Ambiguous Grammars

A CFG is *ambiguous* if one or more terminal strings have multiple leftmost derivations from the start symbol.

- Equivalently: multiple rightmost derivations, or multiple parse trees.

Example

Consider $S \rightarrow AS \mid \epsilon$; $A \rightarrow A1 \mid 0A1 \mid 01$.
The string 00111 has the following two leftmost deriviations from S:

1. $S \underset{lm}{\Rightarrow} AS \underset{lm}{\Rightarrow} 0A1S \underset{lm}{\Rightarrow} 0A11S \underset{lm}{\Rightarrow} 00111S \underset{lm}{\Rightarrow} 00111$

2. $S \underset{lm}{\Rightarrow} AS \underset{lm}{\Rightarrow} A1S \underset{lm}{\Rightarrow} 0A11S \underset{lm}{\Rightarrow} 00111S \underset{lm}{\Rightarrow} 00111$

- Intuitively, we can use $A \rightarrow A1$ first or second to generate the extra 1.

Inherently Ambiguous Languages

A CFL L is *inherently ambiguous* if *every* CFG for L is ambiguous.

- Such things exist; see course reader.

Example

The language of our example grammar is not inherently ambiguous, even though the grammar is ambiguous.

- Change the grammar to force the extra 1's to be generated last.

  $S \rightarrow AS \mid \epsilon$
  $A \rightarrow 0A1 \mid B$
  $B \rightarrow B1 \mid 01$

Why Care?

- Ambiguity of the grammar implies that at least some strings in its language have different structures (parse trees).
  - Thus, such a grammar is unlikely to be useful for a programming language, because two structures for the same string (program) implies two different meanings (executable equivalent programs) for this program.
  - Common example: the easiest grammars for arithmetic expressions are ambiguous and need to be replaced by more complex,

unambiguous grammars (see course reader).

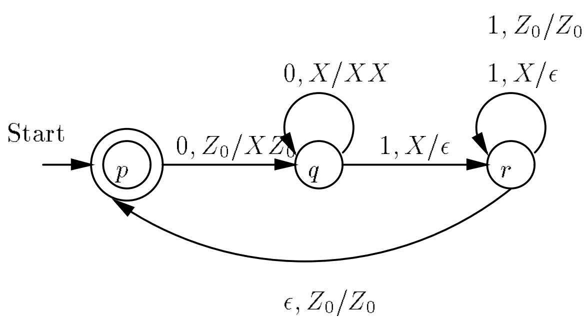
- An inherently ambiguous language would be absolutely unsuitable as a programming language, because we would not have any way of fixing a unique structure for all its programs.

Pushdown Automata

- Add a stack to a FA.
- Typically nondeterministic.
- An automaton equivalent to CFG's.

Example

Notation for "transition diagrams": $a, Z/X_1X +$ $2 \cdots X_k$ = "on input a, with Z on top of the stack, consume the a, make this state transition, and replace the Z on top of the stack by $X_1X_2 \cdots X_k$ (with X_1 at the top).



- p = starting to see a group of 0's and 1's; q = reading 0's and pushing X's onto the stack; r = reading 1's and popping X's until the X's are all popped.
- We can start a new group (transition from r to p) only when all X's (which count the 0's) have been matched against 1's.

Formal PDA

$P = (Q, \Sigma, , , \delta, q_0, Z_0, F)$, where Q, Σ, q_0, and F have their meanings from FA.

- , = stack alphabet.
- Z_0 in , = start symbol = the one symbol on the stack initially.
- δ = transition function takes a state, an input symbol (or ϵ), and a stack symbol and gives you a finite number of choices of:

1. A new state (possibly the same).
2. A string of stack symbols to replace the top stack symbol.

Instantaneous Descriptions (ID's)

For a FA, the only thing of interest about the FA is its state. For a PDA, we want to know its state and the entire content of its stack.

- It is also convenient to maintain a fiction that there is an input string waiting to be read.
- Represented by an ID (q, w, α), where q = state, w = waiting input, and α = stack, top left.

Moves of the PDA

If $\delta(q, a, X)$ contains (p, α), then $(q, aw, X\beta) \vdash (p, w, \alpha\beta)$.

- Extend to $\overset{*}{\vdash}$ to represent 0, 1, or many moves.
- Subscript by name of the PDA, if necessary.
- Input string w is accepted if $(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, \gamma)$ for any accepting state p and any stack string γ.
- $L(P)$ = set of strings accepted by P.

Example

$(p, 0110011, Z_0) \vdash (q, 110011, XZ_0) \vdash (r, 10011, Z_0) \vdash (r, 0011, Z_0) \vdash (p, 0011, Z_0) \vdash (q, 011, XZ_0) \vdash (q, 11, XXZ_0) \vdash (r, 1, XZ_0) \vdash (r, \epsilon, Z_0) \vdash (p, \epsilon, Z_0)$

Acceptance by Empty Stack

Another one of those technical conveniences: when we prove that PDA's and CFG's accept the same languages, it helps to assume that the stack is empty whenever acceptance occurs.

- $N(P)$ = set of strings w such that $(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ for some state p.
  - ❖ Note p need not be in F.
  - ❖ In fact, if we talk about $N(P)$ only, then we need not even specify a set of accepting states.

Example

For our previous example, to accept by empty stack:

1. Add a new transition $\delta(p, \epsilon, Z_0) = \{(p, \epsilon)\}$.
   - ❖ That is, when starting to look for a new 0-1 block, the PDA has the option to pop the last symbol off the stack instead.
2. p is no longer an accepting state; in fact, there *are* no accepting states.

Equivalence of Acceptance by Final State and Empty Stack

A language is $L(P_1)$ for some PDA P_1 if and only if it is $N(P_2)$ for some PDA P_2.

- Given $P_1 = (Q, \Sigma, , , \delta, q_0, Z_0, F)$, construct P_2:
  1. Introduce new start state p_0 and new bottom-of-stack marker X_0.
  2. First move of P_2: replace X_0 by Z_0X_0 and go to state q_0. The presence of X_0 prevents P_2 from “accidentally” emptying its stack and accepting when P_1 did not accept.
  3. Then, P_2 simulates P_1; i.e., give P_2 all the transitions of P_1.
  4. Introduce a new state r that keeps popping the stack of P_2 until it is empty.
  5. If (the simulated) P_1 is in an accepting state, give P_2 the additional choice of going to state r on ϵ input, and thus emptying its stack without reading any more input.
- Given $P_2 = (Q, \Sigma, , , \delta, q_0, Z_0, F)$, construct P_1:
  1. Introduce new start state p_0 and new bottom-of-stack marker X_0.
  2. First move of P_1: replace X_0 by Z_0X_0 and go to state q_0.
  3. Introduce new state r for P_1; it is the only accepting state.
  4. P_1 simulates P_2.
  5. If (the simulated) P_1 ever sees X_0, it knows P_2 accepts, so P_1 goes to state r on ϵ input.