Theory of LSH

Distance Measures

LS Families of Hash Functions

S-Curves

Distance Measures

- Generalized LSH is based on some kind of "distance" between points.
  - Similar points are "close."
- Two major classes of distance measure:
  1. *Euclidean*
  2. *Non-Euclidean*

Euclidean Vs. Non-Euclidean

- A *Euclidean space* has some number of real-valued dimensions and "dense" points.
  - There is a notion of "average" of two points.
  - A *Euclidean distance* is based on the locations of points in such a space.
- A *Non-Euclidean distance* is based on properties of points, but not their "location" in a space.

Axioms of a Distance Measure

- *d* is a *distance measure* if it is a function from pairs of points to real numbers such that:
  1. $d(x,y) \geq 0$.
  2. $d(x,y) = 0$ iff $x = y$.
  3. $d(x,y) = d(y,x)$.
  4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

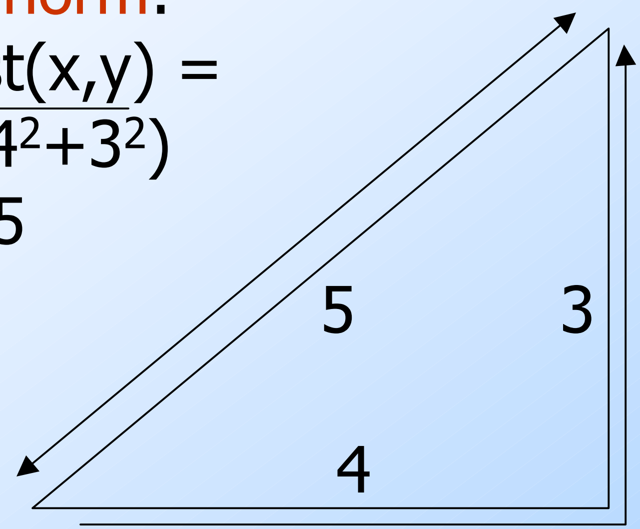
Some Euclidean Distances

- *L_2 norm* : d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.
  - The most common notion of "distance."
- *L_1 norm* : sum of the differences in each dimension.
  - *Manhattan distance* = distance if you had to travel along coordinates only.

Examples of Euclidean Distances

b = (9,8)

L_2-norm:
dist(x,y) =
$\sqrt{(4^2+3^2)}$
= 5



L_1-norm:
dist(x,y) =
4+3 = 7

a = (5,5)

Another Euclidean Distance

- L_∞ *norm* : d(x,y) = the maximum of the differences between *x* and *y* in any dimension.
- Note: the maximum is the limit as *n* goes to ∞ of the L_n norm: what you get by taking the n^{th} power of the differences, summing and taking the n^{th} root.

Non-Euclidean Distances

- *Jaccard distance* for sets = 1 minus Jaccard similarity.
- *Cosine distance* = angle between vectors from the origin to the points in question.
- *Edit distance* = number of inserts and deletes to change one string into another.
- *Hamming Distance* = number of positions in which bit vectors differ.

Jaccard Distance for Sets (Bit-Vectors)

- Example: p_1 = 10111; p_2 = 10011.
- Size of intersection = 3; size of union = 4, Jaccard similarity (not distance) = 3/4.
- d(x,y) = 1 – (Jaccard similarity) = 1/4.

Why J.D. Is a Distance Measure

- $d(x,x) = 0$ because $x \cap x = x \cup x$.
- $d(x,y) = d(y,x)$ because union and intersection are symmetric.
- $d(x,y) \geq 0$ because $|x \cap y| \leq |x \cup y|$.
- $d(x,y) \leq d(x,z) + d(z,y)$ trickier – next slide.

Triangle Inequality for J.D.

$$1 - \frac{|x \cap z|}{|x \cup z|} + 1 - \frac{|y \cap z|}{|y \cup z|} \geq 1 - \frac{|x \cap y|}{|x \cup y|}$$

- Remember: $|a \cap b|/|a \cup b|$ = probability that minhash(a) = minhash(b).
- Thus, $1 - |a \cap b|/|a \cup b|$ = probability that minhash(a) $\neq$ minhash(b).

