Why Study Infinite Sets?

- Occasionally useful — sometimes in CS you reason about infinite sequences of events or other infinite things.
- Intellectually challenging.
- Fun and interesting.
- Something you’re expected to know.

Counting and Cardinality

- The cardinality of a set is the number of elements in that set.
- Two sets are equipotent if and only if they have the same cardinality.
- The existence of a one-to-one correspondence between two sets proves that they are equipotent.
- Counting is really just creating a one-to-one correspondence between a set and the set of integers from 1 to some number $n$.

Example

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Finite and Infinite Sets

- Can you create a one-to-one correspondence between a set and a proper subset of itself? If so, you have a solution to the equation $x = x + y$, where $x$ is the cardinality of the set and $y \geq 1$ is the cardinality of the stuff you left out.
• No finite \( x \) can satisfy that equation, but an “infinite” value can.

• This gives the technical definition of an infinite set: it is a set where there exists a one-to-one correspondence between the set itself and a proper subset.

Example

Let \( \mathbb{N} \) be the set of integers greater than 0. Clearly, \( \mathbb{N} - \{1\} \) is a proper subset of \( \mathbb{N} \). We can create a one-to-one correspondence between these two sets by matching each element \( x \in \mathbb{N} \) with element \( x + 1 \in \mathbb{N} - \{1\} \). Therefore, \( \mathbb{N} \) is an infinite set.

Countable Infinity

• Once we have an infinite set, we can prove another set infinite by creating a one-to-one correspondence between the known-infinite set and a subset (possibly the whole set) of the other set.

• For example, the set of all integers \( \mathbb{Z} \) contains \( \mathbb{N} \), which is obviously in one-to-one correspondence with \( \mathbb{N} \) itself, so \( \mathbb{Z} \) is infinite, too.

• Surprisingly, \( \mathbb{Z} \) and \( \mathbb{N} \) are actually equipotent. For example, a one-to-one correspondence between \( \mathbb{N} \) and \( \mathbb{Z} \) matches any \( x \in \mathbb{N} \) to \((x \div 2)\) if \( x \) is odd and to \(- (x \div 2)\) if \( x \) is even.

• Similarly, the \( \mathbb{Z} \) is equipotent with the set of even integers.

• Even more surprising, the set \( \mathbb{N} \) is equipotent with the set of pairs of positive integers:

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• Therefore, the set of rational numbers \( \mathbb{Q} \) is also equipotent with \( \mathbb{Z} \) and \( \mathbb{N} \), since every rational number can be represented as a pair of integers.

• Many common infinite sets are equipotent with the set of integers. This cardinality is written \( \aleph_0 \) (pronounced “aleph zero”), and a set with this cardinality is said to be \textit{countably infinite} because we can put its elements in one-to-one correspondence with \( \mathbb{N} \).

\textbf{Uncountable Infinity}

• Clearly the set \( \mathbb{R} \) of real numbers is infinite, since it contains all the integers. Is it countably infinite?

• \( \mathbb{R} \) is equipotent with the set of real numbers between 0 and 1 (or any other interval) by the following construction:

Mathematically, the one-to-one correspondence maps any real number \( x \) to \( y = (\arctan(x) + (\pi/2))/\pi \), which is always between 0 and 1, and inversely, maps any real number \( y \in (0,1) \) to \( x = \tan(\pi y - (\pi/2)) \).

• Suppose there exists a one-to-one correspondence between the real numbers from 0 to 1 and \( \mathbb{N} \):
\begin{tabular}{|c|c|}
\hline
\textbf{n} & \textbf{Decimal Representation} \\
\hline
1 & 1 1 2 3 5 \ldots \\
2 & 1 4 1 5 9 \ldots \\
3 & 0 1 9 6 7 \ldots \\
4 & 9 9 9 9 9 \ldots \\
5 & 1 2 3 4 5 \ldots \\
\vdots & \vdots \\
\hline
\end{tabular}

- We can always generate another real number not on the list. Therefore, no one-to-one correspondence exists.

- Therefore, there are more real numbers than there are integers. Sets with cardinality greater than \( \aleph_0 \) are said to be \emph{uncountably infinite}.

**Proving and Disproving Equipotency**

- Two sets are equipotent if \textbf{there exists} a one-to-one correspondence. If you find a one-to-one correspondence between to sets, you have proven them equipotent. If you can’t find a one-to-one correspondence, you neither proven nor disproven anything.

- To disprove equipotence, you must prove that no one-to-one correspondence is possible. The diagonalization technique given above is one way to do this.

- Alternatively, if you can prove one set countably infinite and the other set uncountably infinite, you’ve also proven that the two sets are not equipotent.