CS109A Notes for Lecture 3/10/95

Why Study Infinite Sets?

- Occasionally useful — sometimes in CS you reason about infinite sequences of events or other infinite things.
- Intellectually challenging.
- Fun and interesting.
- Something you're expected to know.

Counting and Cardinality

- The *cardinality* of a set is the number of elements in that set.
- Two sets are *equipotent* if and only if they have the same cardinality.
- The existence of a one-to-one correspondence between two sets proves that they are equipotent.
- Counting is really just creating a one-to-one correspondence between a set and the set of integers from 1 to some number n.

Example

$\square$	1
$\diamond$	2
$\triangle$	3
$\bigcirc$	4
$\heartsuit$	5

Finite and Infinite Sets

- Can you create a one-to-one correspondence between a set and a proper subset of itself? If so, you have a solution to the equation $x = x + y$, where x is the cardinality of the set and $y \geq 1$ is the cardinality of the stuff you left out.

- No **finite** x can satisfy that equation, but an "infinite" value can.

- This gives the technical definition of an infinite set: it is a set where there exists a one-to-one correspondence between the set itself and a proper subset.

Example

Let $\mathbf{N}$ be the set of integers greater than 0. Clearly, $\mathbf{N} - \{1\}$ is a proper subset of $\mathbf{N}$. We can create a one-to-one correspondence between these two sets by matching each element $x \in \mathbf{N}$ with element $x + 1 \in \mathbf{N} - \{1\}$. Therefore, $\mathbf{N}$ is an infinite set.

Countable Infinity

- Once we have an infinite set, we can prove another set infinite by creating a one-to-one correspondence between the known-infinite set and a subset (possibly the whole set) of the other set.

- For example, the set of all integers $\mathbf{Z}$ contains $\mathbf{N}$, which is obviously in one-to-one correspondence with $\mathbf{N}$ itself, so $\mathbf{Z}$ is infinite, too.

- Surprisingly, $\mathbf{Z}$ and $\mathbf{N}$ are actually equipotent. For example, a one-to-one correspondence between $\mathbf{N}$ and $\mathbf{Z}$ matches any $x \in \mathbf{N}$ to $(x \mathrm{div} 2)$ if x is odd and to $-(x \mathrm{div} 2)$ if x is even.

- Similarly, the $\mathbf{Z}$ is equipotent with the set of even integers.

- Even more surprising, the set $\mathbf{N}$ is equipotent with the set of pairs of positive integers:

$$
\begin{array}{|lllll}
 & & \vdots & & \\
(1,4) & (2,4) & (3,4) & (4,4) & \\
(1,3) & (2,3) & (3,3) & (4,3) & \dots \\
(1,2) & (2,2) & (3,2) & (4,2) & \\
(1,1) & (2,1) & (3,1) & (4,1) & \\ \hline
\end{array}
$$

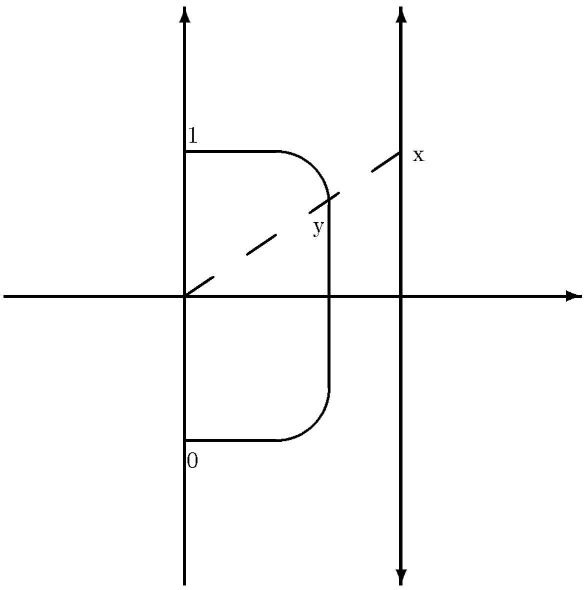
- Therefore, the set of rational numbers **Q** is also equipotent with **Z** and **N**, since every rational number can be represented as a pair of integers.

- Many common infinite sets are equipotent with the set of integers. This cardinality is written $\aleph_0$ (pronounced "aleph zero"), and a set with this cardinality is said to be *countably infinite* because we can put its elements in one-to-one correspondence with **N**.

Uncountable Infinity

- Clearly the set $\mathcal{R}$ of real numbers is infinite, since it contains all the integers. Is it countably infinite?

- $\mathcal{R}$ is equipotent with the set of real numbers between 0 and 1 (or any other interval) by the following construction:



Mathematically, the one-to-one correspondence maps any real number x to $y = (\text{Arctan}(x) + (\pi/2))/\pi$, which is always between 0 and 1, and inversely, maps any real number $y \in (0, 1)$ to $x = \tan(\pi y - (\pi/2))$.

- Suppose there exists a one-to-one correspondence between the real numbers from 0 to 1 and **N**:

n	Decimal Representation					
1	1	1	2	3	5	...
2	1	4	1	5	9	...
3	0	1	9	6	7	...
4	9	9	9	9	9	...
5	1	2	3	4	5	...
$\vdots$			$\vdots$			

- We can always generate another real number not on the list. Therefore, no one-to-one correspondence exists.

- Therefore, there are more real numbers than there are integers. Sets with cardinality greater than $\aleph_0$ are said to be *uncountably infinite.*

Proving and Disproving Equipotency

- Two sets are equipotent if **there exists** a one-to-one correspondence. If you find a one-to-one correspondence between to sets, you have proven them equipotent. If you can't find a one-to-one correspondence, you neither proven nor disproven anything.

- To disprove equipotence, you must prove that no one-to-one correspondence is possible. The diagonalization technique given above is one way to do this.

- Alternatively, if you can prove one set countably infinite and the other set uncountably infinite, you've also proven that the two sets are not equipotent.