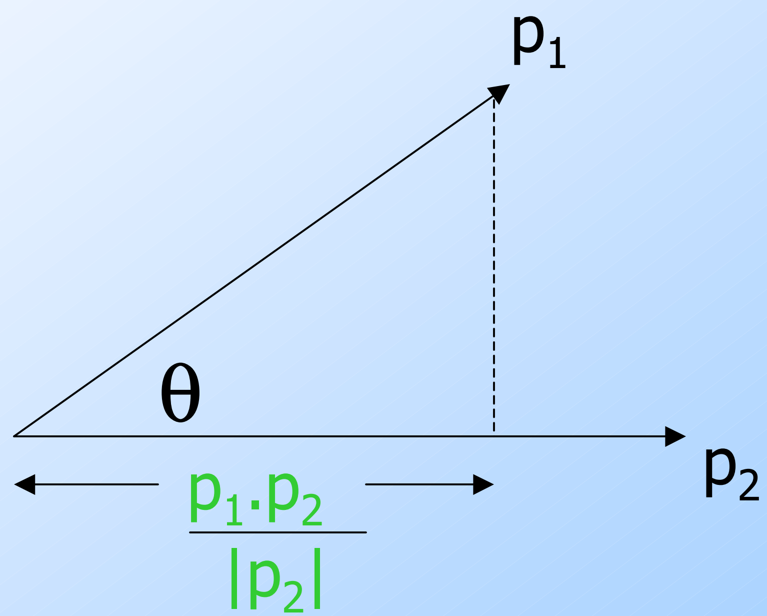
Triangle Inequality – (2)

- Claim: prob[minhash(x) ≠ minhash(y)] $\leq$ prob[minhash(x) ≠ minhash(z)] + prob[minhash(z) ≠ minhash(y)]
- Proof: whenever minhash(x) ≠ minhash(y), at least one of minhash(x) ≠ minhash(z) and minhash(z) ≠ minhash(y) must be true.

Cosine Distance

- Think of a point as a vector from the origin (0,0,…,0) to its location.
- Two points' vectors make an angle, whose cosine is the normalized dot-product of the vectors: $p_1.p_2/|p_2||p_1|$.
  - Example: p_1 = 00111; p_2 = 10011.
  - $p_1.p_2 = 2$; $|p_1| = |p_2| = \sqrt{3}$.
  - $\cos(\theta) = 2/3$; θ is about 48 degrees.

Cosine-Measure Diagram



$d(p_1, p_2) = \theta = \arccos(p_1.p_2/|p_2||p_1|)$

Why C.D. Is a Distance Measure

- d(x,x) = 0 because arccos(1) = 0.
- d(x,y) = d(y,x) by symmetry.
- d(x,y) $\geq$ 0 because angles are chosen to be in the range 0 to 180 degrees.
- Triangle inequality: physical reasoning. If I rotate an angle from *x* to *z* and then from *z* to *y*, I can't rotate less than from *x* to *y*.

Edit Distance

- The *edit distance* of two strings is the number of inserts and deletes of characters needed to turn one into the other. Equivalently:
- $d(x,y) = |x| + |y| - 2|LCS(x,y)|$.
  - LCS = *longest common subsequence* = any longest string obtained both by deleting from x and deleting from y.

Example: LCS

- x = *abcde* ; y = *bcduve*.
- Turn x into y by deleting *a*, then inserting *u* and *v* after *d*.
  - Edit distance = 3.
- Or, LCS(x,y) = *bcde*.
- Note: |x| + |y| - 2|LCS(x,y)| = 5 + 6 –2*4 = 3 = edit distance.

Why Edit Distance Is a Distance Measure

- d(x,x) = 0 because 0 edits suffice.
- d(x,y) = d(y,x) because insert/delete are inverses of each other.
- d(x,y) $\geq$ 0: no notion of negative edits.
- Triangle inequality: changing *x* to *z* and then to *y* is one way to change *x* to *y*.

Variant Edit Distances

- Allow insert, delete, and *mutate*.
  - Change one character into another.
- Minimum number of inserts, deletes, and mutates also forms a distance measure.
- Ditto for any set of operations on strings.
  - Example: substring reversal OK for DNA sequences

Hamming Distance

- *Hamming distance* is the number of positions in which bit-vectors differ.
- Example: p_1 = 10101; p_2 = 10011.
- $d(p_1, p_2) = 2$ because the bit-vectors differ in the 3rd and 4th positions.

