes correspond to the orderings of the 12 items, e.g. `*TT**TTT*T*T` corresponds to no 1's, two 2's, no 3's, three 4's, a 5 and a 6.
  - □ This is an "orders with identical items" problem; the count is $12!/(5!7!) = 792$.
- **General Rule**: The number of ways to assign n items to m bins = the number of orders of $m-1$ markers and n tokens = $(n+m-1)!/\big((m-1)!n!\big) = \binom{n+m-1}{n}$.
  - □ The combinations expression can also be explained by saying: "we are picking n out of the possible $n+m-1$ positions for the tokens."

Several Kinds of Items, Order Within Bin Irrelevant

- *Warning*: the section of FCS starting on p. 182 ("Distributing Distinguishable Objects") is deceptive at best. It cares about orders within bins, so, e.g., in Example 4.17 Kathy receiving an apple then a pear is not the same as her receiving a pear then an apple.
- If we have items of k colors, with i_j items of the jth color, then the number of distinguishable assignments to m bins is:

$\prod_{j=1}^{k} \binom{m+i_j-1}{i_j}$.

Example: 4 red dice, 3 blue dice, and 2 green dice.

- □ Therefore 6 bins.

- Number of distinct tosses is $\binom{9}{4}\binom{8}{3}\binom{7}{2} =$ 148,176.

Several Kinds of Items, Order Within Bin Relevant

- This problem uses the formula on p. 182, l. −6.

- Trick: If there are m bins into which i_j items of the jth type are placed, and order within bins matters, imagine $m-1$ markers separating the bins and i_j tokens of the jth type representing that type of item. By "orders with identical items":

  $$(n+m-1)!/(m-1)!\prod_{j=1}^{k} i_j!$$

  distinct placements.

  - □ Here k = number of item types; n = total number of items.

Example: Suppose we have 4 red balls, 3 blue balls, and 2 green balls. In random order, we throw them into 6 holes, each the diameter of the balls.

- Number of outcomes $= (9+6-1)!/(5!4!3!2!)$ $= 2{,}522{,}520$.

Counting Trick #1

- Describe the desired objects by a sequence of steps in which a choice is made from some number of options.

  - □ Answer = product of the numbers of options.

Example: The California lottery has announced that every ticket with 3 out of 6 numbers correct will win a "chance to win a" trip to Mexico. What are the odds of selecting exactly 3 numbers (from

1 to 53) correct out of 6?

- Basic probability technique: count the total number of possibilities, count the number of successful possibilities, and take their quotient.
  - □ Here, "possibility" = sequence of 6 numbers (1 . . 53) on a card.
- Total number of possibilities = $\Pi(53, 6)$ = $53!/47!$ = 16,529,385,600.
- 3-right-out-of-6 possibilities described by:
  - a) Select the three positions for the right guesses.
  - b) Select the 3 of the 6 numbers drawn in the lottery that appear there, in order.
  - c) Select the 3 out of 47 wrong numbers that appear in the other 3 positions.
- (a) = $\binom{6}{3}$ = 20; (b) = $\Pi(6, 3)$ = 120; (c) = $\Pi(47, 3)$ = 97,290.
- Number of successful outcomes = $20 \times 120 \times 97290$ = 233,496,000.
- Probability of winning =

  $$233{,}496{,}000/16{,}529{,}385{,}600 = 1.4\%$$
- An equivalent solution recognizes that the order of selected numbers is irrelevant and uses combinations instead of ordered selections.

Counting Trick #2

- Describe the desired objects as the union of several disjoint sets (sets without elements in common).
  - □ Answer is sum of counts for the disjoint sets.

Example: Continuing previous "lottery" example: you win if you get 3 or more correct. How many winning picks are there?

- We counted the number of selections in which you get exactly 3 right.
- Similar calculations give us the number of selections with exactly 4, exactly 5, and exactly 6.
  - □ Each is much easier than trying to count 3–6 correct as a single problem.
- Answer is the sum of the 4 simple calculations.

Counting Trick #3

- Describe the desired objects as the set difference of two sets.
  - □ Answer is the difference of the counts.

Example: Suppose we deal 3 cards. What is the probability that two are of one suit and the third is of another?

- To start, the probability is the ratio of the number of deals meeting the condition and the total number of deals.
- Selections of 3 cards, in order = $\Pi(52, 3) =$ 132,600.
- Count the number of "2-and-1" deals as the difference between 132,600 and the sum of the number of "all-3-the-same" deals and the "all-3-different" deals.
  - □ Each of the latter is a use of trick #1 (product).
- For "all-3-different," describe by:
  - a) Pick the first card (52 choices).
  - b) Pick the second from the other 3 suits (39 choices).
  - c) Pick the third from the remaining 2 suits (26 choices).
  - □ Number of "all-3-different" deals = $52 \times 39 \times 26 =$ 52,728.

- “All-3-the-same” deals described by:
  - a) Pick the suit (4 choices).
  - b) Pick the 3 cards of that suit, in order ($\Pi(13, 3)$ choices).
  - □ Number of “all-3-the-same” deals $= 4 \times 13 \times 12 \times 11 =$ 6,864.
- Number of “2-and-1” deals = 132,600 — 52,768 — 6,864 = 73,008.
  - □ i.e., 73,008/132,600 or 55% of the time you get two of one suit, one of another.
- Yes; one could solve this particular problem directly by trick #1, making a sequence of 4 choices (the two suits and the cards of each suit).