Probability Space

Set of points, each with an attached probability (nonnegative, real number), such that the sum of the probabilities is 1.

- We simplify in two ways:
  a) Number of points is finite, \( n \).
  b) Probability of each point is the same, \( 1/n \).

Experiments

An experiment is the selection of a point in a probability space.

- Under our "equally likely" assumption, all points have the same chance of being chosen.

Events

An event is any set of points in a probability space.

- \( \text{PROB}(E) \), the probability of an event \( E \) is the fraction of the points in \( E \).

Example: The event \( E_4 = \) a 4 is dealt. \( \text{PROB}(E_4) = 4/52 = 1/13 \).

The event \( E_\heartsuit = \) a heart is dealt. \( \text{PROB}(E_\heartsuit) = 13/52 = 1/4 \).

- Generally, computing the probability of an event \( E \) involves two counts: the entire probability space and the number of points in \( E \).

Conditional Probability

Given that the outcome of an experiment is known to be in some event \( E \), what is the probability that the outcome is also in some other event \( F \)?

- Known as the conditional probability of \( F \) given \( E \), or \( \text{PROB}(F/E) \).
- = the fraction of the points in \( E \) that are also in \( F \).
Example:

- \( E = \) “a number card is selected” = 36 points corresponding to ranks 2–10.
- \( F = \) “the card is less than 7” = 24 points corresponding to ranks 2–6.
- Of the 36 points in \( E \), 20 are also in \( F \) (those for ranks 2–6).
- Thus, \( \text{PROB}(F|E) = \frac{20}{36} = \frac{5}{9} \).

Independent Events

\( F \) is independent of \( E \) if \( \text{PROB}(F|E) = \text{PROB}(F) \).
- Intuitively, \( F \) does not depend on whether \( E \) occurs.

Example: (Cards) \( E = \) “suit is hearts.” \( F = \) “rank is 5.”
- \( \text{PROB}(F|E) = \frac{1}{13} \) while \( \text{PROB}(F) = \frac{4}{52} = \frac{1}{13} \).

Independence Goes Both Ways

If \( F \) is independent of \( E \) then \( E \) is independent of \( F \). Consider:

\[
\text{PROB}(F|E) = \frac{n_3}{n_3 + n_4} = \text{PROB}(F) = \frac{n_2 + n_3}{n_1 + n_2 + n_3 + n_4}
\]

- Swap left-denominator with right-numerator — preserves truth of the equality.

\[
\frac{n_3}{n_2 + n_3} = \frac{n_3 + n_4}{n_1 + n_2 + n_3 + n_4}
\]

- Left is \( \text{PROB}(E|F) \); right is \( \text{PROB}(E) \) — shows \( E \) independent of \( F \).

Example: Continuing cards example above, \( \text{PROB}(E|F) = \frac{1}{4} \); \( \text{PROB}(E) = \frac{13}{52} = \frac{1}{4} \).
**Complement Events**

If $E$ is an event, $\bar{E}$ is the event "$E$ does not occur."

- $\text{PROB}(\bar{E}) = 1 - \text{PROB}(E)$.
- If $F$ is independent of $E$, then $F$ is independent of $\bar{E}$.

**Expected Value**

- $f$ is some function of points in a probability space.
- $\text{EV}(f) =$ average over points $p$ of $f(p)$.

**Example:** Space = cards. $f =$ Blackjack value (Ace = 11; pictures = 10).
- $\text{ev}(f) = (4 \times 11 + 4 \times 2 + 4 \times 3 + \cdots + 4 \times 9 + 16 \times 10)/52 = 7.31$.

**Randomized Algorithms**

Instead of an exact answer by a slow algorithm, we may be able to get a "close" answer fast by one that takes random "guesses."

**Example:** Video data compression uses a technique (MPEG) involving matching sections of one frame to sections of the previous frame.
- e.g., pieces of a moving car match themselves displaced somewhat from previous frame.
- In our gross simplification, imagine that "sections" are $n \times n$ squares of black-or-white pixels, and we only ask whether this square matches exactly any of the $n \times n$ squares of the previous frame.
- To test exactly requires comparing $n^2$ pixels, an $O(n^2)$ process.

**A Randomized Algorithm for Matching**

Suppose we pick corresponding pixels in the squares to compare at random. What is the expected number of comparisons needed before we discover that two unrelated squares are different.
• Assumption: if squares are unrelated, probability is $1/2$ that any corresponding pixels match.

☐ Dubious: think of a cloudless sky with a small plane.

• If so, half the time we need only one comparison, half of the remaining half we need two, and so on.

• Let $f =$ number of comparisons made. $\text{EV}(f) = 1 \times (1/2) + 2 \times (1/4) + 3 \times (1/8) + \cdots$

$= \sum_{i=1}^{\infty} i2^{-i} = \sum_{i=0}^{\infty} 2^{-i}$ (tricky “triangular” argument explained in class) $= 2$.

☐ Note formula for expected value involves computing fraction of points in probability space leading to each value of $f$; e.g., half have $f(p) = 1$.

• Thus, average running time is $O(1)$, vs. $O(n^2)$ for the exact algorithm.

**Even More Randomness**

We can devise a “Monte-Carlo” algorithm that doesn’t always give the correct answer (it may say “they match” when they don’t), but takes $O(1)$ time rather than $O(n^2)$ even for correct matches.

• Key idea: Make no more than 20 tests. If they all succeed, say the squares match.

• Probability of all 20 matching even though the squares are unrelated $= 2^{-20}$ or about one in a million.