Why Hamming Distance Is a Distance Measure

- d(x,x) = 0 since no positions differ.
- d(x,y) = d(y,x) by symmetry of "different from."
- d(x,y) $\geq$ 0 since strings cannot differ in a negative number of positions.
- Triangle inequality: changing *x* to *z* and then to *y* is one way to change *x* to *y*.

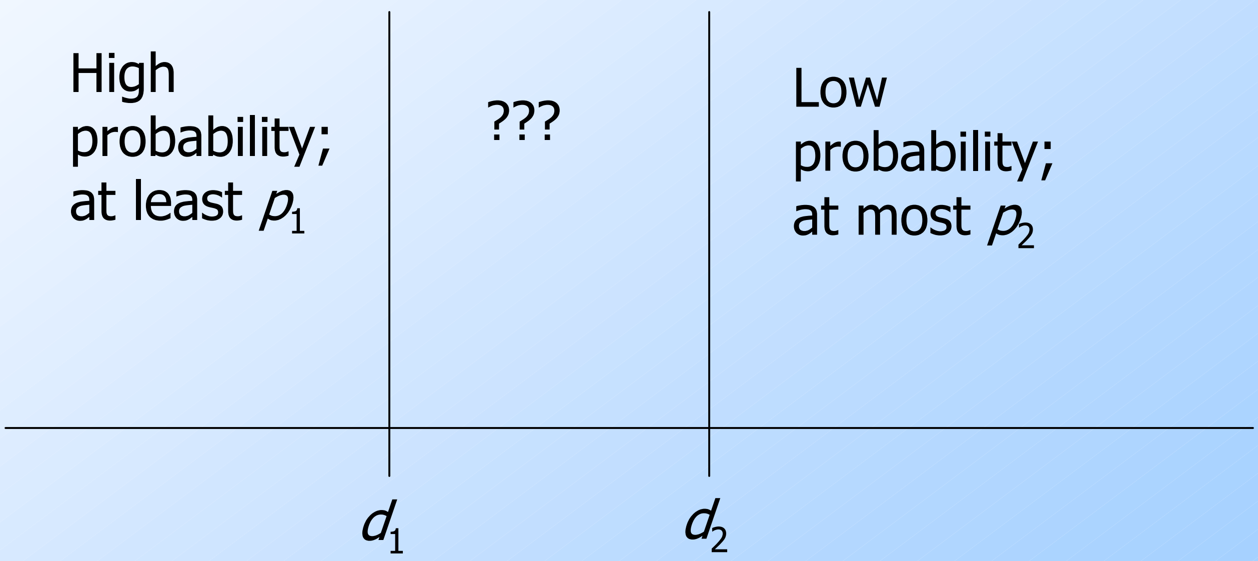
Families of Hash Functions

1. A "hash function" is any function that takes *two* elements and says whether or not they are "equal" (really, are candidates for similarity checking).
   - Shorthand: h(x) = h(y) means "*h* says *x* and *y* are equal."
2. A *family* of hash functions is any set of functions as in (1).

LS Families of Hash Functions

- Suppose we have a space *S* of points with a distance measure *d*.
- A family **H** of hash functions is said to be *(d_1,d_2,p_1,p_2)-sensitive* if for any *x* and *y* in *S* :
  1. If $d(x,y) \leq d_1$, then prob. over all *h* in **H**, that $h(x) = h(y)$ is at least p_1.
  2. If $d(x,y) \geq d_2$, then prob. over all *h* in **H**, that $h(x) = h(y)$ is at most p_2.

LS Families: Illustration



Example: LS Family

- Let *S* = sets, *d* = Jaccard distance, **H** is formed from the minhash functions for all permutations.
- Then Prob[h(x)=h(y)] = 1-d(x,y).
  - Restates theorem about Jaccard similarity and minhashing in terms of Jaccard distance.

Example: LS Family – (2)

- Claim: **H** is a (1/3, 2/3, 2/3, 1/3)-sensitive family for *S* and *d*.

If distance $\leq$ 1/3
(so similarity $\geq$ 2/3)

Then probability
that minhash values
agree is $\geq$ 2/3

Comments

1. For Jaccard similarity, minhashing gives us a $(d_1,d_2,(1\text{-}d_1),(1\text{-}d_2))$-sensitive family for any $d_1 < d_2$.
2. The theory leaves unknown what happens to pairs that are at distance between d_1 and d_2.
   - Consequence: no guarantees about fraction of false positives in that range.

