Decision Properties of Regular Languages

Given a (representation, e.g., RE, FA, of a) regular language L, what can we tell about L?

- Since there are algorithms to convert between any two representations, we can choose the rep that makes the test easiest.

Membership

Is string w in regular language L?

- Choose DFA representation for L.
- Simulate the DFA on input w.

Emptiness

Is $L = \emptyset$?

- Use DFA representation.
- Use a graph-reachability algorithm to test if at least one accepting state is reachable from the start state.

Finiteness

Is L a finite language?

- Note every finite language is regular (why?), but a regular language is not necessarily finite.

DFA method:

- Given a DFA for L, eliminate all states that are not reachable from the start state and all states that do not reach an accepting state.
- Test if there are any cycles in the remaining DFA; if so, L is infinite, if not, then L is finite.

RE method: Almost, we can look for a $*$ in the RE and say its language is infinite if there is one, finite if not. However, there are exceptions, e.g. $\mathbf{0}\epsilon^*\mathbf{1}$ or $\mathbf{0}^*\emptyset$. Thus:

1. Find subexpressions equivalent to $\emptyset$ by:
   - (Basis) $\emptyset$ is; ϵ and $\mathbf{a}$ are not.
   - (Induction) $E+F$ is iff both E and F are; EF is if either E or F are; E^* never is.
2. Eliminate subexpressions equivalent to $\emptyset$ by:
   - Replace $E+F$ or $F+E$ by F whenever E is and F isn't.
   - Replace E^* by ϵ whenever E is equivalent to $\emptyset$.

3. Now, find subexpressions that are equivalent to ϵ by:

   - ✥ (Basis) ϵ is; **a** isn't.
   - ✥ (Induction) $E+F$ is iff both E and F are; ditto EF; E^* is iff E is.

4. Now, we can tell if $L(R)$ is infinite by looking for a subexpression E^* such that E is not equivalent to ϵ.

Example

Consider $(\mathbf{0} + \mathbf{1}\emptyset)^* + \mathbf{1}\emptyset^*$.

- Step 1: $\emptyset$ (twice) and $\mathbf{1}\emptyset$ are subexpressions equivalent to $\emptyset$.
- Step 2: $\mathbf{0}^* + \mathbf{1}\epsilon$ remains.
- Step 3: only subexpression ϵ is equivalent to ϵ.
- Since **0** is starred, language is infinite.

Minimization of States

- Real goal is testing equivalence of (reps of) two regular languages.
- Interesting fact: DFA's have unique (up to state names) minimum-state equivalents.
  - ✥ But proof in course reader doesn't quite get to that point.

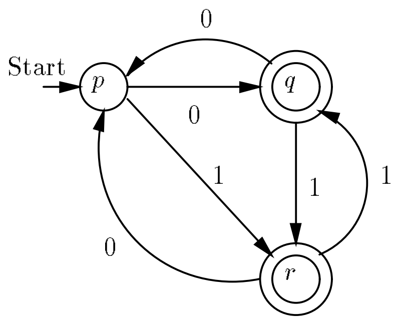
Distinguishable States

Key idea: find states p and q that are *distinguishable* because there is some input w that takes exactly one of p and q to an accepting state.

- Basis: any nonaccepting state is distinguishable from any accepting state ($w = \epsilon$).
- Induction: p and q are distinguishable if there is some input symbol a such that $\delta(p, a)$ is distinguishable from $\delta(q, a)$.
  - ✥ All other pairs of states are indistinguishable, and can be merged into one state.

Example (Very Simple)

Consider:



- p is distinguishable from q and r by basis.

Can we distinguish q from r?

- No string beginning with 0 works, because both states go to p, and therefore any string of the form $0x$ takes q and r to the same state.

- No string beginning with 1 works.

  - ✥ Technically, $\delta(q, 1) = r$ and $\delta(r, 1) = q$ are not distinguishable. Thus, induction does not tell us q and r are distinguishable.

  - ✥ What happens is that, starting in either q or r, as long as we have inputs 1, we are in one of the accepting states, and when a 0 is read, we go to the same state forever after.

Constructing the Minimum-State DFA

- For each group of indistinguishable states, pick a "representative."

  - ✥ Note a group can be large, e.g., $q_1, q_2, \ldots, q_k$, if all pairs are indistinguishable.

  - ✥ Indistinguishability is transitive (why?) so indistinguishability partitions states.

- If p is a representative, and $\delta(p, a) = q$, in minimum-state DFA the transition from p on a is to the representative of q's group (to q itself if q is either alone in a group or a representative).

- State state is representative of the original start state.

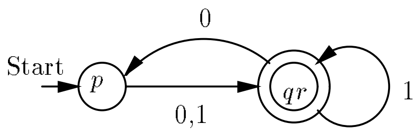
- Accepting states are representatives of groups of accepting states.

  - ✥ Notice we could not have a "mixed" (accepting + nonaccepting) group (why?).

- Delete any state that is not reachable from the start state.

Example

For the DFA above, p is in a group by itself; $\{q, r\}$ is the other group.



Why Above Minimization Can't be Beaten

Suppose we have a DFA A, and we minimize it to construct a DFA M. Yet there is another DFA N that accepts the same language as A and M, yet has fewer states than M. Proof contradiction that this can't happen:

- Run the state-distinguishability process on the states of M and N together.
- Start states of M and N are indistinguishable because $L(M) = L(N)$.
- If $\{p, q\}$ are indistinguishable, then their successors on any one input symbol are also indistinguishable.
- Thus, since neither M not N could have an inaccessible state, every state of M is indistinguishable from at least one state of N.
- Since N has fewer states than M, there are two states of M that are indistinguishable from the same state of N, and therefore indistinguishable from each other.
- But M was designed so that all its states *are* distinguishable from each other.
- We have a contradiction, so the assumption that N exists is wrong, and M in fact has as few states as any equivalent DFA for A.
- In fact (stronger), there must be a 1-1 correspondence between the states of any other minimum-state N and the DFA M, showing that the minimum-state DFA for A is unique up to renaming of the states.