CS109A Notes for Lecture 2/23/96

Probability Space

Set of points, each with an attached *probability* (nonnegative, real number), such that the sum of the probabilities is 1.

- We simplify in two ways:
  a) Number of points is finite, n.
  b) Probability of each point is the same, $1/n$.

Experiments

An *experiment* is the selection of a point in a probability space.

- Under our "equally likely" assumption, all points have the same chance of being chosen.

Events

An *event* is any set of points in a probability space.

- $\text{PROB}(E)$, the *probability of an event* E is the fraction of the points in E.

Example: The event E_4 = a 4 is dealt. $\text{PROB}(E_4) = 4/52 = 1/13$.

The event $E_\heartsuit$ = a heart is dealt. $\text{PROB}(E_\heartsuit) = 13/52 = 1/4$.

- Generally, computing the probability of an event E involves two counts: the entire probability space and the number of points in E.

Conditional Probability

Given that the outcome of an experiment is known to be in some event E, what is the probability that the outcome is also in some other event F?

- Known as the conditional probability of F given E, or $\text{PROB}(F/E)$.
- = the fraction of the points in E that are also in F.

Example:

- E = "a number card is selected" = 36 points corresponding to ranks 2–10.
- F = "the card is less than 7" = 24 points corresponding to ranks 2–6.
- Of the 36 points in E, 20 are also in F (those for ranks 2–6).
- Thus, $\text{PROB}(F/E) = 20/36 = 5/9$.

Independent Events

F is independent of E if $\text{PROB}(F/E) = \text{PROB}(F)$.

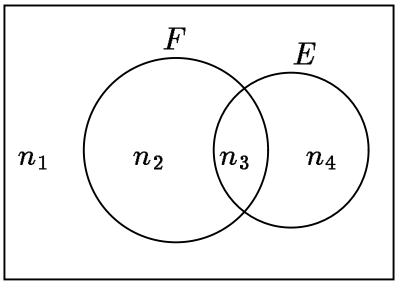
- Intuitively, F does not depend on whether E occurs.

Example: (Cards) E = "suit is hearts." F = "rank is 5."

- $\text{PROB}(F/E) = 1/13$ while $\text{PROB}(F) = 4/52 = 1/13$.

Independence Goes Both Ways

If F is independent of E then E is independent of F. Consider:



$$\text{PROB}(F/E) = \frac{n_3}{n_3 + n_4} = \text{PROB}(F) = \frac{n_2 + n_3}{n_1 + n_2 + n_3 + n_4}$$

- Swap left-denominator with right-numerator — preserves truth of the equality.

$$\frac{n_3}{n_2 + n_3} = \frac{n_3 + n_4}{n_1 + n_2 + n_3 + n_4}$$

- Left is $\text{PROB}(E/F)$; right is $\text{PROB}(E)$ — shows E independent of F.

Example: Continuing cards example above, $\text{PROB}(E/F) = 1/4$; $\text{PROB}(E) = 13/52 = 1/4$.

Complement Events

If E is an event, $\bar{E}$ is the event "E does not occur."

- $\text{PROB}(\bar{E}) = 1 - \text{PROB}(E)$.
- If F is independent of E, then F is independent of $\bar{E}$.

Expected Value

- f is some function of points in a probability space.
- $\text{EV}(f)$ = average over points p of $f(p)$.

Example: Space = cards. f = Blackjack value (Ace = 11; pictures = 10).

- $ev(f) = (4 \times 11 + 4 \times 2 + 4 \times 3 + \cdots + 4 \times 9 + 16 \times 10)/52 = 7.31$.

Randomized Algorithms

Instead of an exact answer by a slow algorithm, we may be able to get a "close" answer fast by one that takes random "guesses."

Example: Video data compression uses a technique (MPEG) involving matching sections of one frame to sections of the previous frame.

- e.g., pieces of a moving car match themselves displaced somewhat from previous frame.
- In our gross simplification, imagine that "sections" are $n \times n$ squares of black-or-white pixels, and we only ask whether this square matches exactly any of the $n \times n$ squares of the previous frame.
- To test exactly requires comparing n^2 pixels, an $O(n^2)$ process.

A Randomized Algorithm for Matching

Suppose we pick corresponding pixels in the squares to compare at random. What is the expected number of comparisons needed before we discover that two unrelated squares are different.

- Assumption: if squares are unrelated, probability is $1/2$ that any corresponding pixels match.
  - □ Dubious: think of a cloudless sky with a small plane.
- If so, half the time we need only one comparison, half of the remaining half we need two, and so on.
- Let f = number of comparisons made. $\mathrm{EV}(f)$ $= 1 \times (1/2) + 2 \times (1/4) + 3 \times (1/8) + \cdots$ $= \sum_{i=1}^{\infty} i2^{-i} = \sum_{i=0}^{\infty} 2^{-i}$ (tricky "triangular" argument explained in class) $= 2$.
  - □ Note formula for expected value involves computing fraction of points in probability space leading to each value of f; e.g., half have $f(p) = 1$.
- Thus, average running time is $O(1)$, vs. $O(n^2)$ for the exact algorithm.

Even More Randomness

We can devise a "Monte-Carlo" algorithm that doesn't always give the correct answer (it may say "they match" when they don't), but takes $O(1)$ time rather than $O(n^2)$ even for correct matches.

- Key idea: Make no more than 20 tests. If they all succeed, say the squares match.
- Probability of all 20 matching even though the squares are unrelated $= 2^{-20}$ or about one in a million.