Amplifying a LS-Family

- The "bands" technique we learned for signature matrices carries over to this more general setting.
- Goal: the "S-curve" effect seen there.
- AND construction like "rows in a band."
- OR construction like "many bands."

AND of Hash Functions

- Given family **H**, construct family **H'** consisting of *r* functions from **H**.
- For $h = [h_1,\ldots,h_r]$ in **H'**, h(x)=h(y) if and only if $h_i(x)=h_i(y)$ for all *i*.
- Theorem: If **H** is (d_1,d_2,p_1,p_2)-sensitive, then **H'** is $(d_1,d_2,(p_1)^r,(p_2)^r)$-sensitive.
- Proof: Use fact that h_i's are independent.

OR of Hash Functions

- Given family **H**, construct family **H'** consisting of b functions from **H**.
- For $h = [h_1,\ldots,h_b]$ in **H'**, h(x)=h(y) if and only if $h_i(x)=h_i(y)$ for some i.
- Theorem: If **H** is (d_1,d_2,p_1,p_2)-sensitive, then **H'** is $(d_1,d_2,1-(1-p_1)^b,1-(1-p_2)^b)$-sensitive.

Effect of AND and OR Constructions

- AND makes all probabilities shrink, but by choosing *r* correctly, we can make the lower probability approach 0 while the higher does not.
- OR makes all probabilities grow, but by choosing *b* correctly, we can make the upper probability approach 1 while the lower does not.

Composing Constructions

- As for the signature matrix, we can use the AND construction followed by the OR construction.
  - Or vice-versa.
  - Or any sequence of AND's and OR's alternating.

AND-OR Composition

- Each of the two probabilities *p* is transformed into $1-(1-p^r)^b$.
  - The "S-curve" studied before.
- Example: Take **H** and construct **H'** by the AND construction with $r = 4$. Then, from **H'**, construct **H''** by the OR construction with $b = 4$.

Table for Function $1-(1-p^4)^4$

p	$\mathbf{1-(1-p^4)^4}$
.2	.0064
.3	.0320
.4	.0985
.5	.2275
.6	.4260
.7	.6666
.8	.8785
.9	.9860

Example: Transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.8785,.0064)-sensitive family.

OR-AND Composition

- Each of the two probabilities *p* is transformed into $(1-(1-p)^b)^r$.
  - The same S-curve, mirrored horizontally and vertically.
- Example: Take **H** and construct **H'** by the OR construction with *b* = 4. Then, from **H'**, construct **H''** by the AND construction with *r* = 4.

Table for Function $(1-(1-p)^4)^4$

p	$\mathbf{(1-(1-p)^4)^4}$
.1	.0140
.2	.1215
.3	.3334
.4	.5740
.5	.7725
.6	.9015
.7	.9680
.8	.9936

Example: Transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9936,.1215)-sensitive family.

Cascading Constructions

- Example: Apply the (4,4) OR-AND construction followed by the (4,4) AND-OR construction.
- Transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9999996,.0008715)-sensitive family.
- Note this family uses 256 of the original hash functions.

General Use of S-Curves

- For each S-curve $1-(1-p^r)^b$, there is a *threshhold* *t*, for which $1-(1-t^r)^b = t$.
- Above *t*, high probabilities are increased; below *t*, they are decreased.
- You improve the sensitivity as long as the low probability is less than *t*, and the high probability is greater than *t*.
  - Iterate as you like.

Use of S-Curves – (2)

- Thus, we can pick any two distances $x < y$, start with a (x, y, (1-x), (1-y))-sensitive family, and apply constructions to produce a (x, y, p, q)-sensitive family, where p is almost 1 and q is almost 0.
- The closer to 0 and 1 we get, the more hash functions must be used.

LSH for Cosine Distance

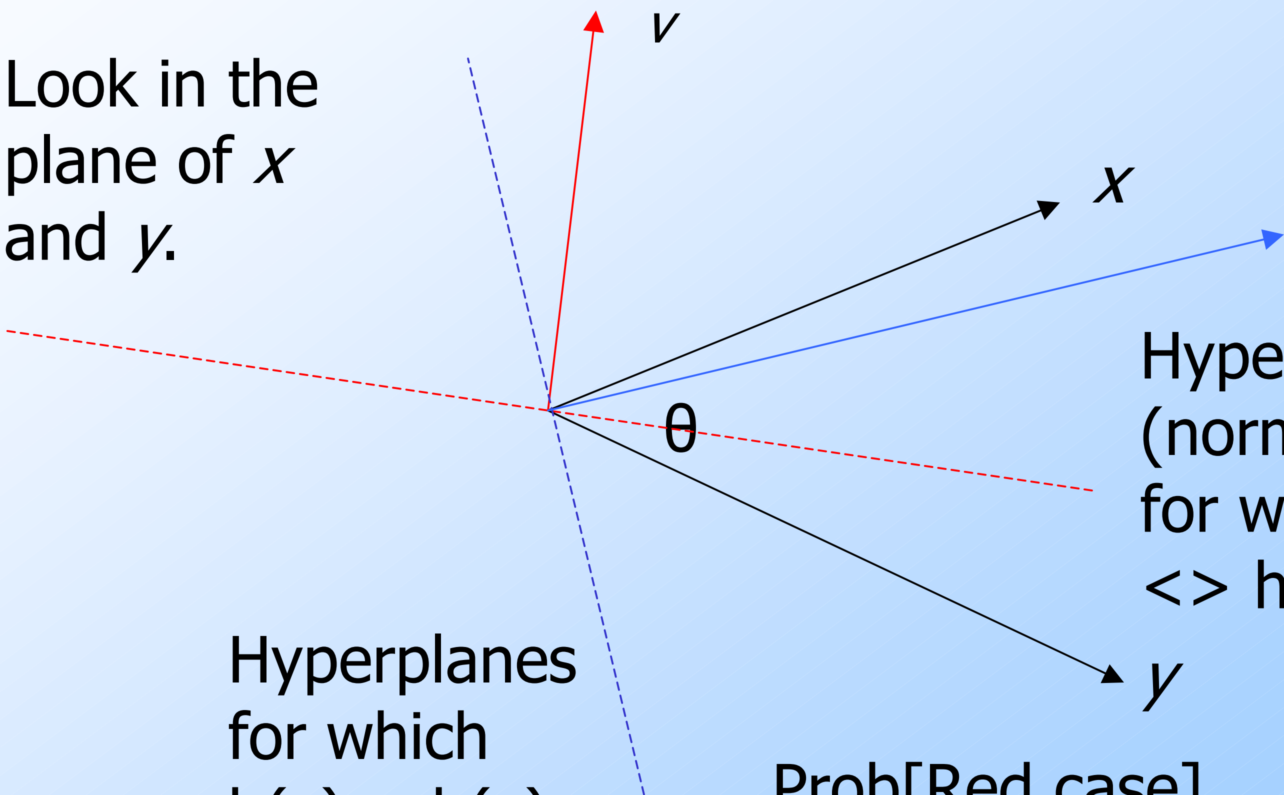
- For cosine distance, there is a technique analogous to minhashing for generating a $(d_1,d_2,(1\text{-}d_1/180),(1\text{-}d_2/180))$- sensitive family for any d_1 and d_2.
- Called *random hyperplanes*.

Random Hyperplanes

- Pick a random vector *v*, which determines a hash function h_v with two buckets.
- $h_v(x) = +1$ if $v.x > 0$; $= -1$ if $v.x < 0$.
- LS-family **H** = set of all functions derived from any vector.
- Claim: $Prob[h(x)=h(y)] = 1 –$ (angle between *x* and *y* divided by 180).

Proof of Claim

Look in the plane of *x* and *y*.



Hyperplanes (normal to *v*) for which h(x) <> h(y)

Hyperplanes for which h(x) = h(y)

Prob[Red case] = θ/180

Signatures for Cosine Distance

- Pick some number of vectors, and hash your data for each vector.
- The result is a signature (*sketch*) of +1's and –1's that can be used for LSH like the minhash signatures for Jaccard distance.
- But you don't have to think this way.
- The existence of the LS-family is sufficient for amplification by AND/OR.

Simplification

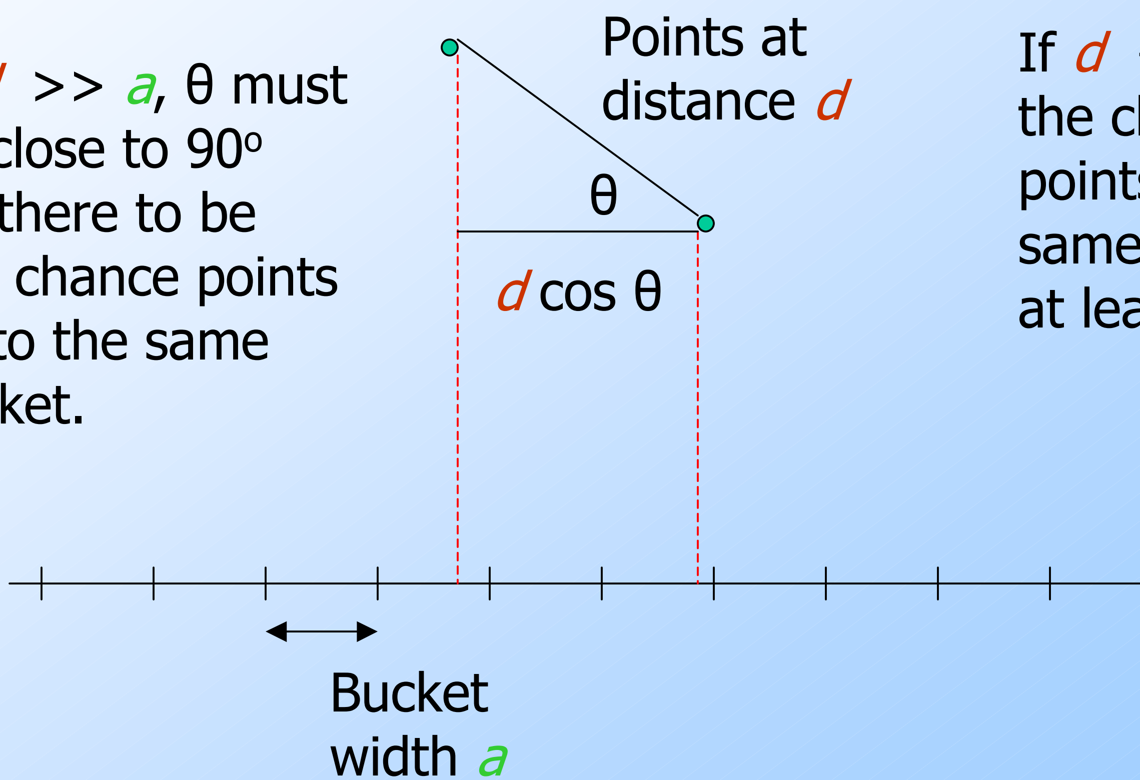
- We need not pick from among all possible vectors v to form a component of a sketch.
- It suffices to consider only vectors v consisting of +1 and –1 components.

LSH for Euclidean Distance

- Simple idea: hash functions correspond to lines.
- Partition the line into buckets of size *a*.
- Hash each point to the bucket containing its projection onto the line.
- Nearby points are always close; distant points are rarely in same bucket.

Projection of Points

If *d* >> *a*, θ must be close to 90º for there to be any chance points go to the same bucket.



If *d* << *a*, then the chance the points are in the same bucket is at least 1 – *d* / *a*.

An LS-Family for Euclidean Distance

- If points are distance $\geq 2a$ apart, then $60 \leq \theta \leq 90$ for there to be a chance that the points go in the same bucket.
  - I.e., at most 1/3 probability.
- If points are distance $\leq a/2$, then there is at least ½ chance they share a bucket.
- Yields a (a/2, 2a, 1/2, 1/3)-sensitive family of hash functions.

Fixup: Euclidean Distance

- For previous distance measures, we could start with an (x, y, p, q)-sensitive family for any $x < y$, and drive p and q to 1 and 0 by AND/OR constructions.
- Here, we seem to need $y \geq 4x$.

Fixup – (2)

- But as long as $x < y$, the probability of points at distance x falling in the same bucket is greater than the probability of points at distance y doing so.
- Thus, the hash family formed by projecting onto lines is an (x, y, p, q)-sensitive family for some $p > q$.
  - Then, amplify by AND/OR constructions.